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CONSISTENCY AND ASYMPTOTIC NORMALITY OF THE *ML* ESTIMATOR OF LIMITED DEPENDENT VARIABLE MODELS WITH HETEROSCEDASTICITY*

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I. INTRODUCTION

The limited dependent variable (LDV) model developed by Tobin (1958) and commonly known as Tobit model has been recognized in the literature as very useful for representation of certain types of economic situations. Specifically, the model can be used in situations when, in the context of a regression model, we do not observe the dependent variable over the entire range. Obviously for those values of the dependent variable which are unobservable, the dependent variable takes the value zero. The relationship between household expenditures on automobiles and household incomes, studied by Tobin himself, is an example of this kind of a situation. In this case the value of the dependent variable i.e., expenditure on automobiles is zero for all households not having any car. Obviously, ordinary least squares method of estimation will give biased and inconsistent estimators of the parameters of such a model. Tobin (1958) had in his original paper suggested using maximum likelihood method of estimation for obtaining consistent estimates. In fact, models of this type require different

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estimation procedures depending upon whether the sample data is censored or truncated.[1]

Tobit model has been extensively used by economists even though the formulation of the problem is quite often somewhat more complicated than the one considered by Tobin. Thus in the context of labour market, for instance, a person is in the labour force if the reservation wage is less than or equal to the market wage, otherwise he is not in the labour force. Some other recent examples of the use of LDV model in economics can be found in the works of Ashenfelter and Ham (1979), Reece (1979) and Adams (1980). Ashenfelter and Ham (1979) have, for example, used LDV model to explain the ratio of unemployed hours to employed hours in terms of years of schooling and working experience. Among others are Reece (1979) who has estimated charitable contributions as being dependent on price of contribution and income, and Adams (1980) who was interested in the relationship between inheritance on the one hand and income, marital status and number of children on the other.

In all these applications as well as in the theoretical studies by Cragg (1971), Amemiya (1973), Olsen (1978) etc., the disturbances in the regression equation have always been assumed to be homoscedastic. Obviously, this assumption restricts the scope of application of the model, particularly when one is working with cross-section data on microeconomic units or data in the form of grouped averages. In such cases, one has to tackle the problem of heteroscedasticity in the data. In the context of LDV models this is all the more important since it has been shown by Maddala and Nelson (1975) and later more explicitly by Hurd (1979) and Arabmazar and Schmidt (1981) that the maximum likelihood estimator (MLE)[2] which assumes homoscedasticity, produces inconsistent estimates when the disturbances are in fact heteroscedastic. Furthermore, Hurd (1979) and Arabmazar and Schmidt (1981) have shown, on the basis of a simple constant—term—only model and for two distinct variances, and under the assumption of normality, that the severity of the inconsistency would depend upon the degree of heteroscedasticity as also on whether the sample is truncated or censored and the degree of censoring, if the sample is censored.[3] And obviously, the usual test satisfies will in that case become invalid. For a good summary of these works one may see Amemiya (1984).

1. By a consored sample we mean a sample in which some observations on the dependent variable corresponding to some known values of the independent variables are not observable. A truncated sample, on the other hand, is one in which independent variables corresponding to unobservable values of the dependent variable are also not observed. Tobin really considered the case of truncated samples only.

2. By MLE we mean a root of the normal equations defined as a solution (of the normal equations) corresponding to a local maximum of the log-likelihood function.

3. Hurd (1979) has arrived at the same conclusion for another model where instead of only one constant term he considered one independent variable and a constant term.

In the light of these findings, it seems important to generalize the standard LDV model by explicitly incorporating heteroscedasticity in the disturbances and consider the problems of estimation and statistical inference in the generalized model. This paper aims at doing that. In other words, we have here considered a generalization of LDV model by explicitly introducing a quite general form of heteroscedasticity (in the disturbances) and then proved the strong consistency and asymptotic normality of the MLE of the parameters of this generalized model (to be henceforth referred to as heteroscedastic limited dependent variable (HLDV) model). This result is similar, though not obvious, to those of Amemiya (1973). As will be seen subsequently, the likelihood function for this model assumes a somewhat peculiar form and the standard theorems on the consistency and asymptotic normality of the MLE are not, in general, applicable. This paper provides a formal proof of the strong consistency and asymptotic normality of the MLE of the parameters of this generalized model. This will obviously enhance the usefulness of LDV models by its effective applicability to situations with heteroscedastic disturbances.

It may lastly be stated that although we have in this paper proved the results for a specific but very general assumption about the nature of heteroscedasticity (cf. section II), we have checked that these hold for some other standard forms of heteroscedasticity as well. We have, in fact, shown this for one such form towards the end of Section III.

So far as the actual estimation of HLDV model is concerned, one can use the standard methods of obtaining solutions to nonlinear equations. Alternatively, one can extend Amemiya's method of obtaining an initial consistent estimator of the parameter vector and then use this in Newton-Raphson method which has the property that the second-round estimator will have the same asymptotic distribution as a consistent root of the normal equations under general conditions.

II. THE HETEROSCEDASTIC LIMITED DEPENDENT VARIABLE (HLDV) MODEL AND THE ASSUMPTIONS

We define the HLDV model[4] as

$$\begin{aligned} y_t &= \beta_0' x_t + \epsilon_t \quad \text{if R.H.S.} > 0 \\ &= 0 \qquad\qquad\quad \text{otherwise} \qquad\qquad (t = 1, 2, \ldots, n) \end{aligned} \tag{1}$$

4. We can as well assume a more general model

$$\begin{aligned} y_t &= \beta' x_t + \epsilon_t \quad \text{if R.H.S.} > \alpha_t \\ &= 0 \qquad\qquad\quad \text{otherwise} \end{aligned}$$

where α_t's are known constants. But as Amemiya (1973) has noted, such a model can easily be analysed with slight modification where instead of y_t, x_t' and β' we now have

$$y_t^* = y_t - \alpha_t, \; x_t^{*\prime} = (x_t', \alpha_t) \text{ and } \beta^{*\prime} = (\beta_0', -1).$$

where x_i is a $(k \times 1)$ column vector of the ith observation on k fixed regressors, β_0' is the $(1 \times k)$ row vector of associated regression coefficients and ϵ_i's are independent disturbances following normal distributions with zero mean and variance σ_{0i}^2 $(i = 1, 2, \ldots, n)$. We make a very general assumption about the structure of σ_{0i}^2 :

$$\sigma_{0i}^2 = \sigma_0^2 (E(y_i))^{\delta_0} = \sigma_0^2 \mu_{0i}^{\delta_0}, \quad i = 1, 2, \ldots, n$$

where $\mu_{0i} = E(y_i) = \beta_0' x_i$ and σ_0^2 and δ_0 are unknown parameters.[5]

Let $\theta_0 = (\beta_0', \sigma_0^2, \delta_0)'$ denote the entire set of parameter vector, s the set of observations for which $y_i = 0$ i.e., $s = \{i : y_i = 0\}$ and $\bar{s}$ the complement set of s. We can then write, like Tobin and Amemiya, the log-likelihood function L as[6]

$$L = \text{const.} + \sum_{s} \ln (1 - F_i) - \frac{1}{2} \sum_{\bar{s}} \ln \sigma_i^2 - \frac{1}{2} \sum_{\bar{s}} \frac{(y_i - \beta' x_i)^2}{\sigma_i^2} \quad (2)$$

where $F_i = F_i(\beta' x_i, \sigma_i^2) = \int_{-\infty}^{\beta' x_i} \frac{1}{\sqrt{2\pi\sigma_i^2}} e^{-\frac{1}{2}(\alpha/\sigma_i)^2} d\alpha.$

We make the following assumptions :

Assumption 1. The parameter space Ψ of $\theta = (\beta', \sigma^2, \delta)'$ is compact. It does not contain the region $\sigma^2 \leqslant 0$, but contains an open neighbourhood of θ_0.

Assumption 2. x_i is bounded and the empirical distribution function, say H_n, defined as $H_n(x) = j/n$ where j is the number of points $x_1, x_2 \ldots, x_n$ less than or equal to x, converges to a distribution function, say H.

Assumption 3. $\lim_{n \to \infty} \frac{1}{n} \sum_{i=1}^{n} x_i x_i'$ is positive definite.

Assumption 4. μ_{0i}'s are positive and bounded away from zero.

Now, under these assumptions, we make use of the lemmas stated and proved in Jennrich (1969) and Amemiya (1973) and the expressions for the first two raw moments of truncated distributions to show, following Amemiya, that $(1/n)L$ converges *a.e.* uniformly for all θ in Ψ to Q defined as

5. One can obviously consider simpler forms for σ_{0i}^2; say, for instance, $\sigma_{0i}^2 = \sigma_0^2 m_i^{\delta_0}$ where m_i's are exogenously given [cf. Kmenta (1971)].

6. We are here considering the censored sample case only. The truncated sample case can be similarly tackled with minor changes in the algebra.

$$Q = \lim_{n\to\infty} \frac{1}{n} \sum_{1}^{n} \Big[\{\ln (1 - F_i)\} (1 - F_{0i}) - \tfrac{1}{2} (\ln \sigma_i^2) F_{0i}$$
$$- \frac{1}{2\sigma_i^2} \{((\beta_0 - \beta)' x_i)^2 F_{0i} + 2\sigma_{0i}^2 (\beta_0 - \beta)' x_i f_{0i}$$
$$+ \sigma_{0i}^2 (F_{0i} - \beta_0' x_i f_{0i})\} \Big]. \tag{3}$$

III. STRONG CONSISTENCY AND ASYMPTOTIC NORMALITY

We now prove the strong consistency and asymptotic normality of the maximum likelihood estimator under our assumption about the variance i.e.,

$\sigma_i^2 = \sigma^2 \mu_i^\delta$ where $\mu_i = \beta' x_i$, $i = 1, 2, \ldots, n$.

We make use of Assumptions 1, 2 and 4 and lemma 1 [Amemiya (1973), p. 1002] and evaluate the first-order derivatives of Q as

$$\frac{\partial Q}{\partial \beta} = \lim_{n\to\infty} \frac{1}{n} \sum_{1}^{n} \Big[- \frac{1 - F_{0i}}{1 - F_i} F_i x_i \Big(1 - \frac{\delta}{2}\Big) - \frac{\delta x_i F_{0i}}{2\mu_i}$$
$$+ \frac{1}{2\sigma^2} \mu_i^{-\delta} \{2(\beta_0 - \beta)' x_i F_{0i} x_i + 2\sigma_{0i}^2 x_i f_{0i}\}$$
$$+ \frac{\delta}{2\sigma^2} \mu_i^{-(\delta+1)} x_i \{((\beta_0 - \beta)' x_i)^2 F_{0i}$$
$$+ 2\sigma_{0i}^2 (\beta_0 - \beta)' x_i f_{0i} + \delta \sigma_{0i}^2 (F_{0i} - \beta_0' x_i f_{0i})\} \Big] \tag{4}$$

$$\frac{\partial Q}{\partial \sigma^2} = \frac{1}{2\sigma^2} \lim_{n\to\infty} \frac{1}{n} \sum_{1}^{n} \Big[\frac{1 - F_{0i}}{1 - F_i} \beta' x_i f_i - F_{0i}$$
$$+ \frac{1}{\sigma^2} \mu_i^{-\delta} \{((\beta_0 - \beta)' x_i)^2 F_{0i} + 2\sigma_{0i}^2 (\beta_0 - \beta)' x_i f_{0i}$$
$$+ \sigma_{0i}^2 (F_{0i} - \beta_0' x_i f_{0i})\} \Big] \tag{5}$$

and
$$\frac{\partial Q}{\partial \delta} = \lim_{n\to\infty} \frac{1}{n} \sum_{1}^{n} \Big[\frac{1 - F_{0i}}{1 - F_i} \frac{\ln \mu_i}{2} \beta' x_i f_i - \frac{1}{2} (\ln \mu_i) F_{0i}$$
$$+ \frac{\mu_i^{-\delta}}{2\sigma^2} \ln \mu_i \{((\beta_0 - \beta)' x_i)^2 F_{0i} + 2\sigma_{0i}^2 (\beta_0 - \beta)' x_i f_{0i}$$
$$+ \sigma_{0i}^2 (F_{0i} - \beta_0' x_i f_{0i})\} \Big] \tag{6}$$

From (4), (5) and (6) it easily follows that

$$\frac{\partial Q(\theta_0)}{\partial \theta} = 0.$$

Now, defining w_i as

$$w_i = \frac{\beta_0' x_i}{\sigma_{0i}} = \frac{\mu_{0i}}{\sigma_{0i}}$$

and writing the standard normal density and distribution functions evaluated at w_i as ϕ_i ($\equiv \sigma_{0i} f_{0i}$) and Φ_i ($\equiv F_{0i}$) respectively, we can show that the matrix of second-order derivatives is

$$\frac{\partial^2 Q(\theta_0)}{\partial\theta\partial\theta'} = -\lim_{n\to\infty} \frac{1}{n} \begin{pmatrix} \sum_1^n a_i x_i x_i' & \sum_1^n d_i x_i & \sum_1^n e_i x_i \\ \sum_1^n d_i x_i' & \sum_1^n b_i & \sum_1^n k_i \\ \sum_1^n e_i x_i' & \sum_1^n k_i & \sum_1^n c_i \end{pmatrix}$$

$$\overset{\text{def}}{=} -A, \text{ say,}$$

where

$$\left.\begin{aligned}
a_i &= -\frac{1}{\sigma_{0i}^2}\left(\left\{1-\frac{\delta_0}{2}\right)^2\left(w_i\phi_i - \frac{\phi_i^2}{1-\Phi_i}\right) - \left(\frac{\delta_0\phi_i}{w_i} + \Phi_i + \frac{\delta_0^2\Phi_i}{2w_i^2} - \frac{\delta_0^2\phi_i}{4w_i}\right)\right\} \\
b_i &= -\frac{1}{4\sigma_0^4}\left(w_i^3\phi_i + w_i\phi_i - \frac{w_i^2\phi_i^2}{1-\Phi_i} - 2\Phi_i\right) \\
c_i &= -\left(\frac{\ln \mu_{0i}}{2}\right)^2\left(w_i^3\phi_i + w_i\phi_i - \frac{w_i^2\phi_i^2}{1-\Phi_i} - 2\phi_i\right) \\
d_i &= \frac{1}{2\sigma_0^2\sigma_{0i}}\left\{\left(1-\frac{\delta_0}{2}\right)\left(w_i^2 - \phi_i \frac{w_i\phi_i^2}{1-\Phi_i} + \phi_i\right) + \frac{\delta_0\Phi_i}{w_i}\right\} \\
e_i &= \frac{\ln \mu_{0i}}{2\sigma_{0i}}\left\{\left(1-\frac{\delta_0}{2}\right)\left(w_i^2\phi_i - \frac{w_i\phi_i^2}{1-\Phi_i} + \phi_i\right) + \frac{\delta_0\Phi_i}{w_i}\right\}
\end{aligned}\right\} \quad (7)$$

and $k_i = \dfrac{\ln \mu_{0i}}{4\sigma_0^2}\left(\dfrac{w_i^2\phi_i^2}{1-\Phi_i} + 2\Phi_i - w_i\phi_i - w_i^3\phi_i\right)$.

We next show that $\partial Q(\theta_0)/\partial\theta\,\partial\theta'$ is negative definite. For this purpose we prove that A is positive definite (p.d.).

Now, for any k-component vector p and scalars q and r, $P'AP$ can be reduced to

$$P'AP = \lim_{n\to\infty} \frac{1}{n} \sum_{1}^{n} S_i' B_i S_i \tag{8}$$

where $P' = (p', q, r)$, $S_i' = (p'x_i, q, r)$ and

$$B_i = \begin{pmatrix} a_i & d_i & e_i \\ d_i & b_i & k_i \\ e_i & k_i & c_i \end{pmatrix}$$

Now, in order to show A to be p.d., let us first study B_i.

LEMMA 1. *Under Assumptions 1, 2 and 4, B_i is positive semidefinite for all i.*

Proof. We write a_i, b_i, c_i, d_i, e_i and k_i as

$$\left.\begin{aligned} a_i &= -\frac{\overline{a_i}}{\sigma_{0i}^2},\ b_i = -\frac{\overline{b_i}}{4\sigma_0^4},\ c_i = -\left(\frac{\ln \mu_{0i}}{2}\right)^2 \overline{b_i}, \\ d_i &= \frac{\overline{d_i}}{2\sigma_{0i}\sigma_0^2},\ e_i = \frac{(\ln \mu_{0i})\,\overline{d_i}}{2\sigma_{0i}} \text{ and } k_i = -\frac{(\ln \mu_{0i})\,\overline{b_i}}{4\sigma_0^2} \end{aligned}\right\} \tag{9}$$

where $\overline{a_i}$, $\overline{b_i}$ and $\overline{d_i}$ are accordingly defined. Now, from (7) and (9)

$$\begin{aligned} \overline{q_i} &= \left(1 - \frac{\delta_0}{2}\right)^2 \left(w_i\phi_i - \frac{\phi_i^2}{1-\Phi_i}\right) - \left(\frac{\delta_0\phi_i}{w_i} + \Phi_i + \frac{\delta_0^2\Phi_i}{2w_i^2} - \frac{\delta_0^2\phi_i}{4w_i}\right) \\ &= \left(1 - \frac{\delta_0}{2}\right)^2 \left\{\phi_i\left(w_i - \frac{\phi_i}{1-\Phi_i}\right)\right\} - \frac{\delta_0^2}{4w_i^2}(2\Phi_i - w_i\phi_i) \\ &\quad - \left(\frac{\delta_0\phi_i}{w_i} + \Phi_i\right). \end{aligned}$$

Now, $\phi_i/1 - \Phi_i > w_i$ (if w_i is bounded and this is ensured by Assumptions 2 and 4)[7] and $(2\Phi_i - w_i\phi_i) > 0$.[8]

It is obvious that $(\delta_0\phi_i/w_i + \Phi_i)$ is positive for $\delta_0 > 0$ since by Assumption 4, $w_i > 0$. For $\delta_0 < 0$ also, it can be seen that $(\delta_0\phi_i/w_i + \Phi_i) > 0$.[9] It thus

7. See Feller (1972), p. 175 or, Amemiya (1973), p. 1007.

8. $2\Phi_i - w_i\phi_i \to 0$ as $w_i \to -\infty$ and its derivative with respect to w_i i.e., $\phi_i + w_i^2\phi_i$ is always positive. Hence $2\Phi_i - w_i\phi_i$ is positive.

9. $\delta_0\phi_i/w_i + \Phi_i \to 0$ as $w_i \to -\infty$ for all finite values of δ_0 and its derivative with respect to w_i i.e., $-\delta_0\phi_i(1 + 1/w_i^2) + \phi_i$ is positive for $\delta_0 < 0$. Hence $\delta_0\phi_i/w_i + \Phi_i$ is positive for all $\delta_0 < 0$.

follows that a_i is negative and hence a_i is positive.

Again, since $(w_i\phi_i - 2\Phi_i) < 0$ (cf. footnote 8), we have

$$\overline{b_i} = \left\{w_i^2\phi_i\left(w_i - \frac{\phi_i}{1-\Phi_i}\right) + (w_i\phi_i - 2\Phi_i)\right\} < 0$$

and therefore from (9), both b_i and c_i are positive.

Let us now look at the second-order principal minors. $\begin{vmatrix} a_i & d_i \\ d_i & b_i \end{vmatrix}$ can, after substituting the values for a_i, b_i and d_i from (7), be reduced to

$$\frac{D_i}{4\sigma_0^4\sigma_{Ui}^2}(w_i^2\Phi_i + 2\phi_i + w_i\Phi_i) + \frac{1}{4\sigma_0^4\sigma_{Ui}^2}(2\Phi_i^2 - \phi_i^2 - w_i\phi_i\Phi_i) \tag{10}$$

where $D_i = \phi_i\left(\frac{\phi_i}{1-\Phi_i} - w_i\right) > 0$, (cf. footnote 7).

Since $(w_i^2\Phi_i + 2\Phi_i + w_i\phi_i)$ is obviously positive and $(2\Phi_i^2 - \phi_i^2 - w_i\phi_i\Phi_i)$ is also positive[10], $\begin{vmatrix} a_i & d_i \\ d_i & b_i \end{vmatrix} > 0$.

One can similarly check that all other second-order principal minors are either greater than or equal to zero. It is also clear that $|B_i| = 0$. Hence B_i is positive semidefinite for all i. Q.E.D.

LEMMA 2. *Under Assumptions 1 through 4, $\partial 2Q(\theta)/\partial\theta\,\partial\theta'$ is negative definite.**

Proof. We have from (8),

$$P'AP = \lim_{n\to\infty} \frac{1}{n}\sum_{i=1}^{n} S_i'B_iS_i \tag{11}$$

Now, $S_i'B_iS_i$ can be written as

$$S_i'B_iS_i = \sum_{k=1}^{3} \lambda_{ik}\gamma_{ik}^2 \quad [\textit{vide}\ \text{Rao (1974), page 40}] \tag{12}$$

where λ_{i1}, λ_{i2} and λ_{i3} are the three characteristic roots of B_i, γ_{ik}'s are the ele-

10. $2\Phi_i^2 - \phi^2 - w_i\phi_i\Phi_i \to 0$ as $w_i \to -\infty$. Its derivative with respect to w_i is $3\phi_i\Phi_i + w_i\phi^2 + w^2\phi_i\Phi_i$, and clearly this is positive.

*The author is grateful to the referee for his suggestions leading to an improvement in the proof of this lemma.

ments of the vector $T_i'S_i$, T_i' is the transpose of the matrix of characteristic vectors for β_i corresponding to the three roots.

Now, since $a_i > 0$, $b_i > 0$ and $c_i > 0$,

$$\lambda_{i_1} + \lambda_{i_2} + \lambda_{i_3} = \text{trace}\,(B_i) > 0. \tag{13}$$

But each of λ_{i1}, λ_{i2} and $\lambda_{i3} \geqslant 0$ (as B_i is positive semidefinite), and hence it follows from (13) that at least one of the characteristic roots is greater than zero. Thus, we have from (12), $S_i'B_iS_i > 0$ for $\gamma_i = (\gamma_{i1}, \gamma_{i2}, \gamma_{i3})' \neq 0$, $i = 1, 2, \ldots, n$. But obviously because of Assumption 3, γ_i cannot be a null vector for all $i = 1, 2, \ldots, n$ unless $P = 0$.

It therefore follows from (11) that $P'AP > 0$ for $P \neq 0$ and hence the result.

Q.E.D.

Since $\partial^2 Q/\partial\theta\,\partial\theta'$ is continuous, Lemma 2 implies there is a closed set

$$G(\theta_0) = \{\theta : |\theta - \theta_0| \leqslant \eta\},\ G \in \Psi$$

such that $\partial^2 Q/\partial\theta\,\partial\theta'$ is negative definite for all θ in $G(\theta_0)$.

We can, therefore, following Amemiya (1973, pp. 1008-1010), state the following theorem:

THEOREM. *Under Assumptions 1 to 4, the normal equations have a strongly consistent root, say $\hat{\theta}_n$, and*

$$\sqrt{n}\,(\hat{\theta}_n - \theta_0) \rightarrow N\left[0, \left(-\frac{\partial^2 Q(\theta_0)}{\partial\theta\,\partial\theta'}\right)^{-1}\right].$$

One can, as a special case, consider a simpler structure of σ_i^2 viz., $\sigma_i^2 = \sigma_i^2 m_i^2$, m_i's are exogenously given [cf. Kmenta (1971)]. To prove strong consistency and asymptotic normality of the MLE under this structure for the variance, μ_{0i} in Assumption 4 is to be replaced by m_i.

The expression for $\partial Q/\partial\beta$ will now be

$$\frac{\partial Q}{\partial\beta} = \lim_{n\to\infty} \frac{1}{n} \sum_1^n \left[-\frac{1 - F_{0i}}{1 - F_i} f_i x_i + \frac{m_i^{-2}}{\sigma^2} \{(\beta_0 - \beta)' x_i F_{0i} x_i + \sigma_{0i}^2 f_{0i} x_i\} \right]$$

and those for $\partial Q/\partial\sigma^2$ and $\partial Q/\partial\delta$ will remain the same as given in (5) and (6) with μ_i's being replaced by m_i. Obviously, we then have $\partial Q(\theta_0)/\partial\theta = 0$ for this case also.

As for the second-order derivatives and hence the matrices A and B_i, a_i, d_i, and e_i will now change to

$$a_i = -\frac{1}{\sigma_{0i}^2}\left(w_i\phi_i - \frac{\phi_i^2}{1 - \Phi_i} - \Phi_i\right)$$

$$d_i = \frac{1}{2\sigma_{0i}\,\sigma_0^2}\left(w_i^2\phi_i + \phi_i - \frac{w_i\phi_i^2}{1 - \Phi_i}\right)$$

and $e_i = \frac{\ln m_i}{2\sigma_{0i}} \left(w_i^2 \phi_i + \phi_i - \frac{w_i \phi_i^2}{1 - \Phi_i} \right)$,

and b_i, c_i and k_i will remain the same as given in (7) excepting for m_i replacing μ_{0i}. Following exactly the same types of derivations and results it can easily be checked that the theorem holds for this assumption of the structure for variance as well.

IV. CONCLUSIONS

In this paper we have considered a generalization of the limited dependent variable model originally due to Tobin by incorporating heteroscedasticity. We have proved the strong consistency and asymptotic normality of the maximum likelihood estimator of this generalized model for a very general form of heteroscedastic structure viz., $\sigma_i^2 = \sigma^2(E(y_i))^2$. It can easily be seen that the theorem holds for other standard heteroscedastics structures as well. In this paper we have, in fact, checked this for $\sigma_i^2 = \sigma^2 m_i^2$.

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