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On Material Balances When There Are Capacity Limitations

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A method for the computation of a consistent set of material balances is proposed when there are capacity limitations in domestic production and exports, and it is related to certain planning practices in East European countries as reported by Montias. Although the computational procedure solves a linear programming problem, it has nothing to do with standard linear programming algorithms. Instead it is a direct generalization of the input–output type of balancing method. *J. Comp. Econ.*, June 1984, 8(2), pp. 159–167. Indian Statistical Institute, New Delhi, India.

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In this paper we propose a method for the computation of a consistent set of material balances when there are capacity limitations in domestic production and exports. The underlying model is due to Chenery (1953) who used it for studying the industrial structure of Italy. The formal properties of this model were first reported by Arrow (1954), who also provided a rigorous justification for the computational procedure or algorithm used by Chenery for solving the model. In some variant or other, the model has also cropped up a number of times in the literature on short-term economy-wide planning in East European countries. See, for example, Montias (1962) and Kalecki (1963).

We carry out some further analysis of this model and offer an alternative computational procedure. We also touch upon certain planning practices in East European countries as reported by Montias and relate these to the proposed procedure. Although both the Arrow–Chenery algorithm and the one to be proposed here solve linear-programming problems, these have nothing to do with the standard linear-programming algorithms. Instead, these are direct generalizations of the input–output type of methods.

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We begin with a description of the model and note some of its analytic properties. The task of describing the model can be split into three parts dealing, respectively, with (domestic) production, foreign trade (or trade for short), and the planning objective.

As mentioned earlier the production conditions are assumed to be described by a simple input–output model with specific capacity limitations on the production of each sector, n in number. There are no other primary resources in the model. That is, there is a single activity for producing each commodity i ($i = 1, \ldots, n$) with one upper bound, say $\bar{x}_i > 0$, on the *permissible level* of its operation. We denote the input matrix by $\bar{A} = (\bar{a}_{ij})$ and the net output matrix by $A = I - \bar{A}$. The production vector (also called production program) is denoted by x.

As for trade, it is assumed that each good can be exported or imported at given respective export and import prices (expressed in some common foreign-exchange unit), with a ceiling on the permissible export levels representing the maximum world-market demand for the respective products from the country considered.[2] No parallel restrictions are imposed on imports so that the country is assumed to face a competitive world market so far as the supply of the import requirements is concerned. The import price of each commodity is taken to be larger than or equal to its export price. The excess represents the trade transport margin (the export prices are f.o.b. and the import prices c.i.f.). Finally, the production and trade conditions are connected by the crucial assumption that each sector has a nonnegative value added at international prices regardless of the actual source of its inputs and destination of its output. Since import prices are larger than or equal to export prices this amounts to the assumption that the export price of each good is greater than or equal to the unit intermediate input cost of its production when all the inputs are valued at their respective import prices. Formally, by letting p^e and p^m be the vectors of export and import prices the two assumptions boil down to

$$p^m \geqq p^e \geqq p^m \bar{A}. \tag{H}$$

We shall assume throughout that $\bar{A}$ is indecomposable. Assumption (H) then implies that $\bar{A}$ is a productive input–output matrix.

We shall denote the export and import vectors by x^e and x^m, respectively, and the given vector of export ceilings by $\bar{x}^e \geqq 0$.

We now turn to the specification of the planning objective, which is of an extremely simple form. The domestic final use of each commodity i, denoted by $\bar{y}_i$, is treated as a fixed target and the only flexibility is with regard to

[2] The defense put forward by Arrow for this assumption is as follows: "This is an approximation to a declining demand curve, and, in view of the wide-spread quantitative restrictions on imports, may have a certain degree of realism. Some such assumption is widely made by planning authorities."

trade balance $p^e x^e - p^m x^m$. The trade balance is assumed to be maximized subject to the constraints imposed by the given targets regarding domestic final use, the given capacities of production, and the given export ceilings.

The model may be fully expressed by means of the linear-programming problem[3]:

(P) maximize $(p^e x^e - p^m x^m)$, subject to

$$Ax + x^m - x^e = \bar{y}, \tag{1}$$

$$x \leqq \bar{x}, \tag{2}$$

$$x^e \leqq \bar{x}^e, \tag{3}$$

and

$$x, x^m, x^e \geqq 0. \tag{4}$$

Note that (P) always has a feasible solution ($x^m = \bar{y}$, $x = x^e = 0$) and the value of its objective function on the set of feasible solutions is bounded above. Hence the existence of a solution of (P) is guaranteed.

The constraints of (P) can be, however, simplified by recognizing that in view of (H) one can always find a solution of (P) where no good is simultaneously exported and imported. Hence $x_i^m = 0$ whenever $x_i^e > 0$, and $x_i^e = 0$ whenever $x_i^m > 0$, so that defining

$$z = \bar{x}^e - x^e + x^m \geqq 0, \tag{5}$$

we can simply read off (x^e, x^m) from z by the rule

$$\begin{aligned} (x_i^e, x_i^m) &= (\bar{x}_i^e - z_i, 0) \qquad \text{if} \qquad z_i \leqq \bar{x}_i^e \\ &= (0, z_i - \bar{x}_i^e) \qquad \text{otherwise.} \end{aligned} \tag{6}$$

Hence writing $d = \bar{y} + \bar{x}^e$, (1), (3), and (4) can be rewritten as

$$Ax + z = d \tag{7}$$

$$(x, z) \geqq 0. \tag{8}$$

Henceforth, we shall speak of (x, z) as being a feasible solution of (P) if it satisfies (2), (7), and (8), it being understood that z refers to (x^e, x^m) as defined in (6). We call z and d the *trade vector* (or trade program) and the *maximum demand vector*, respectively.

A third simple result concerning the optimum solution of (P) follows from the first inequality in (H). Suppose that (x, z) is a feasible solution of (P), with $x_i < \bar{x}_i$. Then the production of commodity i can certainly be increased by importing all the intermediate inputs for some additional output. If x_i^e

[3] In Arrow's paper, the objective function is stated as the minimization of trade deficit rather than the maximization of trade surplus and (1) is stated in a weak inequality form with the left-hand side greater than or equal to the right-hand side. However, as proved by Arrow, one can always find a solution of (P) with equalities in (1) and this is the form we start with here.

$< \bar{x}_i^e$, then some additional output can be entirely exported resulting in a positive or at least zero trade surplus by the assumption just referred to. It follows that one cannot find an optimum solution of (P) where in some sector there is simultaneously an excess capacity (i.e., $\bar{x}_i > x_i$) and an unutilized potential for increasing export (i.e., $z_i > 0$). Hence for the optimum solution of (P) one need only search over (x, z) satisfying

$$z_i(\bar{x}_i - x_i) = 0 \qquad (i = 1, \ldots, n). \tag{9}$$

We shall refer to (9) as the *optimality condition* and to (2), (7), (8), and (9) as defining the *optimum system*. Notice that the optimum system does not involve the export and import prices. Arrow proves that the optimum system has a unique feasible solution, which must therefore be the optimum solution of (P). We give here an alternative and simple proof of this result by using some well-known properties of input–output systems.[4]

THEOREM 1. *The optimum program of production and trade is independent of export and import prices so long as these satisfy* (H).

Proof. We show that the optimum system has a unique feasible solution. Suppose this is not true. Let (x, z) and (x', z') be two feasible solutions, then (7) implies

$$A(x' - x) = -(z' - z). \tag{10}$$

The optimality condition (9) implies

$$\text{if} \quad x'_i > x_i \quad \text{then} \quad z_i = 0 \leqq z'_i \quad (\text{since } x'_i > x_i \Rightarrow x_i < \bar{x}_i)$$

$$\text{if} \quad x'_i < x_i \quad \text{then} \quad z'_i = 0 \leqq z_i \quad (\text{since } x'_i < x_i \Rightarrow x'_i < \bar{x}_i).$$

This means $(x'_i - x_i)(z'_i - z_i) \geqq 0$ for $i = 1, \ldots, n$, i.e., the vectors $(x' - x)$ and $(z' - z)$ have the same sign. Equality (10) then means that the matrix A reverses the sign of the vector $(x' - x)$. But this is a contradiction to the assumption that A is productive (see Theorem 3 of Parthasarathy, 1983). Hence the theorem.

At this point, it is convenient to introduce some notation. We shall denote the set of sectors in the model by N, identifying each sector by a positive integer (its number) ranging from 1 to n, i.e., $N = \{1, 2, \ldots, n\}$. For any $J \subset N$, $\hat{J}$ will denote the complementary set $\{i \in N: i \notin J\}$, matrix A_{JJ} will denote the restriction of A to its rows and columns whose indices are in J and $\hat{J}$, respectively, arranged in any definite order. Given any $x \in R^n$, x_J and $x_{\hat{J}}$ will denote the restrictions of x to the components in $\{i \in N: i \in J\}$ and $\{i \in N: i \in \hat{J}\}$, respectively, arranged in the same order as the rows and

[4] This proof was suggested to the author by an anonymous referee of this journal.

columns of $A_{\hat{J}\hat{J}}$. Hence, given any $J \subset N$ and $x \in R^n$, the product Ax can be written after a suitable rearrangement of its rows and columns as

$$\begin{pmatrix} A_{JJ} A_{J\hat{J}} \\ A_{\hat{J}J} A_{\hat{J}\hat{J}} \end{pmatrix} \begin{pmatrix} x_J \\ x_{\hat{J}} \end{pmatrix} = \begin{pmatrix} A_{JJ}x_J + A_{J\hat{J}}x_{\hat{J}} \\ A_{\hat{J}J}x_J + A_{\hat{J}\hat{J}}x_{\hat{J}} \end{pmatrix}.$$

Returning to the analysis, let (x, z) be the solution of the optimum system, and K the set of sectors operated at full capacity in this solution. I shall denote by K and $\hat{K}$ the sets of *bottleneck* and *free sectors*, respectively. Then it follows from (9) that $z_{\hat{K}} = 0$ while $x_K = \bar{x}_K$, by definition of K. Hence from (7)

$$A_{KK}\bar{x}_K + A_{K\hat{K}}x_{\hat{K}} = d_K - z_K,$$

$$A_{\hat{K}K}\bar{x}_K + A_{\hat{K}\hat{K}}x_{\hat{K}} = d_{\hat{K}},$$

that is,

$$z_K = d_K - A_{KK}\bar{x}_K - A_{K\hat{K}}x_{\hat{K}}, \tag{11}$$

$$x_{\hat{K}} = A_{\hat{K}\hat{K}}^{-1}(d_{\hat{K}} - A_{\hat{K}K}\bar{x}_K). \tag{12}$$

Thus once K is known, the solution of the optimum system is fully determined. In economic terms, output in the bottleneck sectors is *capacity-determined* and in the free sectors *demand-determined*, where demand includes the input requirements for the products of the free sectors in the bottleneck sectors, besides the maximum final demand. The computational procedure proposed here works upon this basic principle and sets out to find K in an iterative fashion, simultaneously with (x, z).

Mathematically, the procedure consists of constructing a sequence of vectors $\{x^t, z^t\}$ through the following sequential relations. Given x^t, z^t is computed from

$$z^t = d - Ax^t \tag{13}$$

and a set J^t defined there from

$$J^t = \{i \in N: z_i^t > 0\}. \tag{14}$$

And x^{t+1} is then computed from

$$x_J^{t+1} = \bar{x}_J, \tag{15}$$

$$x_{\hat{J}}^{t+1} = A_{\hat{J}\hat{J}}^{-1}(d_{\hat{J}} - A_{\hat{J}J}\bar{x}_J), \tag{16}$$

where we have put $J^t = J$ and $\hat{J}^t = \hat{J}$ for notational simplicity. The procedure is initiated by putting

$$x^0 = \bar{x} \tag{17}$$

and is terminated at step T satisfying

$$J^T = J^{T-1}. \tag{18}$$

Thus the proposed solution at the terminal step is (x^T, z^T) as determined by (15), (16), and (13).

In economic terms it is seen that J^t and $\hat{J}^t$ are precisely the sets of bottleneck and free sectors, respectively, at step $(t + 1)$. The procedure begins with all sectors being treated as bottleneck sectors, and finds which of these have exports in excess of the prescribed ceilings as indicated by a negative component in the trade vector. These sectors are then taken to be free and their outputs are determined by solving the input–output subsystem consisting of the free sectors, treating the input requirements of their products in the bottleneck sectors as part of their final demands. Since the excess exports of these sectors are cut out, their output levels are smaller than their previous levels, with a corresponding reduction in the flow of inputs from the bottleneck sectors to the free sectors. This entails a larger availability of the outputs of bottleneck sectors for final uses and gives rise to the emergence of fresh excess exports, the corresponding sectors being then taken to be free for the next round, and so on. The process continues till there are no excess exports. These assertions are borne out, inter alia, in the following algebraic discussion.

To begin with, it is easily seen that the sequence is well defined, for A being a productive input–output matrix, each principal submatrix of it is also a productive input–output matrix by the Hawkin–Simon condition. Hence A_{JJ} has a nonnegative inverse. We now prove the convergence of our procedure.

THEOREM 2. *The computational procedure as defined by Eqs. (13)–(17) converges in a finite number of steps $T \leqq n$.*

LEMMA. *There exists a finite T such that $J^T = J^{T-1}$, that is, (18) is satisfied.*

Proof. Advancing the step index in (13) by one and retaining only those equations in it with indices belonging to $\hat{J}^t$, one obtains

$$A_{\hat{J}^t\hat{J}^t} x^{t+1}_{\hat{J}^t} + A_{\hat{J}^t J^t} x^{t+1}_{J^t} = d_{\hat{J}^t} - z^{t+1}_{\hat{J}^t}. \tag{19}$$

From (15), (16), and (19) it follows at once that

$$z^{t+1}_{\hat{J}^t} = 0. \tag{20}$$

Hence if $z^t_j \leqq 0$ then $z^{t+1}_j = 0$, i.e.,

$$\hat{J}^{t-1} \subset \hat{J}^t \qquad \text{which implies that} \qquad J^{t-1} \supset J^t. \tag{21}$$

Since J^t is a finite set (containing at most n elements) which contracts at each step, it must become stationary at some finite t. This proves the lemma.

Since J^t uniquely determines (x^{t+1}, z^{t+1}) (see (15), (16), and (13)), the lemma implies $(x^{T+1}, z^{T+1}) = (x^T, z^T)$.

Proof of Theorem 2. By the Lemma above, there exists a finite T such that $J^{T-1} = J^T$ and $(x^{T+1}, z^{T+1}) = (x^T, z^T)$. We show that (x^{T+1}, z^{T+1}) satisfies

(2), (7), (8), and (9), i.e., (x^{T+1}, z^{T+1}) is the feasible solution of the optimal system.

It follows from (13) that (x^t, z^t) satisfies (7) at each t. From (15) and (20) it follows that the optimality condition (9) is also satisfied at each t (since $i \in J^{t-1} \Rightarrow x_i^t = \bar{x}_i$ and $i \notin J^{t-1} \Rightarrow z_i^t = 0$). Next, since $A_{\bar{J}J} \leqq 0$ and $A_{\bar{J}\bar{J}}^{-1} \geqq 0$, it follows from (16) that $x_{\bar{J}}^t \geqq 0$, hence $x^t \geqq 0$, for all t. As for z^T, since $J^{T-1} = J^T$, $z_i^{T+1} \leqq 0$ only if $z_i^T \leqq 0$. But as in (20), $z_i^{T+1} = 0$ for all i such that $z_i^T \leqq 0$. Hence $z^{T+1} \geqq 0$. Thus (x^{T+1}, z^{T+1}) satisfies (8). It remains to prove that x^{T+1} satisfies (2). For this, one notes from (19), putting back the step-index to t but keeping to $J = J^t$ that

$$x_{\bar{J}}^t = A_{\bar{J}\bar{J}}^{-1}(d_{\bar{J}} - A_{\bar{J}J}x_J^t) - A_{\bar{J}\bar{J}}^{-1}z_{\bar{J}}^t$$

$$\geqq A_{\bar{J}\bar{J}}^{-1}(d_{\bar{J}} - A_{\bar{J}J}x_J^t) \text{ (since } z_{\bar{J}}^t \leqq 0 \text{ by (14) and } A_{\bar{J}\bar{J}}^{-1} \geqq 0)$$

$$\geqq A_{\bar{J}\bar{J}}^{-1}(d_{\bar{J}} - A_{\bar{J}J}\bar{x}_J) \text{ (by (15) and (21)).}$$

Thus,

$$x_{\bar{J}}^t \geqq x_{\bar{J}}^{t+1} \text{ (by using (16)).} \quad (22)$$

This proves that $\{x^t\}$ is a nonincreasing sequence. Since $x^0 = \bar{x}$, x^t must satisfy (2) for all t. This completes the proof.

Thus the computational procedure as defined by (13)–(18) indeed computes the solution of (P) in a finite number of steps. Computationally, the major task in the procedure is the inversion of a sequence of input–output matrices of an increasing order, for $\bar{J}^t$ expands with each step. Clearly, the smaller the number of sectors in $\bar{J}^T$, the smaller the computational task, for the matrices to be inverted are then of a smaller order and the convergence is also speedier. Also, it appears possible to improve the speed of convergence if there is some a priori knowledge about the set of bottleneck sectors. Thus if J^0 is an initial guess regarding the set of bottleneck sectors, then the algorithm can begin straightaway with (15) and (16)—with $t = 0$ and $J = J^0$—instead of with (17). Since J^t contracts at each step and (17) was required only to prove that $x_J^T = \bar{x}_J$, this modified procedure will also terminate with a solution (x, z), provided that $K \subset J^0$. Thus, overestimates of K in the initial "guess" can always be corrected in the subsequent steps, though not underestimates. This imparts a certain flexibility in the initiation of the procedure although at this stage this hinges entirely upon an a priori judgment.

We compare the present procedure with the original Arrow–Chenery algorithm which represents, in some sense, a polar opposite approach. It suffices to quote Arrow on this

> At the initial stage, we set the output of each industry equal to its capacity or the final demand for its product, including maximum possible exports (i.e., our maximum final demand), whichever is smaller. At each subsequent stage, the output of each industry is expanded to meet the derived demand of the previous round until capacity limits are reached; any remaining demand is met from imports. (Arrow, 1954)

Thus, the sequence of production vectors x^t in the Arrow–Chenery algorithm approaches the solution vector x *from below* with a corresponding expansion of the set of *bottleneck sectors* over the steps. However, the sequence is an infinite one, converging asymptotically, and does not make any direct use of (11) and (12). That is, it goes on comparing successive rounds of desired demand and raising the output levels even after the correct partition of N into K and $\bar{K}$ has been obtained. In fact, the Arrow–Chenery algorithm is a direct generalization of the iterative method for solving an input-output system and dispenses with the necessity of computing an inverse matrix at any stage.[5] However, the attractiveness of this dispensation is largely a matter of the sizes of the matrices to be inverted which in turn depends directly upon the size of $\hat{J}^T$. Hence the Arrow–Chenery algorithm appears to have an advantage when $\hat{J}^T$ is large. However, the Arrow–Chenery algorithm will generally have a slower speed of convergence not only because the convergence is asymptotic, but also because the set of bottleneck sectors does not necessarily change at every step even before K is obtained—it also lacks any flexibility regarding initiation. This is as far as it appears possible to compare the two algorithms on a priori grounds short of specific numerical trials.

Finally, it is interesting to note that the proposed procedure appears to resemble a certain approach to physical planning at the national level, as reported by Montias (1959, 1962) and by Kalecki (1963) in their work on short-term planning in Poland.[6] Speaking about the method of material balances in the presence of bottleneck problems and foreign trade, treated exogenously, Montias (1962), for example, writes

> From experience, or by means of trial and error methods, the planners should have a definite notion as to which sectors present bottleneck problems and which do not. The planner will generally operate the bottleneck sectors at full capacity, if the supply of other exogenous inputs (including foreign exchange) permits.

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[5] Algebraically, $\{x^t\}$ is constructed as follows:

$$x_i^0 = d_i \quad \text{if} \quad d_i < \bar{x}_i$$
$$= \bar{x}_i \quad \text{otherwise}$$
$$x_i^{t+1} = x_i^t + \sum_{j=1}^{n} \bar{a}_{ij}(x_j^t - x_j^{t-1}) \qquad \text{if this is less than } \bar{x}_i$$
$$= \bar{x}_i \quad \text{otherwise.}$$

[6] This may be true also in the case of certain other East European countries. See, for example, Hare (1981) and Cave and Hare (1981) who give a most penetrating account of the operation of the Hungarian economy.

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