# A Polynomial Algorithm for Testing the Nonnegativity of Principal Minors of Z-Matrices

K. G. Ramamurthy
*Indian Statistical Institute*
*7, S.J.S.S. Marg*
*New Delhi-110016, India*

Submitted by Richard A. Brualdi

ABSTRACT

A Z-matrix is a square matrix with nonpositive off-diagonal elements. We give a polynomial algorithm for testing the nonnegativity of principal minors of Z-matrices.

## 1. INTRODUCTION

Real square matrices with nonnegative (positive) principal minors are called *$P_0$-matrices* (*P-matrices*). A real square matrix with nonpositive off-diagonal elements is called a Z-matrix. In this paper, we use the terminology of Berman and Plemmons [2, Chapter 6] with regard to $M$-matrices. An *M-matrix* is a Z-matrix which is also a $P_0$-matrix. A square matrix $A$ is an $M$-matrix if and only if it can be written in the form $A = sI - B$, with $B \geq 0$ and $s \geq \rho(B)$, the spectral radius of $B$. A nonsingular $M$-matrix is a $P$-matrix, and its inverse is a nonnegative matrix.

In Section 2, we introduce the notation and briefly discuss certain required preliminary results. In Section 3, we consider the problem of testing whether a given Z-matrix is also an $M$-matrix. A polynomial algorithm is proposed for this purpose. The proposed algorithm is essentially based on certain known results of the *linear complementarity problem* and Lemke's algorithm.

## 2. NOTATION AND PRELIMINARIES

All the vectors are column vectors unless explicitly stated otherwise. The components of a vector $x \in R^n$ are denoted by $x_1, x_2, \ldots, x_n$. The notation




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$x \geqq 0$ means $x_j \geqq 0$ for all $j$, $x \geq 0$ means $x \geqq 0$ and $x \neq 0$, and $x > 0$ means $x_j > 0$ for all $j$. We shall denote by $e$ the vector in $R^n$ whose components are all equal to unity. Let $A$ be an $m \times n$ matrix. We shall denote by $A'$, $a_{ij}$, and $A_{i\cdot}$ and $A_{\cdot j}$, respectively its transpose, $(i, j)$th element, $i$th row, and $j$th column. For any nonempty subsets $\alpha \subseteq \{1,2,\ldots,m\}$ and $\beta \subseteq \{1,2,\ldots,n\}$, we shall denote by $A_{\alpha\beta}$ the submatrix of $A$ containing those rows and columns whose indices are in $\alpha$ and $\beta$ respectively. If any one of the sets $\alpha$ and $\beta$ is a singleton, say $\beta = \{j\}$, then the corresponding submatrix is also written as $A_{\alpha j}$. The identity matrix of order $n$ is denoted by $I$. The cardinality of a set $\alpha$ is denoted by $|\alpha|$.

The following simple results are used in the sequel.

LEMMA 1. *Let $A$ be a square matrix which can be written in the partitioned form*

$$A = \begin{pmatrix} B & C \\ 0 & D \end{pmatrix}$$

*where $B$ and $D$ are square matrices. Then $A$ is a $P_0$-matrix if and only if $B$ and $D$ are $P_0$-matrices.*

*Proof.* The required assertion is a trivial consequence of the definition of $P_0$-matrices.

LEMMA 2. *Let $A$ be a Z-matrix of order $n$ such that the principal submatrix $A_{\alpha\alpha}$ is a nonsingular M-matrix, where $\alpha = \{1,2,\ldots,n-1\}$. The matrix $H$ obtained by replacing the last column of $A$ by $e$ is then nonsingular, and $(H^{-1})_{n\cdot} \geq 0$. Moreover, if $x = H^{-1}A_{\cdot n}$, then:*

(i) $x_n > 0 \Leftrightarrow$ *$A$ is a nonsingular M-matrix.*[1]
(ii) $x_n > 0 \Rightarrow x_\alpha < 0$.

*Proof.* We can write $A$ and $H$ in the partitioned form

$$A = \begin{pmatrix} A_{\alpha\alpha} & A_{\alpha n} \\ A_{n\alpha} & a_{nn} \end{pmatrix} \quad \text{and} \quad H = \begin{pmatrix} A_{\alpha\alpha} & e_\alpha \\ A_{n\alpha} & 1 \end{pmatrix}.$$

[1] It is possible to assert

(i′) $x_n = 0 \Leftrightarrow A$ is a singular $M$-matrix.

Let $f=(a_{nn}-A_{n\alpha}A_{\alpha\alpha}^{-1}A_{\alpha n})$ and $g=(1-A_{n\alpha}A_{\alpha\alpha}^{-1}e_\alpha)$. By Schur's determinantal formula we have

$$\det A = f\det A_{\alpha\alpha}, \tag{1}$$

$$\det H = g\det A_{\alpha\alpha}. \tag{2}$$

The hypothesis implies $\det A_{\alpha\alpha}>0$ and $g \geqq 1$. The nonsingularity of $H$ follows from (2). It is easy to verify that

$$(H^{-1})_{n\cdot}=\left(-g^{-1}A_{n\alpha}A_{\alpha\alpha}^{-1}, g^{-1}\right)\geq 0. \tag{3}$$

From (3) we get

$$x_n = -g^{-1}A_{n\alpha}A_{\alpha\alpha}^{-1}A_{\alpha n}+g^{-1}a_{nn}=g^{-1}f. \tag{4}$$

It is known (see [4, Theorem (4,3)]) that a Z-matrix is a nonsingular $M$-matrix if and only if its leading principal minors are positive. Therefore the hypothesis implies that $A$ is a nonsingular $M$-matrix if and only if $\det A>0$. The validity of assertion (i) follows from (1) and (4). By making use of the observation $A_{\cdot n}=Hx$, we get

$$x_\alpha = A_{\alpha\alpha}^{-1}(A_{\alpha n}-e_\alpha x_n).$$

The validity of assertion (ii) follows easily.

LEMMA 3. *Let $A$ be an $M$-matrix of order $n$ ($\geq 2$) such that the matrix $H$ obtained by replacing the last column of $A$ by $e$ is nonsingular. If $x=H^{-1}A_{\cdot n}$, then $x_j \leqq 0$ for all $j\neq n$ and $x_n \geqq 0$.*

*Proof.* For any $s\in R$, let $A(s)=sI+A$ and $H(s)$ denote the matrix obtained by replacing the last column of $A(s)$ by $e$. We note that $A(s)\to A$ and $H(s)\to H$ as $s\to 0$. Since $A$ is an $M$-matrix, it follows (see [4, Theorem (5,1)]) that $A(s)$ is a nonsingular $M$-matrix for all $s>0$. By hypothesis and Lemma 2 we observe that $H(s)$ is nonsingular for all $s\geq 0$. Since $[H(s)]^{-1}\to H^{-1}$ as $s\to 0^+$, the required assertions follow from Lemma 2 as the limiting case when $s\to 0^+$.

For a given square matrix $A$ of order $n$ and $q\in R^n$, the problem of finding a solution (if any) to the system $(A,q)$ of constraints

$$w-Az=q,$$

$$w\geqq 0,\qquad z\geqq 0,\qquad w^tz=0.$$

is known as the *linear complementarity problem* (LCP). *The complementary pivot method* proposed by Lemke for this purpose is referred to as Lemke's algorithm in this paper. For the terminology used in this context and other details we refer to standard texts such as [1] and [7]. It is known that Lemke's algorithm terminates in finitely many steps, finding either a *complementary basic feasible solution* or a *secondary ray*, under the assumption of nondegeneracy or by incorporating a degeneracy resolving procedure like the lexicographic rule.

We shall denote by $L(A,q)$ the application of Lemke's algorithm for a given instance $(A,q)$ of the LCP. We shall assume throughout this paper that for initiating $L(A,q)$ we introduce the artificial variable $z_0$ by using the vector $e$ and write the initial system as

$$w - Az - ez_0 = q,$$

$$w \geqq 0, \qquad z \geqq 0, \qquad z_0 \geqq 0,$$

$$w'z = 0.$$

This does not involve any loss of generality as far as the results of this paper are concerned.

We note that $L(A,q)$ terminates in a secondary ray when $(A,q)$ has no solution. The results of Lemma 4 which are based on this observation play a key role in this paper.

LEMMA 4. *Suppose $A$ is a Z-matrix of order $n$ and $q \in R^n$ is such that $(A,q)$ has no solution. Let $\mathscr{R} = \{(\bar{w}, \bar{z}, \bar{z}_0) + \lambda(\hat{w}, \hat{z}, \hat{z}_0) : \lambda \geqq 0\}$ denote the secondary generated by $L(A,q)$, and also let $\alpha = \{j : \hat{z}_j \neq 0\}$ and $\beta = \{j : \hat{z}_j = 0\}$. If $\hat{z}_0 \neq 0$, then $A$ is not an M-matrix. On the other hand, if $\hat{z}_0 = 0$, then*:

(i) $\hat{w} = 0$, $\hat{z} \geq 0$, *and* $A\hat{z} = 0$,
(ii) $A_{\beta\alpha} = 0$ *when $\beta$ is nonempty*,
(iii) $A_{\alpha\alpha}$ *is an M-matrix*.

*Proof.* It is well known [5, p. 686] that $\hat{w} \geqq 0$, $\hat{z} \geq 0$, $\hat{w}'\hat{z} = 0$, and $\hat{w} - A\hat{z} - e\hat{z}_0 = 0$. If $\hat{z}_0 \neq 0$, then $(\hat{w}, \hat{z})$ is a nontrivial solution to $(A, e\hat{z}_0)$. Hence (see [3, Lemma 3, p. 620]) $A$ is not a $P_0$-matrix. Suppose now $\hat{z}_0 = 0$. In this case we have $\hat{w} = A\hat{z}$. We note that $\hat{z}_\alpha > 0 \Rightarrow \hat{w}_\alpha = 0$ and $\hat{z}_\beta = 0 \Rightarrow \hat{w}_\beta = A_{\beta\alpha}\hat{z}_\alpha$. The facts that $\hat{w}_\beta \geqq 0$, $\hat{z}_\alpha > 0$, and $A_{\beta\alpha} \leqq 0$ imply $\hat{w}_\beta = 0$ and $A_{\beta\alpha} = 0$. Since $\hat{z}_\alpha > 0$ and $A_{\alpha\alpha}\hat{z}_\alpha = 0$, it follows (see [4, Theorem (5,4)]) that $A_{\alpha\alpha}$ is an $M$-matrix.

LEMMA 5. *Let A be a Z-matrix of order n, and $q \in R^n$ be such that $q < 0$. Then $(A, q)$ has a solution if and only if A is a nonsingular M-matrix.*

*Proof.* See [6, Theorem 4.1] and [2, Theorem 2.14, p. 273].

LEMMA 6. *Let B be an almost complementary basis matrix of $L(A, q)$ with $\alpha = \{j : I_{.j}$ is a column of $B\}$, $\beta = \{j : -A_{.j}$ is a column of $B\}$, and $\{I_{.s}, -A_{.s}\}$ being the left-out pair of complementary columns. If A is an M-matrix and $y = B^{-1}(-A_{.s})$, then:*

(i) $B_{.i} = -e \Rightarrow y_i \geq 0$,
(ii) $j \in \beta$ *and* $B_{.i} = -A_{.j} \Rightarrow y_i \leq 0$.

*Proof.* Let $\Pi$ be the permutation matrix such that

$$\Pi A \Pi' = \begin{pmatrix} A_{\alpha\alpha} & A_{\alpha\beta} & A_{\alpha s} \\ A_{\beta\alpha} & A_{\beta\beta} & A_{\beta s} \\ A_{s\alpha} & A_{s\beta} & a_{ss} \end{pmatrix}$$

We shall assume without loss of generality that the columns of $B$ are so arranged that

$$\Pi B = \begin{pmatrix} I_{\alpha\alpha} & -A_{\alpha\beta} & -e_{\alpha} \\ 0 & -A_{\beta\beta} & -e_{\beta} \\ 0 & -A_{s\beta} & -1 \end{pmatrix}$$

We note that

$$y = -B^{-1}A_{.s} = -B^{-1}\Pi'\Pi A_{.s} = [\Pi(-B)]^{-1}(\Pi A_{.s}).$$

Since the matrix

$$\begin{pmatrix} A_{\beta\beta} & A_{\beta s} \\ A_{s\beta} & a_{ss} \end{pmatrix}$$

satisfies the hypothesis of Lemma 3, the required assertions follow.

In Section 3, we make use of the following (slightly) modified version of Lemke's algorithm. We shall denote by $\hat{L}(A, q)$ the application of Lemke's algorithm to $(A, q)$ with the additional proviso that we prematurely terminate the algorithm when at any stage the ratio test indicates that the basic variable to become nonbasic is a $z$-variable. We note that $\hat{L}(A, q)$ has three mutually exclusive types of termination, namely solution, ray, and premature terminations. Some properties of $\hat{L}(A, q)$ are established in Lemma 7.

LEMMA 7. *The following statements are true when A is a Z-matrix of order n:*

(i) $\hat{L}(A, q)$ *terminates in at most* $n+1$ *pivot steps.*
(ii) $(A, q)$ *has a solution* $\Rightarrow$ $\hat{L}(A, q)$ *terminates with a solution to* $(A, q)$.
(iii) $\hat{L}(A, q)$ *terminates prematurely* $\Rightarrow$ *A is not an M-matrix.*

*Proof.* It is easy to see that in $\hat{L}(A, q)$ each row (except the initial pivot row) can be selected at most once for pivoting. If the initial pivot row is selected again for pivoting, then $z_0$ becomes nonbasic and $\hat{L}(A, q)$ terminates with a complementary basic feasible solution. This establishes the validity of (i). We note that premature termination is ruled out [6, Theorem 3.3 and Corollary 3.4] when $(A, q)$ has a solution and thus $\hat{L}(A, q)$ is same as $L(A, q)$. Since Lemke's algorithm processes LCPs with Z-matrices [9], the validity of assertion (ii) follows. The validity of (iii) is a trivial consequence of Lemma 6.

We note that cycling cannot occur in $\hat{L}(A, q)$, and so no degeneracy resolving mechanism is necessary. The known result that Lemke's algorithm is polynomial for $M$-matrices follows trivially from Lemma 7. The initial pivot step of $\hat{L}(A, q)$ does not involve any multiplications or divisions. There is no need to update the columns corresponding to $w$-variables. In view of this, $\hat{L}(A, q)$ requires at most $n(n+1)^2/2$ multiplications and divisions and roughly as many subtractions.

## 3. ALGORITHM-Z-P-ZERO

We shall now consider the problem of checking whether a given Z-matrix of order $n$ is also an $M$-matrix. A finite algorithm to do this is to directly verify the nonnegativity of principal minors. This algorithm has exponential growth rate of *computational complexity* in the sense that an "yes" instance of the problem requires evaluation of $2^n - 1$ determinants. We now give

ALGORITHM-Z-P-ZERO, which is polynomial for testing the nonnegativity of principal minors of Z-matrices.

ALGORITHM-Z-P-ZERO

Step 0. Let $A$ be a given Z-matrix of order $n$. Put $\gamma = \{1, 2, \ldots, n\}$, choose any $q \in R^n$ such that $q < 0$, and go to step 1.

Step 1. Put $\theta = \gamma$, and apply the modified version of Lemke's algorithm to $(A_{\theta\theta}, q_\theta)$.

a. If $\hat{L}(A_{\theta\theta}, q_\theta)$ terminates with a solution, stop; $A$ is an $M$-matrix.

b. If $\hat{L}(A_{\theta\theta}, q_\theta)$ has premature termination, stop; $A$ is not an $M$-matrix.

c. If $\hat{L}(A_{\theta\theta}, q_\theta)$ has ray termination, go to step 2.

Step 2. Let $\mathscr{R} = \{(\bar{w}, \bar{z}, \bar{z}_0) + \lambda(\hat{w}, \hat{z}, \hat{z}_0) : \lambda \geqq 0\} \subseteq R^{2|\theta|+1}$ denote the secondary ray generated by $\hat{L}(A_{\theta\theta}, q_\theta)$. If $\hat{z}_0 \neq 0$, stop; $A$ is not an $M$-matrix. Otherwise go to step 3.

Step 3. Let $\alpha = \{j : \hat{z}_j \neq 0\}$ and $\beta = \{j : \hat{z}_j = 0\}$. If $\beta$ is empty, stop; $A$ is an $M$-matrix. Otherwise put $\gamma = \beta$ and return to step 1.

THEOREM 1. *Let $A$ be a Z-matrix of order $n$. Then* ALGORITHM-Z-P-ZERO *verifies whether $A$ is an M-matrix in time $O(n^4)$.*

*Proof.* At any stage of the algorithm, we note that the principal submatrix $A_{\theta\theta}$ at step 1 is a Z-matrix. Using Lemma 5, we see that termination at step 1a implies $A_{\theta\theta}$ is a nonsingular $M$-matrix. By Lemma 7 we note that termination at step 1b implies $A_{\theta\theta}$ is not an $M$-matrix. By Lemma 4, we note that termination at step 2 implies that $A_{\theta\theta}$ is not an $M$-matrix. Again by Lemma 4, we see that termination at step 3 implies that $A_{\theta\theta}$ is an $M$-matrix. If $\beta$ is nonempty at step 3, we note from Lemma 4 that $A_{\alpha\alpha}$ is an $M$-matrix and $A_{\beta\alpha} = 0$. Therefore there exists a permutation matrix $\Pi \in R^{|\theta| \times |\theta|}$ such that

$$\Pi A_{\theta\theta} \Pi' = \begin{pmatrix} A_{\alpha\alpha} & A_{\alpha\beta} \\ 0 & A_{\beta\beta} \end{pmatrix}.$$

From Lemma 1 we note that $A_{\theta\theta}$ is an $M$-matrix if and only if $A_{\beta\beta}$ is an $M$-matrix. We also observe that $|\beta| < |\theta|$. Therefore ALGORITHM-Z-P-ZERO requires at most $n$ repeated applications of the modified Lemke's algorithm. The required assertion is then immediate.

When $A$ is an irreducible Z-matrix of order $n$, we note that ALGORITHM-Z-P-ZERO requires a single application of the modified version of Lemke's algorithm and is thus $O(n^3)$ in time.[2] This suggests that it should be possible to check in $O(n^3)$ time whether any Z-matrix (irreducible or not) is an $M$-matrix. We show below that a combination of Tarjan's ***depth-first search method*** and ALGORITHM-Z-P-ZERO for this purpose is indeed $O(n^3)$ in time.

Let $A$ be any reducible Z-matrix of order $n$. Then there exists a permutation matrix $\Pi$ such that

$$\Pi A \Pi' = \begin{pmatrix} A_{\theta_1\theta_1} & A_{\theta_1\theta_2} & \cdots & A_{\theta_1\theta_k} \\ 0 & A_{\theta_2\theta_2} & \cdots & A_{\theta_2\theta_k} \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & A_{\theta_k\theta_k} \end{pmatrix},$$

where each $A_{\theta_i\theta_i}$ is either irreducible or a 1-by-1 null matrix. By Lemma 1, $A$ is an $M$-matrix if and only if each $A_{\theta_i\theta_i}$ is an $M$-matrix. This can be checked in $O(|\theta_1|^3 + \cdots + |\theta_k|^3)$ or in $O(n^3)$ time by applying ALGORITHM-Z-P-ZERO to each $A_{\theta_i\theta_i}$.

The partition of $\{1,2,\ldots,n\}$ into the sets $\theta_1, \theta_2, \ldots, \theta_k$ can be done in $O(n^2)$ time by Tarjan's depth-first search algorithm for finding the strongly connected components of a directed graph [10, p. 155]. For this purpose we associate a directed graph $G = (V, \mathscr{E})$ with the matrix $A$ as follows. We take the vertex set $V = \{1,2,\ldots,n\}$, and the arc set $\mathscr{E}$ to be

$$\mathscr{E} = \{(i,j) : i \neq j \text{ and } a_{ij} \neq 0\}.$$

We define the equivalence relation $\sim$ on the set $V$ of vertices by $i \sim j$ if and only if either $i = j$ or there exists a directed path from $i$ to $j$ and also from $j$ to $i$. The distinct equivalence classes under this relation are exactly the sets $\theta_1, \theta_2, \ldots, \theta_k$ which form the output of Tarjan's algorithm.

Finally, let $T$ denote the transition-probability matrix of a homogeneous Markov chain with state set $\{1,2,\ldots,n\}$. For any chain, regardless of its structure, $A = I - T$ is an $M$-matrix [2, p. 226]. We can use

[2] In this case, if $\alpha = \{1,2,\ldots,n-1\}$, then $A$ is an $M$-matrix if and only if $A_{\alpha\alpha}^{-1} \geq 0$ and $a_{nn} - A_{n\alpha}A_{\alpha\alpha}^{-1}A_{\alpha n} \geq 0$. Inversion of an $(n-1)$-by-$(n-1)$ matrix requires as many as $(n-1)^3$ multiplications and divisions with about the same number of subtractions. In comparison, ALGORITHM-Z-P-ZERO requires only half the computational effort.

ALGORITHM-Z-P-ZERO with some modifications to analyse the chain to find its distinct ergodic classes, the set of transient states, and also the stationary probability distribution vectors associated with the ergodic classes [8].

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