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MULTIVALUED SOCIAL CHOICE FUNCTIONS AND STRATEGIC MANIPULATION WITH COUNTERTHREATS

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In this paper, we examine the manipulability properties of social decision rules which select a non-empty subset of the set of alternatives. Assuming that if an individual prefers x to y, then he prefers the outcome set $\{x, y\}$ to $\{y\}$, and also $\{x\}$ to $\{x, y\}$, we show that a wide class of scf's which allow ties even in pairwise choice violates one of the weakest notions of strategyproofness – a single individual can profitably misrepresent his preferences, even when he takes into account the possibility of countercoalitions. This class of scf's also violates *exact consistency* – no equilibrium situation gives the same outcome set as the 'true profile'.

Keywords: Manipulability; multivalued social choice function; consistency.

1. Introduction

In this paper, we examine the manipulability properties of *social choice functions*, i.e. decision rules which do not necessarily select a single alternative in all situations, but merely a non-empty subset of alternatives. Thus, the present paper is in line with the work of Barbera (1977), Gardenfors (1976), Kelly (1977), Pattanaik (1973), and MacIntyre and Pattanaik (1979), amongst others. As has been pointed out by the above writers, the assumption that the decision rule always yields a single outcome as the social outcome is a restrictive one, since under many democratic group decision methods, there are situations where the outcome is a tie which is then broken by some random mechanism to obtain the winning alternative. Hence a study of the manipulability properties of social choice functions is of considerable interest.

The results on manipulability of social choice functions are less well known than similar results for single-valued social decision rules. Perhaps the main reason for this is that there is no 'natural' criterion by which one can determine individuals' preferences over *subsets* of a set X, given their preferences over *elements* of X. Obviously, such criteria will depend, at least partially, on individuals' behaviour under risk. The earliest attempt to derive voters' orderings over subsets of alternatives was by Pattanaik (1973), who assumed that individuals used a class of *maximin*

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rules. In contrast, Gibbard (1977) assumes that individuals are expected utility maximisers. However, as MacIntyre and Pattanaik (1979) remark, "these are rather restrictive assumptions which limit the scope of the theorems proved".

Barbera (1977), on the other hand, proves an interesting result with an extremely weak behavioural assumption. He assumes that if an individual prefers x to y then he must prefer the outcome set $\{x, y\}$ to the outcome set $\{y\}$, and also he must prefer the outcome set $\{x\}$ to the outcome set $\{x, y\}$. However, Barbera's result is extremely limited in scope since it applies only to 'binary' social choice functions. The most elegant result in this respect has recently been proved by MacIntyre and Pattanaik (1979), who relax the assumption of binariness so as to retain the notion of pairwise comparisons only in a very minimal sense, while continuing to use behavioural assumptions which are intuitively almost as plausible as the one made by Barbera.

In this paper, using the same behavioural assumption used by Barbera, we show that a wide class of social choice functions which allow ties even in pairwise choice, violates one of the weakest notions of strategy proofness – a single individual can profitably misrepresent his preferences, *even when he takes into account the possibility of counter-coalitions*. Using a stronger behavioural assumption, we then show that the simple majority rule, if the number of individuals is even, violates a concept which is considerably weaker than strategy proofness – that of *consistency*, which merely requires the set of equilibrium situations to be nonempty, corresponding to any specification of true individual preferences. Lastly, we show that the class of social choice functions considered in the first result violates *exact consistency*, i.e., there exists some specification of true preferences under which no equilibrium situation gives the same outcome set as the 'true' situation. Thus, the main aim of this paper is to weaken the notions of strategyproofness, and see whether any interesting social choice functions satisfy these weaker notions of strategyproofness.

The plan of this paper is as follows. In Section 2, we give the notation and some basic definitions. Section 3 discusses the concepts of strategyproofness and consistency. Section 4 contains the results.

2. The notation and some definitions

$L = \{1, 2, \ldots, n\}$ denotes the society, with $n \geq 2$. X will stand for the set of all feasible alternatives, which may be presented for choice, with $|X| \geq 3$. The set of alternatives actually presented before L for making its choice will be called an *issue*. G, the set of all non-empty subsets of X, is the set of all possible issues that may arise.

For all $i \in L$, we assume that there is a binary preference relation $\bar{R}_i$ defined over X, which represents *sincere* preference of individual i. Since individual i may not actually reveal his preference, the preference actually expressed by i is indicated by

R_i, R_i' etc. Throughout this paper, we assume that $\bar{R}_i, R_i, R_i'$ etc. are orderings over X. Let S be the set of all possible orderings over X. The elements of S^n will be called *situations*, to be denoted by s, s' etc. Thus, $s = (R_1, R_2, \ldots, R_n)$, $s' = (R_1', R_2', \ldots, R_n')$, and so on. A situation $s \in S^n$ is called *sincere* iff $R_i = \bar{R}_i$ for all $i \in L$.

Definition 1. A *social choice function* (henceforth SCF) is a function F which for every $(A, s) \in GXS^n$ specifies exactly one nonempty subset $F(A, s)$ of A.

We now introduce certain properties of an SCF. Some additional notation is required for this purpose. Given F, and any $s \in S^n$ the *base relation* of F is defined as follows:

$$\forall x, y \in X, \quad xR^F y \quad \text{iff } x \in F(\{x, y\}, s).$$

P^F and I^F are the antisymmetric and symmetric components of R^F. (To simplify notation, when there is no ambiguity about the SCF, we simply write $\bar{R}, R', R$ corresponding to situations $\bar{s}, s', s$ etc.) The *binary choice set* $C(A, R)$, corresponding to s and A, is defined as:

$$C(A, R) = \{x \in A \mid xRy, \text{ for all } y \in A\}.$$

For any $x, y \in X$, and $s \in S^n$, $L_{xy} = \{i \in L \mid xP_i y\}$ and $L_{yx} = \{i \in L \mid yP_i x\}$. Similarly, L_{xy}', L_{yx}' correspond to situation s', and so on. Let $B \subseteq X$. For any binary relations Q, Q', if for all $x, y \in B, xQy$ if $xQ'y$, then we say that $Q:B = Q':B$.

Definition 2. Let s, s' be any two situations in S^n, and let x, y be any two alternatives. Let $A \in G$, and let F be the SCF.

(2.1) *Independence of Irrelevant Alternatives* (IIA). If [for all $i \in L$, $(R_i\colon \{x, y\} = R_i'\colon \{x, y\})$], then ($xRy$ iff $xR'y$).

(2.2) *Monotonicity* (M). If $[(L_{xy} \subseteq L_{xy}')$ and $(L_{yx}' \subseteq L_{yx})]$, then $[(xIy \to xR'y)$ and $(xPy \to xP'y)]$.

(2.3) *Neutrality* (NT). For all $x, y, z, w \in X$, if $[(L_{xy} = L_{wz})$ and $(L_{yx} = L_{zw})]$, then (xRy iff wRz).

(2.4) *Absence of Veto.* If $|L_{xy}| \geq n - 1$, then xPy.

(2.5) *Limited Irresoluteness* (LIR). There exists $\hat{s} \in S^n$, and $c \subseteq L$, such that ($x\hat{P}_i y$ for all $i \in c$ and $y\hat{P}_i x$ for all $i \in L - c$ and $x\hat{I}y$).

(2.6) *Weak Binariness* (WB). If $C(A, R) \neq \emptyset$, then $F(A, s) = C(A, R)$.

The properties of IIA, M, NT, AV are all familiar conditions in social choice theory. LIR is satisfied by the entire class of SCF's which declares x to be preferred to y iff $L_{xy} > tn'$ where $\frac{1}{2} \leq t < (n' - 1)/n'$ and $n' = |L_{xy} \cup L_{yx}|$.

Perhaps, the most familiar SCF is F^M, the simple majority rule, whose base relation R^M, is for all $x, y \in X, xR^M y$ iff $|L_{xy}| \geq |L_{yx}|$. Note that F^M satisfies LIR iff n is even.

For any SCF F, let W_F be the set of *winning* coalitions, and B_F be the set of *block-*

ing coalitions. The following lemma stated without proof, explains the terminology.

Lemma 1. *Let the* SCF *satisfy* IIA, M *and* NT. *For all $x, y \in X$, and for all $s \in S^n$ such that R_i is antisymmetric for all $i \in L[(xRy$ iff $L_{xy} \in B_F)$ and $(xPy$ iff $L_{xy} \in W_F)]$.*

Lastly, for any $s \in S^n$, and any $E \subseteq L, s'S^n$ is said to be *E-variant* from s iff for all $i \in L - E, R_i = R_i'$, and for some $i \in E, R_i \neq R_i'$. When $|E| = 1$, we simply write *i*-variant, *j*-variant, etc.

3. Strategyproofness and weaker conditions

In this section, we discuss the concept of strategyproofness and some related concepts which though weaker than strategyproofness, are designed to serve the same intuitive purpose. But, in order to do this, we need assumptions which enables us to derive individuals' preferences over situations given their preference ordering over X.

Assumption B (Barbera (1977)). Suppose $x, y \in X$ and $x\bar{P}_i y$. If $[(F(A,s) = \{x\}$ and $F(A,s') = \{x, y\})$, or $(F(A,s) = \{x, y\}$ and $F(A,s') = \{y\})]$, then, $s >_i^A s'$.

Assumption STP (Sure-thing principle). Let $\bar{R}_i$ be any sincere preference ordering over X. Let $s, s' \in S^n$ and $A \in G$. Then $s >_i^A s'$ if one of the following conditions is satisfied:

(i) $F(A,s) \subset F(A,s')$, and for all $x \in F(A,s)$, for all $y \in F(A,s') - F(A,s)$, $x\bar{R}_i y$, and for some $x \in F(A,s)$ and some $y \in F(A,s')$, $x\bar{P}_i y$.

(ii) $F(A,s') \subset F(A,s)$, and for all $y \in F(A,s')$, for all $x \in F(A,s) - F(A,s')$, $x\bar{R}_i y$, and for some $y \in F(A,s')$ and some $x \in F(A,s) - F(A,s')$, $x\bar{P}_i y$.

(iii) Neither $F(A,s) \subset F(A,s')$, nor $F(A,s') \subset F(A,s)$, but $|F(A,s)| = |F(A,s')|$, and for all $x \in F(A,s) - F(A,s')$, for all $y \in F(A,s') - F(A,s)$, $x\bar{R}_i y$, and for some $x \in F(A,s) - F(A,s')$ and some $y \in F(A,s') - F(A,s)$, $x\bar{P}_i y$.

In the context of strategic voting, a stronger version of Assumption STP was originally introduced by Gardenfors (1976). The sole difference between Assumption STP and Gardenfors' assumption is that we have added the additional condition that $|F(A,s)| = |F(A,s')|$ in part (iii) above. This condition is imposed in view of the recent criticism of Gardenfors' assumption by MacIntyre and Pattanaik (1979). They point out that if for example $x_1 \bar{P}_i x_2 \bar{P}_i, \ldots, \bar{P}_i x_{10}$, and $F(A,s) = \{x_1, x_2, \ldots, x_9\}$ while $F(A,s') = \{x_1, x_{10}\}$, then $s >_i^A s'$ according to Gardenfors' assumption. Note that intuitively this is not a compulsive conclusion – in fact it may be inconsistent with expected utility maximisation depending on the utility function representing $\bar{R}_i$ and the probabilities attached to the alternatives in the two sets. However, the additional condition $|F(A,s) = F(A,s')|$ gets rid of problems of this kind particularly if individuals attach equal probabilities of being chosen to each alternative in $F(A,s)$, $F(A,s')$, etc. Nevertheless, our assumption STP, although intuitively plausible, is stronger than the corresponding assumptions made by Barbera, Kelly or MacIntyre and Pattanaik.

Note that Assumption STP gives only a sufficient condition for preferring one situation to another. However, in one of our results, we also need a necessary condition. The following condition lacks plausibility, but has been made by Gardenfors, as well as MacIntyre and Pattanaik. Essentially, this replaces the 'if' of STP by 'iff'.

Assumption STP*. Let $\bar{R}_i$ be any sincere preference ordering. Let $s, s' \in S^n$ and $A \in G$. Then $s \succ_i^A s'$ *iff* one of conditions (i), (ii) or (iii) of Assumption STP is satisfied.

Definition 3. $s \in S^n$ is a

(i) *Nash-equilibrium* (Type 1) iff there does not exist any $i \in L$, and any i-variant situation s' such that $s' \succ_i^A s$;

(ii) *Nash-equilibrium* (Type 2) iff for all $i \in L$, and all i-variant situations s', if $s' \succ_i^A s$, then there exists an $(L-\{i\})$ variant situation s'' (of s') such that $s \succ_i^A s''$;

(iii) *strict equilibrium* (Type 1) if there does not exist any $E \subseteq L$ and any E-variant situation s' such that $s' \succ_i^A s$ for all $i \in E$;

(iv) *strict equilibrium* (Type 2) iff for all coalitions $E \subseteq L$, and all E-variant situations s' if $s' \succ_i^A s$ for all $i \in E$, then there exists an $(L-E)$-variant situation s'' (of s') such that $s \succ_i^A s''$ for some $i \in E$.

Let $N_a(A,F,\bar{s})$ be the set of Nash-equilibrium (Type a) for $a = 1,2$, corresponding to issue A, SCF F, and the sincere situation $\bar{s}$. Similarly, $S_a(A,F,\bar{s})$ is the set of strict equilibrium (Type a).

Definition 4. An SCF F is Nash *strategyproof* (Type a) (resp. strictly strategyproof (Type a)) iff for all $A \in G$, every possible sincere situation $\bar{s} \in N_a(A,F,\bar{s})$ (resp. $\bar{s} \in S_a(A,F,\bar{s})$).

Under strictly strategyproof (Type 1) SCF's, no group of individuals can profitably misrepresent their sincere preferences, even if the group disregards the formation of any possible countercoalitions. Nash-strategyproof (Type 1) SCF's are not manipulable by single individuals, if each individual disregards the possibility of a counter coalition trying to prevent any such manipulation. The concepts of Type-2 strategyproofness incorporate the possibility that individuals or groups will, before trying to misrepresent their sincere preferences, try to anticipate the reactions of individuals outside the coalition. Note that the notion of Type-2 equilibrium assumes extremely pessimistic behaviour on the part of individuals trying to disrupt any situation – each individual or group expects the rest of the individuals to punish them for the disruption, even if they themselves suffer a loss in the process. Hence, as Pattanaik (1978) has remarked, 'the resultant concept of equilibrium is extremely weak'. Thus, at least two of our negative results with Type-2 equilibrium are of additional significance.

While the concepts of strategyproofness are in a similar spirit since they concentrate on the equilibrium of the sincere situation, Peleg (1978) in a seminal paper, proposed a weaker notion which serves almost the same intuitive purpose.

Definition 5.

(5.1) An SCF F is *consistent* (Type a) iff for all $A \in G$, and all possible situations $\bar{s}$, $S_a(A,F,\bar{s}) \neq \emptyset$.

(5.2) An SCF F is *exactly consistent* (Type a) iff for all $A \in G$, and all possible sincere situations $\bar{s}$, there exists $s \in S_a(A,F,\bar{s})$ such that $F(A,s) = F(A,\bar{s})$.

In the next section, we show that under Assumption STP, a wide class of SCF's violates an even weaker property.

Definition 6. An SCF F is *almost exactly consistent* (Type a) iff for all $A \in G$, and all possible sincere situations $\bar{s}$, there exists $s \in S_a(A,F,\bar{s})$ such that $F(A,s) \subseteq F(A,\bar{s})$.

Thus, under almost exactly consistent SCF's, for some equilibrium situation, no element which is rejected in the sincere situation can figure in the set of outcomes, although some element selected in the sincere situation may now be rejected. As was argued in Dutta (1977), we should be satisfied if no 'non-optimal' alternative is selected. Hence, almost exactly consistent SCF's can be viewed as proper substitutes for the more demanding exactly consistent SCF's. (For results on consistent decision rules, see Peleg (1978), Dutta (1980), Dutta and Pattanaik (1978), Pattanaik (1978)).

4. Impossibility theorems

In this section, we first show that a wide class of SCF's violates *Nash-strategyproofness* (Type 2), under the extremely weak behavioural Assumption B. The negative impact of this result is very significant because this is the weakest notion of strategyproofness discussed in the literature. Moreover, this is the first result which combines the notion of equilibrium with counter threats together with such a weak behavioural assumption. In the second theorem, we show that F^M (with n even) satisfying WB violates consistency (Type 1) under Assumption STP. Again, given that F^M is perhaps the most widely used and certainly the most widely discussed SCF, the negative impact of this result is significant. Theorems 3 and 4 show that the class of SCF's considered in Theorem 1 violates almost exact consistency (Type 1) and (Type 2) under two alternate behavioural assumptions.

Theorem 1. *Let Assumption* B *hold. There is no* SCF *satisfying* IIA, M, NT, AV, LIR, WB *and Nash-strategyproofness* (*Type* 2).

Proof. Recall that B_F is the set of blocking coalitions. Let b^{*} be the smallest blocking coalition, i.e. for all $b\in B_F$, $|b|\geq|b^{*}|$. Note that since F satisfies AV, $|b^{*}|\geq 2$. Suppose $k\in b^{*}$. Partition L into L_1, L_2, L_3 such that $L_1=b^{*}-\{k\}$, $L_2=\{k\}$, and $L_3=L-b^{*}$. Let $A=\{x,y,z\}$. Construct $\bar{s}$ as follows:

$$\text{for all } i\in L_1:\quad x\bar{P}_i y\bar{P}_i z,\qquad z\bar{P}_k x\bar{P}_k y;$$

$$\text{for all } i\in L_3:\quad y\bar{P}_i z\bar{P}_i x.$$

Since F satisfies IIA, M, and LIR, $b^{*}\notin W_F$. Also, $L_1\notin B_F$ since $|L_1|<|b^{*}|$ and b^{*} is the smallest blocking coalition. Hence, $x\bar{I}y$, $y\bar{P}z$ and $z\bar{P}x$ must hold. Since F satisfies WB, $F(A,\bar{s})=C(A,\bar{R})=\{y\}$.

Consider s such that for all $i\in L-\{k\}$, $P_i=\bar{P}_i$ and $xP_k zP_k y$. Since $b^{*}\in B_F$, we now have xIy, xIz and yPz. Hence $F(A,s)=\{x,y\}$. Since $x\bar{P}_k y$, by assumption B, we must have $s>_k^A\bar{s}$.

Note also that since y is the $\bar{P}_k$ worst element in the issue A, there does not exist any s' such that $\bar{s}>_k^A s'$. Hence, $\bar{s}\notin N_2(A,F,\bar{s})$.

This completes the proof of the theorem.

Theorem 2. *Let Assumption* STP *hold. If* F^{M} *satisfies* WB, *and* n *is even, then* F^{M} *violates consistency* (*Type* 1).

Proof. Let $A=\{x,y,z\}$. Partition L into L_1, L_2, L_3 and L_4 such that $|L_1|=|L_2|=\frac{1}{2}n-1$, and $|L_3|=|L_4|=1$.

Construct $\bar{s}$ as follows:

$$\text{for all } i\in L_1:\quad z\bar{P}_i y\bar{P}_i x,$$

$$\text{for all } i\in L_2:\quad x\bar{P}_i z\bar{P}_i y,$$

$$\text{for all } i\in L_3:\quad z\bar{P}_i x\bar{P}_i y,$$

$$\text{for all } i\in L_4:\quad y\bar{P}_i x\bar{P}_i z.$$

Clearly, we have $x\bar{I}z$, $z\bar{P}y$ and $y\bar{I}x$, so that by WB we have $F(A,\bar{s})=\{x,z\}$. We show that $S_1(A,F,\bar{s})=\emptyset$.

Case A. Suppose $C(A,R^{M})=\emptyset$. Let $F(A,s)\neq\{z\}$. Then consider s' such that s' is $(L_1\cup L_3)$-variant from s, and for all $i\in L_1\in L_3$: (zP'_ix and zP'_iy, and $R'_i:\{x,y\}=R_i:\{x,y\}$). Then we must have $F(A,s')=\{z\}$ and $s'>_i^A s$ for all $i\in L_1\cup L_3$.

Suppose $F(A,s)=\{z\}$.

Then consider s' such that s' is $(L_2\cup L_4)$-variant from s, and for all $i\in L_2\cup L_4$: (xP_iz and xP_iy, and $R'_i:\{z,y\}=R_i:\{z,y\}$). This gives $F(A,s')=\{x\}$, and $s'>_i^A s$ for all $i\in L_2\cup L_4$.

Hence, if $C(A,R^M)=\emptyset$, then $s\notin S_1(A,F,S)$.

Case B. Suppose $C(A,R^M)\neq\emptyset$.

First, note that by STP, if $z\notin F(A,s)$, then $s\notin S_1(A,F,S)$ since $L_1\cup L_3\in B_F$, and $(zP_ix$ and $zP_iy)$ for all $i\in L_1\cup L_3$. Also, if $|F(A,s)|=1$, then $s\notin S_1(A,F,S)$. Hence, we only have to consider $s\in S^n$ such that (i) $F(A,s)=\{x,z\}$, (ii) $F(A,s)=\{z,y\}$, or (iii) $F(A,s)=\{x,y,z\}$.

Proposition 1. *If* $F(\{a,b,c\},s)=\{a,b\}$, *and for any* $i\in L$, aP_ib *and* bR_ia, *then* $s\notin N_1(A,F,S)$.

Proof. Suppose the hypothesis of the proposition is true. Then, we have aIb and either aPc or bPc. Consider s' such that s' is i-variant from s, and $aP_i'bP_i'c$. Then, we must have $aP'b$, and $F(\{a,b,c\},s)=\{a\}$. By STP, $s'>_i^A s$. Hence, $s\notin N_1(A,F,S)$ and proposition 1 is true.

Note that by Proposition 1, if $F(A,s)=\{z,y\}$, then $s\notin N_1(A,F,S)$. This holds because we have zPy, and $F(A,s)=\{z,y\}$ implies zIy since F satisfies WB and $C(A,R^M)\neq\emptyset$ by assumption. Hence, for some $i\in L_{zy}$, we have yR_iz.

Now, suppose $F(A,s)=\{x,z\}$, and for all $i\in L$, $R_i:\{x,z\}=R_i:\{x,z\}$. (Of course, if this is not true, then $s\notin S_1(A,F,S)$ by Proposition 1). Let $|\{i\in L_2\cup L_3/yP_iz\}|=k_1$, $|\{i\in L_2\cup L_3/yI_iz\}|=k_2$. Consider s' such that for all $i\in L_2\cup L_3$, $R_i=R_i'$, and for all $i\in L_1\cup L_4$, $[(zP_i'x$ and $yP_i'x)$ and $|\{i|zP_i'y\}|=k_1$, $|\{i|zI_i'y\}|=k_2$, and $|\{i/yP_i'z\}|=\frac{1}{2}n-(k_1+k_2)]$. Noting that for $i\in L_3$, we have $zP_i'x$, $|L_{zx}'|>|L_{xz}'|$.

Then, $yI'z$, $zP'x$ and $yR'x$. Hence, $F(A,s')=\{y,z\}$, and by STP $s'>_i^A s$ for all $i\in L_1\cup L_4$. Therefore, $s\notin S_1(A,F,S)$.

We now have to show that if $F(A,S)=\{x,y,z\}$, then $\sim s\in S_1(A,F,S)$. Noting that $C(A,R^M)\neq\emptyset$, $(F(A,S)=\{x,y,z\})$ implies $(xIy$, yIz and $zIx)$ since F^M satisfies WB by assumption.

Proposition 2. *Suppose* aI^Mb, bI^Mc *and* cI^Ma *and for* $i\in L$, aP_ibP_ic *and* $\sim(aP_ic$ *or* $bP_ic)$. *Then* $\sim s\in N_1(A,F,S)$.

Proof. Without loss of generality, assume cR_ib. Consider s' such that s' is i-variant from s, and $R_i':\{a,b,c\}=R_i:\{a,b,c\}$. Thus, $bP'c$, $bR'a$ and $aR'c$. Hence, $c\notin F(A,s')$ and $b\in F(A,s')$. Therefore, $s'>_i^A s$. This proves the proposition.

By Proposition 2, if xI^My, yI^Mz and zI^Mx, then we only have to consider the case where

for all $i\in L_1$: zP_ix and yP_ix,

for all $i\in L_2$: xP_iy and zP_iy,

for all $i\in L_3$: zP_iy and xP_iy,

for all $i\in L_4$: yP_iz and xP_iz.

Noting that $|L_2 \cup L_3| = \frac{1}{2}n$, we must have yP_izP_ix for all $i \in L_1$, and yP_ixP_iz for all $i \in L_4$. This, in turn implies that for some $j \in L_2 \cup L_3 : zP_jxP_jy$ and for all $i \in L_2 \cup L_3 - \{j\} : xP_izP_iy$.

Consider s' which is $(L_1 \cup L_3)$-variant from s, and for all $i \in L_1 \cup L_3 : zP'_iyP'_ix$. Since $L'_{zy} = L - L_4$, we have $zP'y$. Obviously, $zR'x$. Also, note that

$$|L'_{yx}| = |L_1 \cup L_3 \cup L_4| = \tfrac{1}{2}n + 1.$$

Hence $yP'x$ and $F(A, s') = \{z\}$. By STP $s' >^A_i s$ for all $i \in L_1 \cup L_3$.

Therefore, $S_1(A, F, \bar{s}) = \emptyset$, which completes the proof of the theorem.

Remark. Theorem 2 shows that F^M violates consistency (Type 1) when n is *even*. It is conjectured that this result cannot be proved when n is *odd*. This conjecture is based on Theorem 3 of MacIntyre and Pattanaik (1979), who show that if individual preferences are restricted to be antisymmetric, then every SCF satisfying IIA, M, Schwartz's Rule and violating LIR is *strictly nonmanipulable* (Type 1).

Note that Theorem 2 shows that F^M violates consistency (Type 1). It would be interesting to see whether the result goes through for consistency (Type 2). Unfortunately, we have not been able to prove this stronger result. However, we show below that given Assumption STP, every SCF satisfying IIA, M, AV, LIR and WB violates almost exact consistency (Type 1). Under the stronger behavioural Assumption STP*, this class of SCF's also violates almost exact consistency (Type 2).

Theorem 3. *Suppose Assumption* STP *holds. Then every* SCF *satisfying* IIA, M, NT, AV, WB *and* LIR *violates almost exact consistency* (*Type* 1).

Proof. Let $A = \{x, y, z\}$. Let b^* be the *smallest* blocking coalition, and let $k \in b^*$. Construct $\bar{s}$ as follows:

$$\text{for all } i \in L - b^*: \quad y\bar{P}_iz\bar{P}_ix.$$

$$\text{for all } i \in b^* - \{k\}: \quad x\bar{P}_iy\bar{P}_iz, \;\; z\bar{P}_kx\bar{P}_ky.$$

It can be checked that $x\bar{I}y$, $y\bar{P}z$, $z\bar{P}x$ so that $F(A, \bar{s}) = \{y\}$. We show that there does not exist any $s \in S_1(A, F, \bar{s})$ such that $F(A, s) = \{y\}$.

Suppose $F(A, s) = \{y\}$. Consider s' such that for all $i \in b^*$, we have $xP'_iyP'_iz$, and for all $i \in L - b^*$, $R_i = R'_i$. Since $b^* \in B_F$, we have $xR'z$ and $xR'y$. Also, since $z \notin F(A, s)$, either yPz or xPz. Noting that $L_{xz} \subseteq L'_{xz}$ and $L_{yz} \subseteq L'_{yz}$, yPz implies $yP'z$ and xPz implies $xP'z$. Hence, $x \in F(A, s')$ and $z \notin F(A, s')$. Therefore, $s' >^A_i s$ for all $i \in b^*$.

Hence, $s \notin S_1(A, F, \bar{s})$.

Theorem 4. *Suppose Assumption* STP* *holds. Then every* SCF *satisfying* IIA, M, NT, AV, WB *and* LIR *violates almost exact consistency* (*Type* 2).

Proof. Let $A = \{x, y, z\}$.

Consider the same $\bar{s}$ as specified in Theorem 3 above. We know that if $F(A, s) = \{y\}$, then $s \notin S_1(A, F, \bar{s})$.

Also note that since y is the $\bar{P}_k$ worst element in A, there does not exist any s'' such that $s >_k^A s''$. Further, if s'' is $(L - b^*)$-variant from s', then $x \in F(A, s'')$. Hence, given Assumption STP*, there does not exist any $(L - b^*)$ variant situation s'' from s', such that $s >_i^A s''$ for any $i \in b^* - \{k\}$. Hence, $s \notin S_2(A, F, \bar{s})$.

Theorems 1–4 show that even if we relax the requirement that the social aggregation rule does not necessarily yield a single outcome for all configurations of preferences, the manipulability properties of a wide class of such aggregation rules are very weak indeed. These results are very disturbing particularly because we have used weak behavioural assumptions to determine individuals' preferences over possible sets of outcomes. Of course, Theorem 4 has been proved with the strong and rather implausible Assumption STP*, but the reader can check that the result goes through even if individuals are expected utility maximisers.

References

S. Barbera, The manipulability of social choice mechanisms that do not leave too much to chance, J. Econom. Theory 15 (1977) 266–278.

B. Dutta, Existence of stable situations, restricted preferences and strategic manipulation under democratic group decision rules, J. Econom. Theory 15 (1977) 99–111.

B. Dutta, On the possibility of consistent voting procedures, Rev. Econom. Studies 47 (1980) 603–616.

B. Dutta and P.K. Pattanaik, On nicely consistent voting systems, Econometrica 13 (1976) 163–170.

P. Gardenfors, Manipulation of social choice functions, J. Econom. Theory 13 (1976) 217–228.

A. Gibbard, Manipulation of voting schemes: a general result, Econometrica 45 (1977) 587–601.

A. Gibbard, Manipulation of schemes that combine voting with chance, Econometrica 45 (1977) 665–681.

J. Kelly, Strategyproofness and social choice functions without single valuedness, Econometrica 45 (1977) 439–446.

I. MacIntyre and P.K. Pattanaik, The sure-thing rule and strategic voting under minimally binary group decision functions, J. Econom. Theory, to appear.

P.K. Pattanaik, On the stability of sincere voting situation, J. Econom. Theory 6 (1973) 558–574.

P.K. Pattanaik, Strategy and Group Choice (North-Holland, Amsterdam, 1978).

B. Peleg, Consistent voting systems, Econometrica 46 (1978) 153–162.

M.A. Satterthwaite, Strategyproofness and Arrow's conditions, J. Econom. Theory 10 (1975) 187–217.