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AN ANALOGUE OF THE WIENER-TAUBERIAN THEOREM FOR SPHERICAL TRANSFORMS ON SEMI-SIMPLE LIE GROUPS

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Let G be a semi-simple connected noncompact Lie group with finite center and K a fixed maximal compact subgroup of G. Fix a Haar measure dx on G and let $I_1(G)$ denote those functions in $L^1(G, dx)$ which are biinvariant under K. The purpose of this paper is to prove that if $f \in I_1(G)$ is such that its spherical Fourier transform (i.e., Gelfand transform) $\hat{f}$ is nowhere vanishing on the maximal ideal space of $I_1(G)$ and $\hat{f}$ "does not vanish too fast at ∞", then the ideal generated by f is dense in $I_1(G)$. This generalizes earlier results of Ehrenpreis-Mautner for G=SL(2, $\boldsymbol{R}$) and R. Krier for G of real rank one.

1. **Introduction.** Let f be an L^1-function on $\boldsymbol{R}$ (or more generally on a locally compact abelian group). Then the celebrated Wiener-Tauberian theorem says that if the Fourier transform $\hat{f}$ is a nowhere vanishing function then the ideal generated by f is dense in $L^1(\boldsymbol{R})$. In [1] Ehrenpreis and Mautner observe that the corresponding result is not true if one considers the commutative Banach algebra of K-biinvariant functions on noncompact semi-simple Lie group G, where K is a maximal compact subgroup of G. More precisely, let $G = \mathrm{SL}(2, \boldsymbol{R})$ i.e., the group of 2×2 real matrices of determinant 1, and

$$K = \mathrm{SO}(2) = \left\{\begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix};\quad 0 \leqq \theta \leqq 2\pi\right\} \text{ and let}$$

$I_1(G)$ denote the commutative Banach algebra of K-biinvariant L^1-functions on G. For $f \in I_1(G)$, let $\hat{f}$ denote its spherical Fourier transform (see § 2). Then Ehrenpreis and Mautner observed that there exist functions $f \in I_1(G)$ such that $\hat{f}$ does not vanish anywhere on the maximal ideal space of $I_1(G)$ and yet the algebra generated by f is *not* dense in $I_1(G)$. However they were able to show that if $\hat{f}$ is non vanishing *and* $\hat{f}$ 'does not go to zero too fast at ∞' then the ideal generated by f is indeed dense in $I_1(G)$. (Theorems 6 and 7 of [1].) These results have been generalized by R. Krier [6] in his thesis when G is a noncompact connected semi-simple Lie group of real rank 1. (The author does not know whether Krier's results have been published.) The purpose of this note is to prove a theorem in the spirit of Theorem 7 of [1] without any restriction on the rank of G. While the basic technique we use is that of [1], we have to use the more recent results of Trombi-Varadarajan [7] and some observations of Gangolli-Warner [4] to prove our main theorem. Indeed in [3] Gangolli predicts that a theorem of the Trombi-Varadarajan type would yield a Tauberian type theorem.

2. Notation and preliminaries. (For any unexplained notation and terminology please see [5].) G will denote a connected non-compact semi-simple Lie group with finite center and K a fixed maximal compact subgroup of G. Fix an Iwasawa decomposition $G = KAN$ and let $\mathfrak{a}$ be the Lie algebra of A. Let $\mathfrak{a}^*$ be the real dual of $\mathfrak{a}$ and $\mathfrak{a}^*_c$ its complexification. Let ρ be the half-sum of the positive roots for the adjoint action of $\mathfrak{a}$ on $\mathfrak{g}$ (where $\mathfrak{g}$ is the Lie algebra of G). The Killing form will induce a form $\langle\cdot,\cdot\rangle$ on $\mathfrak{a}^*\times\mathfrak{a}^*$. Then, as is well known, $\langle\cdot,\cdot\rangle$ is positive definite on $\mathfrak{a}^*\times\mathfrak{a}^*$. Extend the form $\langle\cdot,\cdot\rangle$ to a bilinear form on $\mathfrak{a}^*_c\times\mathfrak{a}^*_c$. This bilinear form also will be denoted by $\langle\cdot,\cdot\rangle$. Let W be the Weyl group of the symmetric space G/K. Then there is a natural action of W on $\mathfrak{a}$, $\mathfrak{a}^*$ and $\mathfrak{a}^*_c$ and $\langle\cdot,\cdot\rangle$ is invariant under the action of W.

For each $\lambda\in\mathfrak{a}^*_c$ let ϕ_λ be the elementary spherical function associated with λ. (Recall that ϕ_λ is given by the formula, $\phi_\lambda(x) = \int_K e^{(i\lambda-\rho)(H(xk))}dk$ — see [5] for details.) Then it is known that $\phi_\lambda = \phi_{\lambda'}$ iff $\exists s \in W$ with $s\lambda = \lambda'$. Let $F = \{\lambda;\ \phi_\lambda$ is a bounded function on $G\}$. Then it is known (a theorem of Helgason and Johnson) that:

$$F = \mathfrak{a}^* + iC_\rho \text{ where } C_\rho = \text{convex hull of } \{s\rho : s\in W\}\ .$$

Let $P(\mathfrak{a}^*_c)$ be the symmetric algebra over $\mathfrak{a}^*_c$. Then each $u\in P(\mathfrak{a}^*_c)$ gives rise to a differentiatior $\partial(u)$ on $\mathfrak{a}^*_c$.

Let $I(G)$ be the set of all complex valued spherical functions on G, i.e., $I(G) = \{f; f(k_1xk_2) = f(x),\ k_1, k_2\in K,\ x\in G\}$. Fix a Haar measure dx on G and let $I_1(G) = I(G)\cap L^1(G)$. Then it is well known that $I_1(G)$ is a commutative Banach algebra under convolution (and that the maximal ideal space of $I_1(G)$ can be identified with F/W). We shall denote by $I^\infty(G)$ the space of C^∞-spherical functions and by $I_c^\infty(G)$ the space of compactly supported functions in $I^\infty(G)$.

For $f\in I_1(G)$ define its spherical Fourier transform, $\hat f$ on F by:

$$\hat f(\lambda) = \int_G f(x)\phi_{-\lambda}(x)dx\ ,\quad \lambda\in F\ .$$

Then it is known that $\hat f$ is a W-invariant bounded function on F, holomorphic in F^0(=interior of F) and continuous on F. Also $(f*g)\hat{} = \hat f\cdot\hat g$ for $f, g\in I_1(G)$ where $f*g$ is the convolution of f and g and is given by

$$(f*g)(y) = \int_G f(yx^{-1})g(x)dx\ , \quad y \in G\ .$$

If $f \in I_c^\infty(G)$ then $\hat{f}$ is defined on all of a_c^* (and in fact will be an entire W-invariant function on a_c^* satisfying the Paley-Wiener growth condition—see [2]).

We shall now introduce a space of rapidly decreasing functions in $I^\infty(G)$ which we will denote by $S_1(G)$. (This is the so called L^1-Harish-Chandra-Schwartz space of spherical functions):

Let $x \in G$. Then $x = k \exp X, k \in K, X \in p$ ($g = k + p$ is the Cartan decomposition of the Lie algebra g of G). Put $\sigma(x) = \|X\|$, where $\|\cdot\|$ is the norm induced on p by the restriction of the Killing form. For any left invariant differential operator D on G and any integer $r \geqq 0$, we define for $f \in I^\infty(G)$

$$p_{D,r}(f) = \sup_{x \in G} (1 + \sigma(X))^r |\phi_0(x)|^{-2} |Df(x)|$$

where ϕ_0 is the elementary spherical function corresponding to $\lambda = 0$. Define $S_1(G) = \{f; f \in I^\infty(G) \text{ and } p_{D,r}(f) < \infty \forall r, D\}$. $S_1(G)$ becomes a Frechet-space when equipped with topology induced by the family of semi norms $p_{D,r}$. It is known that $S_1(G) \hookrightarrow I_1(G)$ and $I_c^\infty(G) \hookrightarrow S_1(G)$ are both dense inclusions.

Now let $Z(F)$ be the space of functions f on F satisfying the following conditions: (i) f is holomorphic in F^0 and continuous on F, (ii) If $u \in P(a_c^*)$ and $l \geqq 0$ is any integer, then $q_{u,l}(f) = \sup_{\lambda \in F^0}(1 + \|\lambda\|^2)^l |(\partial(u)f)(\lambda)| < \infty$, (where $\|\lambda\|^2 = \|\lambda_1\|^2 + \|\lambda_2\|^2$, $\lambda = \lambda_1 + i\lambda_2$, $\lambda_1, \lambda_2 \in a^*$ and $\|\lambda_i\|^2 = \langle\lambda_i, \lambda_i\rangle$). Let $\bar{Z}(F)$ denote the subspace of $Z(F)$ consisting of W-invariant functions. $Z(F)$, $\bar{Z}(F)$ are algebras under pointwise multiplication and we topologize them by the family of semi norms $q_{u,l}$. In this topology $Z(F)$, $\bar{Z}(F)$ are Frechet spaces. If $a \in \bar{Z}(F)$ define the 'wave packet' ψ_a on G by:

$$\psi_a(x) = \frac{1}{|W|}\int_{a^*} a(\lambda)\phi_\lambda(x)c(\lambda)^{-1}c(-\lambda)^{-1}d\lambda\ ,$$

($|W|$ is the order of the Weyl group) .

($c(\lambda)$ is the well known c-function of Harish-Chandra and one knows that $c(\lambda)^{-1}c(-\lambda)^{-1}$ is a continuous function on a^* of at most polynomial growth. Further if $d\mu$ is the measure on a^* defined by $d\mu = |W|^{-1}c(\lambda)^{-1}c(-\lambda)^{-1}d\lambda$, then one knows that the map $f \to \hat{f}$ is an isometry of $I(G) \cap L^2(G)$ onto $L^2(a^*, d\mu)^W$ where the superscript W indicates Weyl-group invariants in $L^2(a^*, d\mu)$). We are now finally in a position to state the theorem of Trombi-Varadarajan [7]:

THEOREM 2.1. (i) *If* $f \in S_1(G)$, *then* $\hat{f} \in \bar{Z}(F)$.

(ii) *If $a \in \bar{Z}(F)$ then the integral defining the wave packet ψ_a converges absolutely and in fact $\psi_a \in S_1(G)$ and $\hat{\psi}_a = a$.*

(iii) *The map $f \to \hat{f}$ is a topological linear isomorphism of $S_1(G)$ onto $\bar{Z}(F)$.*

Before closing this section we introduce some more function spaces and state a proposition due to Gangolli-Warner [4]. (As the authors point out in [4] this proposition can be obtained by a careful examination of the proof of Theorem 2.1 of Trombi-Varadarajan.)

Let m, l be nonnegative integers and let us put $\bar{Z}_{m,l}(F)$ for the space of functions f on F such that (i) f is holomorphic in F^0, continuous on F, and invariant under the action of W (ii) If $u \in P(a_c^*)$ and degree $u \leq m$, then

$$q_{m,l}(f) = \sup_{\lambda \in F^0} (1 + \|\lambda\|^2)^l |(\partial(u)f)(\lambda)| < \infty .$$

Put $\bar{Z}_m(F) = \bigcap_{l \geq 0} \bar{Z}_{m,l}(F)$. Then tne following proposition is contained in Proposition 3.3 and Corollary 3.4 of Gangolli-Warner [4].

PROPOSITION 2.2. *Let G be a noncompact connected semi-simple Lie group with finite center. Then $\exists$ an integer m_G (depending only on the group G) such that if $a \in \bar{Z}_{m_G}(F)$, then:*

(i) *The integral defining the wave packet ψ_a converges absolutely.*

(ii) $\psi_a \in I_1(G)$.

3. An analogue of the Wiener-Tauberian theorem. Before we state and prove the main theorem we will first prove a couple of preliminary lemmas which will be used in the proof of the main theorem. The first lemma is a very mild strengthening of Proposition 2.2 and the second lemma is a slight generalization of Lemma 5.2 for the case of $G = SL(2, R)$ in [1].

LEMMA 3.1. *There exists an integer m_G (depending only on the group G) such that if $a \in \bar{Z}_{m_G}(F)$ then all the following conditions are satisfied*

(i) *The integral defining ψ_a (the wave packet) converges absolutely.*

(ii) $\psi_a \in I_1(G)$.

(iii) $\hat{\psi}_a = a$.

Proof. From Proposition 2.2 it follows that we can find an integer m_G such that if $a \in \bar{Z}_{m_G}(F)$ then (i) and (ii) are satisfied. We will show that (iii) is also satisfied. Observe first that if $a \in \bar{Z}_{m_G}(F)$, then since $(\forall l)$ it decays faster than $1/(1 + \|\lambda\|^2)^l$ on $\mathfrak{a}^*$ and since $c(\lambda)^{-1}c(-\lambda)^{-1}$ has at most polynomial growth, a is integrable with respect to the measure $c(\lambda)^{-1}c(-\lambda)^{-1}d\lambda$ on $\mathfrak{a}^*$. To prove that $\hat{\psi}_a = a$, we first show that

$$(*)\qquad \forall b \in \bar{Z}(F),\ \int_{\mathfrak{a}^*} a(\lambda)b(\lambda)c(\lambda)^{-1}c(-\lambda)^{-1}d\lambda = \int_{\mathfrak{a}^*} \hat{\psi}_a(\lambda)b(\lambda)c(\lambda)^{-1}c(-\lambda)^{-1}d\lambda .$$

The integral on the left hand side exists since both a, b decay faster than $1/(1 + \|\lambda\|^2)^l$ on $\mathfrak{a}^*$ and $c(\lambda)^{-1}c(-\lambda)^{-1}$ has at most polynomial growth. The integral on the right hand side exists because $\hat{\psi}_a$ is a bounded function (being the spherical Fourier transform of an integrable function) and b is a rapidly decreasing function. The proof of $(*)$ is a straightforward application of Fubini's theorem keeping in mind the following facts (i) Since $b \in \bar{Z}(F)$, $\psi_b \in S_1(G)$ and is hence integrable and further $\hat{\psi}_b = b$ (ii) ψ_a is an integrable function on G and $a(\lambda)$ is integrable with respect to $c(\lambda)^{-1}c(-\lambda)^{-1}d\lambda$. Since $(*)$ is true $\forall b \in \bar{Z}(F)$ and since $\bar{Z}(F)$ contains 'enough' functions it follows easily that

$$a(\lambda)c(\lambda)^{-1}c(-\lambda)^{-1} = \hat{\psi}_a(\lambda)c(\lambda)^{-1}c(-\lambda)^{-1} \quad \text{a.e. on } \mathfrak{a}^*$$

with respect to Lebesgue measure. But the zeros of $c(\lambda)^{-1}c(-\lambda)^{-1}$ must have zero Lebesgue measure in $\mathfrak{a}^*$ and hence it follows that $a = \hat{\psi}_a$.

LEMMA 3.2. *Let k be a fixed nonnegative integer and let $\phi(z) = e^{\langle z,z\rangle^k}$, $z \in F$. Define X by $X = \{h;\ h \in \bar{Z}(F) \text{ and } h\phi \in \bar{Z}(F)\}$. Then X is a dense linear subspace of $\bar{Z}(F)$.*

Proof. Let $\psi_n(z) = e^{-\langle z,z\rangle^{k+1}/n}$. Then since $\langle\cdot,\cdot\rangle$ is W-invariant, ψ_n, ϕ are W-invariant. It is easy to see that $\psi_n, \phi\psi_n \in \bar{Z}(F)$. (To see this observe that $F = \mathfrak{a}^* + iC_\rho$. Clearly $\psi_n, \phi\psi_n$ are rapidly decreasing on $\mathfrak{a}^*$, but if $z \in F$ the 'imaginary' part of z varies only over a compact set.) Hence if $f \in \bar{Z}(F)$, $f\phi\psi_n \in \bar{Z}(F)$. Now it is easy to see that as $n \to \infty$, $f\psi_n \to f$ in the topology of $\bar{Z}(F)$. But since $f\psi_n\phi \in \bar{Z}(F)$, $f\psi_n \in X$ and the lemma is proved.

We are now in a position to state and prove our main theorem.

THEOREM 3.3. *Let $f \in I_1(G)$ and suppose*

(i) $\hat{f}$ is nowhere vanishing on F.

(ii) $\exists$ a positive integer k such that for every $u \in P(\mathfrak{a}_c^*)$ with degree $u \leqq m_0$ (where m_0 is as in Lemma 3.1) we have

$$\sup_{z\in F^0} |\partial(u)[(\hat{f}(z))^{-1}e^{-\langle z,z\rangle^k}]| < \infty .$$

Then the ideal generated by f is dense in $I_1(G)$.

Proof. (Note: condition (ii) says that '$\hat{f}$ does not vanish too fast at ∞'.) Let X be as in Lemma 3.2. Let $Y = \{\psi_a; a \in X\}$. Since by Lemma 3.2 X is dense in $\bar{Z}(F)$, by Theorem 2.1, Y is dense in $S_1(G)$. Hence since $S_1(G) \hookrightarrow I_1(G)$ is a dense inclusion, Y is a dense subspace of $I_1(G)$. We will show that every $h \in Y$ can be written as $h = f*g$, with $g \in I_1(G)$ and this will prove the theorem. Now if $h \in Y$, $\hat{h} \in X$ and $\hat{h} = \hat{f}\cdot\hat{f}^{-1}\hat{h}$.

(Note that since $\hat{f}$ does not vanish on F, $\hat{f}^{-1}$ is well defined on F.)

Now we claim $\hat{f}^{-1}\hat{h}$ is in $\bar{Z}_{m_0}(F)$. This follows from the definition of X and condition (ii) of Theorem 3.3 (since $\hat{f}(z)^{-1}h(z) = \hat{f}(z)^{-1}$ $e^{-\langle z,z\rangle^k}e^{\langle z,z\rangle^k}\hat{h}(z)$). Hence by Lemma 3.1 $\psi_{\hat{f}^{-1}\hat{h}} \in I_1(G)$ and $\hat{\psi}_{\hat{f}^{-1}\hat{h}} = \hat{f}^{-1}\hat{h}$.

Claim: $h = f*\psi_{\hat{f}^{-1}\hat{h}}$. This is because

$$(f*\psi_{\hat{f}^{-1}\hat{h}})\hat{} = \hat{f}\hat{\psi}_{\hat{f}^{-1}\hat{h}} = \hat{f}\hat{f}^{-1}\hat{h} = \hat{h} .$$

Hence (by the semi simplicity of $I_1(G)$) $f*\psi_{\hat{f}^{-1}\hat{h}} = h$. Thus we have shown that every function h in a dense subspace Y of $I_1(G)$ can be writted as $h = f*g$ and this concludes the proof of our theorem.

(Note: For $G = \mathrm{SL}(2, \boldsymbol{R})$ or more generally for G a real rank one group $m_0 = 2$ (see [1], [6]).)

4. The case of L^p for $1 \leqq p \leqq 2$. For $\varepsilon \geqq 0$, let $F^\varepsilon = \mathfrak{a}^* + i\varepsilon C_\rho$. Then one can introduce the spaces $Z(F^\varepsilon)$, $\bar{Z}(F^\varepsilon)$ just as in § 2. Let $I_p(G) = I(G) \cap L^p(G)$. Then one can define the so called L^p-Harish Chandra-Schwartz subspace of K-biinvariant functions i.e., $S_p(G) \subsetneqq I_p(G)$ (see [7] for details). Actually the theorem of Trombi-Varadarajan is more general than stated in § 2. In fact they show that under the map $f \to \hat{f}$ the spaces $S_p(G)$ and $\bar{Z}(F^\varepsilon)$ where $\varepsilon = 2/p - 1$ are topologically isomorphic ($p \leqq 2$). Also one knows that if $p \geqq 1$ then $S_1(G) \hookrightarrow S_p(G)$ is a dense inclusion. Using this one can modify the arguments in the last section to obtain the following theorem.

THEOREM 4.1. *Let $1 \leqq p < 2$ and $f \in I_p(G) \cap I_1(G)$, such that:*

(i) $\hat{f}$ is nowhere vanishing on F.

(ii) $\exists$ a positive integer k such that for every $u \in P(\mathfrak{a}^*_c)$ with degree $u \leqq m_0$ (m_0 as in Lemma 3.1), we have

$$\sup_{z\in F^0} |\partial(u)[(\hat{f}(z))^{-1}e^{-\langle z,z\rangle^k}]| < \infty .$$

Then the set of functions of the form $g * f$, $g \in I_2^\infty(G)$, is dense in $I_2(G)$.

Finally we observe that the Plancharel theorem for $I_2(G)$ (i.e., the spherical Fourier transform is an isometric isomorphism of $I_2(G)$ onto $L^2(\mathfrak{a}^*, \mu)^W$, where the superscript indicates Weyl-group invariance and μ is the measure on $\mathfrak{a}^*$ defined by $d\mu = |W|^{-1}c(\lambda)^{-1}c(-\lambda)^{-1}d\lambda$) gives us the following fact: Let $f \in I_2(G)$ such that $\hat{f}$ is nonvanishing on $\mathfrak{a}^*$ except possibly on a set of μ-measure zero. Then the set of functions of the form $g * f$, $g \in I_2^\infty(G)$ is dense in $I_2(G)$. (The proof of this fact is exactly as in the case of abelian groups).

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