CHARACTERIZATION OF FORCIBLY LINE-GRAPHIC DEGREE SEQUENCES

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ABSTRACT. Let π be a graphic sequence of positive integers. Call π forcibly line-graphic if every realization of π is a line graph. In this paper we determine the forcibly line-graphic degree sequences. The proof uses the 'laying off' technique developed by Kleitman and Wang to construct a realization of a graphic sequence.

1. *Introduction.*

Let $\pi = (d_1,\ldots,d_p)$ be a nonincreasing sequence of positive integers. Call π *graphic* if there exists a graph with degree sequence π. Let P be an invariant property of graphs, that is, a property depending only on the isomorphic types of graphs. Call a graphic sequence π *potentially* P (*forcibly* P) if at least one (respectively, every) realization of π has the property P. The characterization of forcibly hamiltonian, potentially planar, potentially line-graphic degree sequences are some of the unsolved problems in this area. Characterization of potentially self-complementary degree sequences was obtained by Clapham and Kleitman [1], and that of forcibly self-complementary degree sequences was given in Rao [4]. In this paper we characterize forcibly line-graphic (equivalently, potentially non-line-graphic) degree sequences.

Let $\pi = (d_1,\ldots,d_p)$ be a nonincreasing sequence of positive integers. By the *residual sequence obtained after laying off* d_j *from* π, we mean the nonincreasing rearrangement of the sequence π^*, where

$$\pi^* = \begin{cases} d_1-1,\ldots,d_{d_j}-1, d_{d_j+1},\ldots,d_{j-1}, d_{j+1},\ldots,d_p & \text{if } d_j < j\,, \\ d_1-1,\ldots,d_{j-1}-1, d_{j+1}-1,\ldots,d_{d_j+1}-1, d_{d_j+2},\ldots,d_p & \text{if } d_j \ge j\,. \end{cases}$$

We record here three theorems which are used repeatedly in our discussion.

THEOREM A. (Kleitman and Wang [5,6]). *Suppose* π *is graphic; then the residual sequence obtained after laying off* d_j *from* π *is also graphic for every* j, $1 \le j \le p$.

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Further, a realization of π can be constructed from any realization of π^* by adding a new point adjacent to points of degrees $d_1-1, d_2-1, \ldots, d_{d_j}-1$ if $d_j < j$ and of degrees $d_1-1, \ldots, d_{j-1}-1, d_{j+1}-1, \ldots, d_{d_j+1}-1$ if $d_j \geq j$.

Let $\pi = (d_1, \ldots, d_p)$ be a nonincreasing sequence. For every integer r, $1 \leq r < p$, define

$$EG(r, \pi) = r(r-1) + \sum_{i=r+1}^{p} \min\left\{d_i, r\right\} - \sum_{i=1}^{r} d_i .$$

The theorem of Erdős-Gallai [2, Theorem 6.2] states that a sequence π with even sum is graphic if and only if $EG(r, \pi)$ is non-negative for every r, $1 \leq r < p$.

We use the following mild form of Koren's theorem [3]:

THEOREM B. (Koren). *Let* π *be graphic and* $EG(k, \pi) = 0$ *for some* k, $1 \leq k < p$. *Suppose* $d_{k+1} \leq k$. *In any realization* $G = G(u_1, \ldots, u_p)$, *where the degree of* $u_i = d_i$,

$\langle u_1, \ldots, u_k \rangle$ *is the complete graph; and*

$\langle u_{k+1}, \ldots, u_p \rangle$ *is the empty graph.*

To state Theorem C we need a definition. A triangle of a graph G is called *odd* if there is a point of G adjacent to an odd number of its points.

THEOREM C. (Van Rooij, Wilf [P 74,2]). G *is a line graph if and only if* G *does not have a* $K_{1,3}$ *as an induced subgraph, and if two odd triangles have a common line, then the subgraph induced by their points is* K_4, *that is, the complete graph of order 4.*

For terminology and notation we follow Harary [2].

2. *Characterization.*

We remark that every graphic sequence with maximum degree at most two is forcibly line-graphic. So we assume henceforth that the maximum degree in a graphic sequence is at least three.

LEMMA 1. *Let* $\pi = (d_1, \ldots, d_p)$ *be a nonincreasing graphic sequence with* $d_p \geq 3$. *Then* π *is forcibly line-graphic if and only if one of the following holds:*

(1) $\pi = (4,3,3,3,3)$;

(2) $\pi = (4,4,4,4,4,4)$;

(3) $\pi = (p-1, \ldots, p-1)$.

Proof. Suppose π is one of the sequences (1), (2), or (3). Then π has a unique realization G_i accordingly as π is as in (i), $1 \leq i \leq 3$, where G_1 and G_2 are as in Figure 1 and G_3 is the complete graph

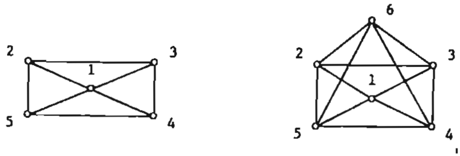

Figure 1

The graphs G_1 and G_2

of order p. Clearly, G_i is a line graph, $1 \leq i \leq 3$. Thus π is forcibly line-graphic.

To prove the necessity, assume that it is false for some value of p and let n be the smallest such p. Let $\pi_0 = (d_1, \ldots, d_n)$ be a graphic sequence of length n different from (1), (2), and (3) with $d_n \geq 3$ and which is forcibly line-graphic. We first derive several properties of this π_0 and then complete the proof of the lemma. Note that $n \geq 5$.

Case I: $d_n \neq 3$. Note that $d_n \geq 4$. Then lay off d_n from π_0 to obtain the residual sequence π_1. By Theorem A, π_1 is graphic. Also the minimum degree in π_1 is at least 3. If π_1 equals (1) or (2), then π_0 equals (2) or (5,5,5,5,4,4,4). Then let G_4 be the graph obtained by joining a new point to the points 1,2,3, of G_2. Note that (1,2,3), (2,3,6) are odd triangles in G_4 with a common line and $\langle 1,2,3,6\rangle \neq K_4$. Hence by Theorem C, G_4 is not a line graph. If π_1 equals (3) with p replaced by $n-1$, then π_0 is the degree sequence of the graph G_5 obtained from K_{n-1} by joining a new point x to d_n points of K_{n-1}. Let a,b be two points adjacent to x in G_5 and let c be a point non-adjacent to x. Since $3 \le d_n \le n-2$ and $n \ge 5$, it follows that (x,a,b), (a,b,c) are odd triangles in G_5 with a common line. Clearly, $\langle x, a,b,c\rangle \neq K_4$. Consequently G_5 is a non-line-graphic realization of π_0. We may assume therefore that π_1 is different from (1), (2), and (3) and the minimum degree of π_1 is at least 3. Then, by definition of n, π_1 has a non-line-graphic realization. But then, by the Wang and Kleitman theorem and the fact that an induced subgraph of a line graph is also a line graph, it follows that π_0 is potentially non-line-graphic, a contradiction.

Case II: $d_3 \neq 3$. Then $d_3 \ge 4$. By (1), we have $d_n = 3$. Now lay off d_n from π_0 to get π_1. If π_1 equals (1), then π_0 is one of the sequences (4,4,4,4,3,3), (5,4,4,3,3,3). The graph G_6 obtained from $K_{3,3}$ with bipartition (u_1,u_2,u_3), (v_1,v_2,v_3) by adding the two lines (u_1,u_2), (v_1,v_2) is a non-line-graphic realization of the former since (u_1,u_2,v_1), (u_1,u_2,v_3) are odd triangles in G_6 with a common line and (v_1,v_3) is not a line. The graph G_7 obtained from $K_{3,3}$ by adding the two lines (u_1,u_2), (u_1,u_3) is a non-line-graphic realization of the latter since $\langle u_1,v_1,v_2,v_3\rangle = K_{1,3}$. If π_1 is (2), then $\pi_0 = (5,5,5,4,4,4,3)$ and the graph G_8 obtained from G_2 by joining a new point x to the points 1,4,6 of G_2 is a non-line-graphic realization of π_0 since $(x,1,4)$, $(x,6,4)$ are odd triangles in G_8 with a common line, but (1,6) is not a line of G_8. If π_1 equals (3), it can be shown as in I that π_0 is potentially non-line-graphic, a contradiction.

Case III: $d_2 \neq 3$ and 4. Then $d_2 \ge 5$. Note that $d_3 = 3$ by II. Lay off d_n from π_0 to get π_1 and then lay off the degree 2 from π_1 to get π_2. If π_2 equals (1) then $\pi_0 = (6,5,3,3,3,3,3)$. Since (4,2,2,2,2,2) is potentially non-line-graphic, so is π_0. If $\pi_2 = (3,3,3,3)$ then $\pi_0 = (5,5,3,3,3,3)$. π_0 is unigraphic and that graph is not a line graph. Otherwise π_0 is not equal to (2) or (3) (with $p \ge 5$). Now, by definition of n, π_2 and hence π_0 is potentially non-line-graphic, a contradiction.

Case IV: $d_2 \neq 3$. Then $d_2 = 4$, the only other possible value by III. Lay off d_1 from π_0 to obtain π_1. Let n_i be the number of terms in π_1 which are equal to i, $i = 2,3$. Clearly, $n_2 + n_3 = n-1$, $n_2 \geq 3$, and n_3 is a positive even integer. The graph H_1 of Figure 2 is a non-line-graphic realization of π_1 where $x = (n_3-2)/2$, $y=n_2-3$. This is a contradiction.

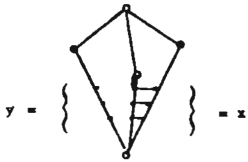

Figure 2.
The graph H_1.

We now complete the proof by showing that $\pi = (d_1, 3,\ldots,3)$ is potentially non-line-graphic. If $n = 5$, then π_0 equals (1). If $n \geq 6$, lay off d_1 to get π_1 and define n_2, n_3 as in IV. If $n_3 = 0$, then the wheel of order $n-1$ is a non-line-graphic realization of π_0 since $n \geq 6$. In the case $n_3 \geq 1$, we proceed as in Case IV to show that π_0 is potentially non-line-graphic, a contradiction and this completes the proof of the lemma.

LEMMA 2. *Let* $\pi = (d_1,\ldots,d_p)$ *be a nonincreasing graphic sequence with* $d_p \leq 2$. *Let* $k(\pi) = k$ *be the largest integer* i, $1 \leq i < p$, *such that* $d_i \geq 3$. *Suppose* $EG(k, \pi) > 0$. *Then* π *is potentially non-line-graphic.*

Proof. The proof is by induction on p. The only graphic sequences of length 4 and $d_4 \leq 2$ are (3,2,2,1), (3,1,1,1), and (3,3,2,2). Further, $EG(k, \pi) = 0$ for each of these three sequences. Thus the lemma is true for $p=4$. Assume that the lemma holds for $p-1$ and let π be a graphic sequence of length p satisfying the conditions of the lemma. In the case $d_2 \le 2$ or $d_1 = 3$, it is not difficult to show, by direct construction, that π is potentially non-line-graphic. Thus we may assume that $d_1 \ge 4$ and $d_2 \ge 3$. We prove the lemma only in the case $d_p = 2$, since the case $d_p = 1$ is similar. Lay off d_p from π to get π_1. If the minimum degree of π_1 is at least 3, π_1 and hence π is potentially non-line-graphic by Lemma 1 unless π_1 is one of (1), (2), or (3) of Lemma 1. Since $EG(k, \pi) > 0$, π_1 is not equal to (3). The condition π_1 equal to (1) or (2) implies that π is one of (5,4,3,3,3,2), (4,4,4,3,3,2), or (5,4,3,3,3,2). The graphs obtained by joining a new point x to 2 and 4 in G_1, G_2 of Figure 1 are non-line-graphic realizations of the second and third sequences respectively since (2,1,5) and (4,1,5) are odd triangles with a common line but (2,4) is not a line. The graph obtained by joining a new point x to 1,2 of G_1 is a non-line-graphic realization of the third since $\langle 1, x, 3, 5\rangle = K_{1,3}$. Thus we may assume that the minimum degree in π_1 is at most two. If now $EG(k_1, \pi_1) > 0$, where $k_1 = k(\pi_1)$, then by the induction hypothesis, π_1 is potentially non-line-graphic. This in turn implies that π is also potentially non-line-graphic. So we may assume that $EG(k_1, \pi_1) = 0$. This in particular shows that $d_2 = 3$, for if $d_2 \ge 4$, then $k=k_1$ and $EG(k, \pi) = EG(k_1, \pi_1) = 0$, contradicting the hypothesis.

Thus $d_2 = 3$ and hence $k_1 = k-1$. Now by Theorem B in every realization of π_1 the k_1 vertices of degree greater than two are complete. Since $d_2 = 3$, we have $k_1 \le 4$. Now $k_1 \ne 4$, for otherwise the vertex of degree d_2-1 in π_1 is joined to two vertices of degree greater than 3, implying that $d_2 \ge 4$, which is clearly false.

Case 1. $k_1 = 3$. Let H be a realization of π_1 in which the point u_2 of degree d_2-1 is adjacent to u_1 of degree d_1-1 and u_3 of degree d_3. Let G be the realization of π obtained from H by joining u_p to the points u_1, u_2 of H. Let u_i be the vertex not equal to u_1, u_3 adjacent to the other vertex u_4 of degree 3 in H. Since $d_i = 2$, it follows that (u_1, u_i) is a line of H and hence one of G. But then $\langle u_1, u_3, u_i, u_p\rangle = K_{1,3}$, which implies that π is potentially non-line-graphic.

Case 2. $k_1 = 2$. Define G as above. Any vertex u_j $(j \ne p)$ adjacent to u_1 is adjacent to u_3 as well. This implies that $d_1 = 4$ and

$\pi = (4, 3, 3, 2, 2)$. Here $k = 3$ and $EG(k, \pi) = 0$, contradicting the hypothesis.

Case 3. $k_1 = 1$. Let H be any realization of π_1 and G be the graph obtained from H by joining u_p to the points u_1, u_2 of H. Let u_i be the vertex adjacent to u_2 in H. Then (u_1, u_i) is a line of H. Since $d_1 \geq 4$ there is at least one more vertex u_j such that (u_1, u_j) is a line of H. Then in $G, \langle u_1, u_i, u_j, u_p \rangle = K_{1,3}$. This implies that π is potentially non-line-graphic. This completes the proof of the lemma.

THEOREM 3. *Let* $\pi = (d_1, \ldots, d_p)$ *be a nonincreasing sequence with even sum and* $d_p \leq 2$. *Let* n_i *be the number of terms in* π *equal to* i, $i = 1, 2$. *Define* $k = p - n_1 - n_2$. *Suppose* $k \geq 4$. *Then* π *is forcibly line-graphic if and only if*

(1) $EG(k, \pi) = 0$,

(2) $d_1 = k$,

(3) $2n_2 + n_1 \leq k$.

Proof. Suppose π is forcibly line-graphic. Then (1) follows from Lemma 2. By Theorem B, in any realization of π, the k points of degree greater than 2 induce a complete graph and the remaining $p-k$ (> 0) points induce the empty graph. Now if $d_1 > k$, then, since $k \geq 4$, any realization of π contains $K_{1,3}$ as an induced subgraph. Consequently π is not forcibly line-graphic. Thus $d_1 = k$, proving (2). To prove (3), we note that $2n_2 + n_1 > k$ implies, by Theorem B, that $d_1 > k$.

Conversely, suppose π is a sequence satisfying (1), (2), and (3). By (1) and (3), π is graphic. By Theorem B, the only realization G of π is the line graph of the connected graph H consisting of a cut vertex with the property that the cut vertex belongs to exactly $k - n_2$ pieces of which n_2 are triangles, n_1 are $K_{1,2}$ and the remaining $k-2n_2-n_1$ are edges, where a piece of G with respect to a cut vertex x is the subgraph induced on $V(C_1) \cup x$, where C_1 is a component of $G - x$. Note that H has $3n_2 + 2n_1 + (k - 2n_2 - n_1) = k + n_2 + n_1 = p$ edges, and $k - 2n_2 - n_1 \geq 0$. This completes the proof of the theorem.

THEOREM 4. *Let* π *be a nonincreasing graphic sequence with* $d_p \le 2$. *Define* k *as in Lemma 2. Suppose* $k \le 3$. *Then* π *is forcibly line-graphic if and only if* π *is one of the following:*

(F_1) (4,3,3,2,2) ,

(F_2) (4,4,4,2,2,2) ,

(F_3) (3,3,3,2,1) ,

(F_4) (3,3,3,1,1,1) ,

(F_5) (3,3,2,2) ,

(F_6) (3,3,2,1,1) ,

(F_7) (3,2,2,1) ,

(F_8) (4,2,2,2,2) .

Proof. Let π be one of (F_1) through (F_8); then π is realizable as a unique graph and this graph is a line graph. Thus π is forcibly line-graphic.

Conversely, let π be forcibly line-graphic. Then by Lemma 2, $EG(k, \pi) = 0$. By hypothesis $k \le 3$.

Case 1. $k = 3$. Let u_1, u_2, u_3 be the vertices of degree greater than 2 in a realization of G of π. By Theorem B, the subgraph induced on the remaining p-3 vertices of G is the empty graph. If $d_1 \ge 5$, then G has $K_{1,3}$ as an induced graph. Thus $d_1 \ge 4$. Suppose $d_1 = 4$, and let u_i, u_j be the points $(i,j > 3)$ adjacent to u_1 in G. If one of u_i, u_j is of degree 1, then G has $K_{1,3}$ as an induced subgraph. Thus we may assume that both u_i, u_j have degree 2. If both u_i, u_j are joined to the same set of points, then again $K_{1,3}$ is an induced subgraph of G. Now in case $p = 5$, π equals (F_1), otherwise $p = 6$ and π equals (F_2). Suppose now $d_1 = 3$, then π is (F_3) or (F_4).

Case 2. $k = 2$. If $d_1 \ge 4$, we get a non-line-graphic realization of π. Thus $d_1 = 3$. But then π equals (F_5) or (F_6).

Case 3. $k = 1$. If $d_1 = 5$, then we have a non-line-graphic realization of π. Thus $d_1 = 4$ or 3. In case $d_1 = 3$, π equals (F_7) and finally if $d_1 = 4$, then π equals (F_8) and this completes the proof of the theorem.

By the above characterization we note the curious and interesting fact that if π is a graphic sequence with at least one degree greater than two and π is not unigraphic, then π has a non-line-graphic realization.

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