BIGRACEFUL GRAPHS-I

T. Gangopadhyay and S. P. Rao Hebbare

ABSTRACT. Characterization of the class of graceful graphs is an open problem. A necessary condition for an eulerian graph with e edges to be graceful is that $e \equiv 0$ or $3 \pmod 4$. In this paper, the authors have studied the class of bigraceful (bipartite and graceful) graphs. Here, some infinite classes of bigraceful graphs, related to products of graphs, are presented.

1. *Introduction.*

Throughout this paper, the word 'graph' will mean a finite, undirected graph without loops and multiple edges. Let $G(N, E)$ be a connected graph with N, E as its node, edge sets respectively. Let $|N| = n$ and $|E| = e$. Then G is an (n, e)-graph.

By *node-numbering* of a graph G, we mean an assignment of distinct non-negative integers, called *node numbers*, to the nodes of G, such that the edges of G receive distinct positive integers, called *edge numbers*, where the edge (i, j) is assigned the edge number $|a_i - a_j|$ if a_i, a_j are the node numbers received by the i^{th} and j^{th} nodes. Every finite graph admits such a node-numbering. We are interested in minimizing the maximal node number. Let $N(G)$ denote the set of all node-numberings of G. We define,

$$O(G) = \min_{N(G)} \{\max_{i \in N} \{a_i\}\} .$$

$O(G)$ is called *optimal number* of G. A node-numbering of G which attains $O(G)$ is called an *optimal node-numbering* of G. Clearly, $O(G) \geq e$. A graph G for which $O(G) = e$ is called a *graceful graph* and the corresponding node-numbering is called a *graceful node-numbering*. A graph G is said to be *non-graceful* if it does not admit a graceful node-numbering. Further, a bipartite graceful graph is called *bigraceful* and a bipartite nongraceful graph is called *non-bigraceful*. For the literature on graceful graphs we refer to [2, 3, 4, 6] and the relevant references given in them.

In the case of eulerian graphs Golomb [2] has proved the following

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THEOREM 1.1. *A necessary condition for an eulerian graph* G *to be graceful is that* $[(e+1)/2]$ *is even.*

Hence, it follows that an eulerian graph with $e \equiv 1$ or 2(mod 4) is non-graceful. Further, if G is bipartite and eulerian, the above necessary condition implies that $e \equiv 0 \pmod 4$. It is not known whether this condition is also sufficient for this class of graphs.

In this paper certain infinite families of bigraceful graphs are presented for the first time. The classes of graphs treated are related to products of graphs. All the theorems are stated without proof. However, the proofs are all constructive in nature and anybody interested is welcome to write to the authors for a more detailed technical report.

2. *Bigraceful Distance Convex Simple Graphs.*

Below, we define a distance convex simple (d.c.s.) graph. A subset S of N is said to be a *distance convex* (d-convex) set of G if for any two distinct nodes u, v of G all the nodes on all u - v goedesics in G are contained in S. A graph G is said to be a d.c.s. graph if the only d-convex sets of G are the empty set, the singleton sets, all the sets with two nodes which are adjacent in G, and N itself. For further details see [4,5].

In this section, two infinite families of graceful d.c.s. graphs are given. We shall first describe a method of construction of a large family of d.c.s. graphs given in [5].

Construction. Let G be a connected graph of order n, with e edges without triangles. Let $G_1, G_2, \ldots, G_\lambda$ be $\lambda \geq 2$ copies of the graph G. Label the vertices of G_i by $V_1^i, \ldots, V_n^i$ $(i = 1,2,\ldots,\lambda)$. Define a $(\lambda n, \lambda^2 e)$-graph $D_\lambda(G)$ consisting of the above λ copies of G with the following additional edges, V_j^i is adjacent to V_l^k if and only if V_j^k is adjacent to V_l^k for each $i, k = 1,\ldots,\lambda$ and $j, l = 1,\ldots,n$.

We conclude the section by stating the following two theorems without proof.

THEOREM 2.1. $D_\lambda(P_n)$ *is graceful for* $\lambda \geq 2$, $n \geq 2$.

THEOREM 2.2. $D_\lambda(C_n)$ *is graceful for* $\lambda \geq 2$ *and* $n \geq 4$, n *even*.

3. *Some Bigraceful Graphs.*

In this section, we consider some families of bigraceful graphs which are subgraphs of $D_\lambda(P_n)$ and $D_\lambda(C_n)$.

(i) Cylindrical graphs: $C_\lambda(P_n)$ *and* $C_\lambda(C_n)$.

Consider λ copies of $P_n: P_n^1, P_n^2, \ldots, P_n^\lambda$, $\lambda \geq 2$.

Label the vertices of P_n^i by $v_1^i, \ldots, v_n^i$ $(i = 1, \ldots, \lambda)$. $C_\lambda(P_n)$ is the graph consisting of $P_n^1, \ldots, P_n^\lambda$ with the following additional edges: v_j^i is adjacent to v_l^{i+1} if and only if v_j^{i+1} is adjacent to v_l^{i+1} in P_n^{i+1} for each $i = 1, \ldots, \lambda - 1$ and $j = 1, \ldots, n$. The graph $C_\lambda(C_n)$ is similarly defined replacing P_n by C_n. The name is derived from the shape of $C_\lambda(C_n)$.

We now state the following two theorems without proof.

THEOREM 3.1. $C_\lambda(P_n)$ *is graceful for* $\lambda \geq 2$ *and* $n \geq 2$.

THEOREM 3.2. $C_\lambda(C_n)$ *is graceful for* $\lambda \geq 4$ *and* $n \equiv 0 \pmod 4$.

As an illustration, a graceful numbering of the graph $C_4(P_4)$ according to Theorem 3.1 is given in Figure 1.

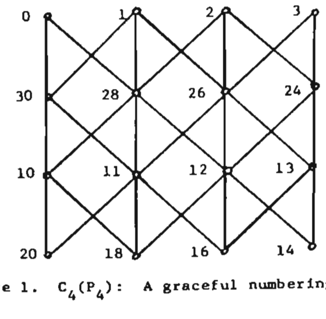

Figure 1. $C_4(P_4)$: A graceful numbering.

(ii) Torus graphs: $T_\lambda(P_n)$ *and* $T_\lambda(C_n)$.

Let $P_n^1, P_n^2, \ldots, P_n^\lambda$ be $\lambda \geq 4$ copies of P_n, $n \geq 2$. Label the vertices of P_n^i by $v_1^i, \ldots, v_n^i$ $(i = 1, \ldots, \lambda)$. $T_\lambda(P_n)$ is the graph consisting of $P_\lambda^1, \ldots, P_n^\lambda$ with the following additional edges: v_j^i is adjacent to v_1^{i+1}, where the superscript is reduced (mod λ), for each $i = 1, \ldots, \lambda$ and $j = 1, \ldots, n$. The graph $T_\lambda(C_n)$ is defined similarly for C_n instead of P_n. $T_\lambda(P_n)$ is a $(\lambda\, n,\ 3\lambda(n-1)$ graph and $T_\lambda(C_n)$ is a $(\lambda n,\ 3\lambda n)$ graph. We remark that $T_\lambda(P_n)$ and $T_\lambda(C_n)$ are d.c.s. graphs for $\lambda = 4$.

We now state the following theorems on torus graphs without proof.

THEOREM 3.3. $T_\lambda(P_n)$ *is graceful for any* $\lambda \geq 4$, *even, and* $n \geq 2$.

THEOREM 3.4. $T_\lambda(P_n)$ *is graceful for* $n \geq 2$ *and* λ *odd.*

THEOREM 3.5. $T_\lambda(C_n)$ *is graceful for* $n \geq 2$, $n \equiv 0 \pmod 4$, *and* λ *even.*

THEOREM 3.6. $T_\lambda(C_n)$ *is graceful for* $n \geq 4$, $n \equiv 0 \pmod 4$, *and* λ *odd.*

Concluding Remarks.

The study of bigraceful graphs has been taken up in this series of papers. Themes of further research may be phased in terms of conjectures. Related to the results of Section 2 is

Conjecture 1. Let G be a bigraceful graph. Then $D_{2t}(G)$, $t \geq 1$, is bigraceful.

We have found at least one graceful node-numbering for cubes Q_n for $n = 2, 3, 4$.

Conjecture 2. All cubes are graceful.

Besides the non-bigraceful graphs implied by Theorem 1.1 no other non-bigraceful graph is known in the literature. Recently such a non-bigraceful graph has been found in Devarajan et al. [1]. P. Erdős has proved the following result mentioned in [2] as unpublished work:

0% of all graphs are graceful.

In the class of bipartite graphs we make the following, similar,

Conjecture 3. 0% of all bipartite non-eulerian graphs with $e \equiv 2 \pmod 4$ are non-graceful.

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Statistics-Mathematics Division
Indian Statistical Institute
Calcutta

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