ON THE EQUIVALENCE OF EFFICIENCY-CONSISTENCY AND ORTHOGONAL FACTORIAL STRUCTURE

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ABSTRACT. Orthogonal factorial structure is shown to be a necessary and sufficient condition for efficiency-consistency in connected and regular disconnected designs. Thus, the result of Lewis and Dean (1985) is generalized and the converse is proved. The equivalence of partial orthogonal factorial structure and partial efficiency-consistency is investigated.

1. Introduction.

Consider a block design, d, whose n-digit treatment labels represent the treatment combinations of an n-factor experiment. Let T denote the set of non-null binary vectors $x = x_1x_2\dots x_n$, $(x_i = 0$ or $1; i = 1,\dots,n)$, and let α^x denote the interaction between those factors for which $x_i = 1$, $i = 1,\dots,n$. Let d_x denote the design formed from d by deleting the i^{th} digit from the treatment labels for all i for which $x_i = 0$, $i = 1,\dots,n$.

The design, d, is defined to have orthogonal factorial structure if the best linear unbiased estimators of the estimable contrasts belonging to different factorial spaces are uncorrelated after adjusting for block effects (see, for example, [4]). Lewis and Dean (1985) established that for an equireplicate, connected, n-factor design, d, orthogonal factorial structure is sufficient for efficiency-consistency, where efficiency-consistency is defined as follows:

DEFINITION 1 (LEWIS AND DEAN (1985)): An n-factor design, d, is *efficiency-consistent* if the efficiencies of all the estimable contrasts corresponding to α^x are equal to the efficiencies of the equivalent contrasts in d_x, $x \in T$.

The purpose of this paper is firstly to generalize the result of Lewis and Dean (1985) to disconnected designs, and secondly to establish that orthogonal factorial structure is a necessary condition for efficiency-consistency. Thus it is shown that efficiency-consistency provides a characterization for orthogonal factorial structure. The equivalence of partial orthogonal factorial structure and partial efficiency-consistency is also investigated.

UTILITAS MATHEMATICA 30(1986), pp.145-151

2. Main results.

Let the treatment labels of the design, d, correspond to the treatment combinations of an n-factor experiment, where the n factors have $m_1, m_2, \ldots, m_n$ levels respectively. Let τ denote the vector of the $v = m_1 m_2 \ldots m_n$ factorial treatment effects listed in lexicographical order. The estimable space corresponding to the interaction α^x, $x \in T$, can be represented by a vector space, V^x, of dimension q^x, where $0 \leq q^x \leq \Pi(m_i - 1)^{x_i}$. Let P^x be a $q^x \times v$ matrix whose rows form an orthonormal basis for V^x. Then $P^x \tau$ denotes a full set of orthonormal estimable contrasts corresponding to α^x. If α^x is totally confounded then $q^x = 0$ and P^x is the null matrix. Let A denote the intrablock matrix of d, and V_A the row space of A. We shall consider only connected and regular disconnected designs, that is designs for which $V_A = \oplus V^x$ where $\oplus$ denotes direct sum over $x \in T$, (see [8]).

Let $\bar{P}^x$ be a matrix whose rows form an orthonormal basis for $\oplus V^y$ where $\oplus$ denotes direct sum over $y \in T$, $y \neq x$. Then, if $q^x > 0$, the variance-covariance matrix for $P^x \hat{\tau}$ is given by

$$\sigma^2 [P^x A P^{x\prime} - (P^x A \bar{P}^{x\prime})(\bar{P}^x A \bar{P}^{x\prime})^{-1}(\bar{P}^x A P^{x\prime})]^{-1} = \sigma^2 G_x, \tag{2.1}$$

say, where $\hat{\tau}$ is an intrablock estimator for τ and σ^2 is the common variance of the errors, (cf. Mukerjee (1980, 5.7)). To avoid trivialities it has been assumed here that $\bar{P}^x$ is non-null. Note that the matrix inverses in (2.1) are well-defined, since for a connected or regular disconnected design, PAP' is positive definite, where $P = (P^{x\prime}, \bar{P}^{x\prime})'$.

For a fixed $x \in T$, consider the subdesign d_x. Let $v_x = \Pi m_i^{x_i}$ and let τ_x denote the vector of the v_x factorial treatment effects in d_x. The intrablock matrix, A_x, of d_x is given by

$$A_x = S^x A S^{x\prime}, \tag{2.2}$$

where

$$S^x = S_1^{x_1} \otimes S_2^{x_2} \otimes \ldots \otimes S_n^{x_n}, \tag{2.3}$$

where $\otimes$ is Kronecker product and where $S_i^{x_i}$ is an $m_i \times m_i$ identity matrix if $x_i = 1$ and a row vector of m_i unit elements if $x_i = 0$, $i = 1, \ldots, n$.

Define $T_x = \{y : y \in T,\ y_i \leq x_i,\ i = 1, \ldots, n\}$ and note that for each $y \in T - T_x$, $P^y S^{x\prime} = 0$ (cf. [6], [8]). Hence it can be shown that for a connected or a regular disconnected design, d, the row space of A_x equals the direct sum of the row spaces of $P^y S^{x\prime}$, the direct sum being over $y \in T_x$. Now defining

$$Q_x^y = (v_x/v)^{1/2} P^y S^{x\prime}, \tag{2.4}$$

it follows that $Q_x^y \tau_x$ represents a full set of estimable orthonormal contrasts corresponding to α^y in d_x (cf. [2], [7]), for all $y \in T_x$. Since the contrast matrix P^y has the property that

$$P^y S^{x\prime} S^x = (v/v_x) P^y \text{ for } y \in T_x, \tag{2.5}$$

(cf. [6], [8]), it follows from (2.2)-(2.5) that for $y, z \in T_x$,

$$Q_x^y A_x Q_x^{z\prime} = (v/v_x) P^y A P^{z\prime}. \tag{2.6}$$

Therefore, if $q^x > 0$, the variance-covariance matrix of $Q_x^x \hat{\tau}_x$, where $\hat{\tau}_x$ is the intrablock estimator of τ_x, is given by $\sigma^2 H_x$, where

$$H_x = \begin{cases} (v_x/v)(P^x A P^{x\prime})^{-1}, \text{ if } \alpha^x \text{ represents a main effect,} & (2.7a) \\ (v_x/v)[P^x A P^{x\prime} - (P^x A \bar{P}_x^{x\prime})(\bar{P}_x^x A \bar{P}_x^{x\prime})^{-1}(\bar{P}_x^x A P^{x\prime})]^{-1}, & \\ \quad \text{otherwise,} & (2.7b) \end{cases}$$

where $\bar{P}_x^x$ is a matrix whose rows form an orthonormal basis for $\oplus V^y$ where $\oplus$ denotes direct sum over $y \in T_x$, $y \neq x$. The fact that $T_x = \{x\}$ when α^x represents a main effect explains (2.7a). Note that by (2.4) and the discussion preceding it, the connectedness or regular disconnectedness of d implies that of d_x and hence, as in (2.1), the matrix inverses in (2.7a, b) exist.

If the treatment labels in d are each replicated r times then the treatment labels in d_x are replicated $r_x = (rv/v_x)$ times. The intrablock estimator of a typical estimable contrast $k' P^x \tau$, $(k \neq 0)$, has efficiency $r^{-1} k'k/(k' G_x k)$ in d whilst that of the equivalent contrast $k' Q_x^x \tau_x$ has efficiency $r_x^{-1} k'k/(k' H_x k)$ in d_x. Hence using Definition 1, the design, d, is efficiency-consistent if and only if $r^{-1} k'k/(k' G_x k) = r_x^{-1} k'k/(k' H_x k)$, for all $k \neq 0$ and for all $x \in T$, that is if and only if $G_x = (v/v_x) H_x$ for $x \in T$. Defining $T_{(u)} = \{x : x \in T,\ x \text{ contains exactly } u \text{ unit digits}\}$, it now follows from (2.1) and (2.7) that d is efficiency-consistent if and only if

$$(P^x A \bar{P}^{x\prime})(\bar{P}^x A \bar{P}^{x\prime})^{-1}(\bar{P}^x A P^{x\prime}) = \begin{cases} 0 \text{ for } x \in T_{(1)} & (2.8a) \\ (P^x A \bar{P}_x^{x\prime})(\bar{P}_x^x A \bar{P}_x^{x\prime})^{-1}(\bar{P}_x^x A P^{x\prime}) & \\ \quad \text{for } x \in T - T_{(1)} & (2.8b) \end{cases}$$

For brevity we shall denote the left and right hand sides of (2.8a, b) as L and R respectively.

Mukerjee [8] proved that the design, d, has orthogonal factorial structure if and only if

$$P^x A P^{y'} = 0, \text{ for all } x, y \in T, \; x \neq y. \tag{2.9}$$

In Theorem 1, we show that (2.8) holds for an n-factor design, d, if and only if (2.9) holds for d. This extends the result of Lewis and Dean (1985) to disconnected designs and proves the converse. The following lemma is easy to prove.

LEMMA 1. *Let M_1, M_2 be matrices such that M_1 is positive definite and the row space of M_2 is a subspace of the row space of M_1. Then $M_2 M_1^{-1} M_2' = 0$ if and only if $M_2 = 0$.*

COROLLARY 1. *$L = 0$ if and only if $P^x A \bar{P}^{x'} = 0$ for $x \in T$.*

THEOREM 1. *An n-factor design, d, is efficiency-consistent if and only if it has orthogonal factorial structure.*

PROOF: Sufficiency. Assume that d has orthogonal factorial structure, so that (2.9) holds. Thus $L = 0$ and $R = 0$ for all $x \in T$. Hence (2.8) holds. (Note: This result also follows from [7] since it can be shown that, if d is disconnected but regular, then a factorial treatment contrast is estimable in d if and only if the equivalent contrast is estimable in d_x, —see (2.4) and the discussion preceding it.)

Necessity. Assume that d is efficiency-consistent so that (2.8) holds. If $x \in T_{(1)}$, then by (2.8a), $L = 0$. Hence from Corollary 1, it follows that

$$P^x A P^{y'} = 0 \text{ for } y \in T, \; y \neq x. \tag{2.10}$$

We prove that (2.10) holds for all $x \in T$ by induction. Assume that (2.10) holds for $x \in T_{(1)}, T_{(2)}, \ldots, T_{(g)}$, $(1 \leq g < n)$ and consider $x \in T_{(g+1)}$. If $y \in T_x$ and $y \neq x$ then $y \in \cup T_{(i)}$, the union being over $i = 1, \ldots, g$. Reversing the roles of x and y in (2.10), it follows that $P^x A \bar{P}_x^{x'} = 0$. Hence in (2.8b) $L = R = 0$ and from Corollary 1, (2.10) holds for $x \in T_{(g+1)}$. Hence by induction (2.9) holds.

THEOREM 2. *The efficiency of a contrast corresponding to α^x in d cannot exceed the efficiency of the equivalent contrast in d_x.*

PROOF: The rows of $\bar{P}_x^x$ are a subset of the rows of $\bar{P}^x$, hence $L - R$ is non-negative definite (see, for example, Kunert (1983, Proposition 2.3)). Hence from (2.1), (2.7) and the definition of efficiency, the result follows.

3. Partial efficiency-consistency.

Partial orthogonal factorial structure of order t was defined by Mukerjee (1980), and a more general definition of partial orthogonal factorial structure was given by Chauhan and Dean (1985). In this section, we investigate the equivalence of such properties and their efficiency-consistency counterparts. We consider only connected and regular disconnected equireplicate designs.

DEFINITION 2 (MUKERJEE (1980)): An n-factor design, d, has *orthogonal factorial structure of order t* if $\text{Cov}(P^x\tau, P^y\tau) = 0$ for all $y \in T$, $y \neq x$, and all $x \in \cup T_{(i)}$, where the union is over $i = 1, \ldots, t$.

Similarly, an n-factor design, d, is defined to be partially efficiency-consistent of order t if the condition of Definition 1 holds for all $x \in \cup T_{(i)}$ for $i = 1, \ldots, t$ rather than for $i = 1, \ldots, n$.

THEOREM 3. *An n-factor design, d, is partially efficiency-consistent of order t if and only if it has orthogonal factorial structure of order t.*

PROOF: Follows exactly the same arguments as the proof of Theorem 1 replacing $x \in T$ by $x \in \cup T_{(i)}$ for $i = 1, \ldots, t$, and using Theorem 2.2 of Mukerjee (1980) to obtain the condition (2.9) for $x \in \cup T_{(i)}$ for $i = 1, \ldots, t$.

DEFINITION 3 (CHAUHAN AND DEAN (1985)): An n-factor design, d, has *partial orthogonal factorial structure (POFS) with respect to* α^x, if for a fixed $x \in T$, $\text{Cov}(P^x\tau, P^y\tau) = 0$ for all $y \in T$, $y \neq x$.

Similarly, an n-factor design, d, is partially efficiency-consistent for α^x if the condition of Definition 1 holds for a fixed $x \in T$. Theorem 1 of Chauhan and Dean (1985) provides (2.9) for a fixed $x \in T$, from which it follows that (2.8) holds for this x, (see also [3]). However the converse of this result is not true, unless $x \in T_{(1)}$, and therefore in the most general case it becomes apparent that POFS is the stronger condition.

DEFINITION 4: For a fixed $x \in T$, and n-factor design, d, has

(i) *external POFS with respect to* α^x *if*

$$\text{Cov}(P^x\hat{\tau}, P^y\hat{\tau}) = 0 \text{ for all } y \in T - T_x,$$

(ii) *internal POFS with respect to* α^x *if*

$$\text{Cov}(P^x\hat{\tau}, P^y\hat{\tau}) = 0 \text{ for all } y \in T_x,\ y \neq x.$$

THEOREM 4. *For a fixed $x \in T$, an n-factor design, d, has external POFS with respect to α^x if and only if it has partial efficiency-consistency for α^x.*

PROOF:

(i) Suppose that $x \in T_{(1)}$. If d has POFS with respect to α^x, then from Chauhan and Dean (1985), $P^x A P^{y'} = 0$ for all $y \in T$, $y \neq x$. Hence (2.8a) holds. Conversely, using Corollary 1, (2.8a) implies (2.10), which in this case is equivalent to external POFS with respect to α^x.

(ii) Suppose that $x \in T - T_{(1)}$. Writing $\bar{P}^{x'} = [\bar{P}_x^{x'}, \bar{\bar{P}}_x^{x'}]'$, a straightforward multiplication of matrices shows that (2.8b) gives

$$L - R = Q'\Lambda^{-1}Q = 0$$

where

$$\Lambda = (\bar{\bar{P}}_x^{x} A \bar{\bar{P}}_x^{x'}) - (\bar{\bar{P}}_x^{x} A \bar{P}_x^{x'})(\bar{P}_x^{x} A \bar{P}_x^{x'})^{-1}(\bar{P}_x^{x} A \bar{\bar{P}}_x^{x'})$$

and where Q is defined in (3.1) below.

Now Λ^{-1} is a diagonal submatrix of the positive definite matrix $(\bar{P}^x A \bar{P}^{x'})^{-1}$, and therefore Λ^{-1} is also positive definite. Hence $L - R = 0$ implies

$$Q = (P^x A \bar{\bar{P}}_x^{x'}) - (P^x A \bar{P}_x^{x'})(\bar{P}_x^{x} A \bar{P}_x^{x'})^{-1}(\bar{P}_x^{x} A \bar{\bar{P}}_x^{x'}) = 0. \qquad (3.1)$$

Thus (3.1) provides an alternative definition of efficiency-consistency. The variance-covariance matrix of $\Delta\hat{\tau} = [\hat{\tau}'\bar{P}_x^{x'}, \hat{\tau}'P^{x'}, \hat{\tau}'\bar{\bar{P}}_x^{x'}]'$ is $\sigma^2(\Delta A \Delta')^{-1}$, (cf. Mukerjee (1980)). Inverting this as a partitioned matrix shows that $\text{Cov}(P^x\hat{\tau}, \bar{\bar{P}}_x^{x}\hat{\tau}) = 0$ if and only if (3.1) holds.

Note that an alternative proof can be obtained for Theorems 1 and 3 using an inductive argument when Theorem 4 holds for all $x \in \cup T_{(i)}$ for $i = 1, \ldots, m$ where $m = n$ and t respectively.

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Received January 15, 1986; Revised March 24, 1986