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On a Conjecture on Singular Martingales

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A conjecture by Luis Baez-Duarte on singular martingales is disproved.

1. INTRODUCTION AND NOTATION

Let $(\Omega, \mathscr{F}, P)$ be a probability space and $\{\mathscr{F}_n\}_{n\geq 1}$ an increasing sequence of sub σ-algebras of $\mathscr{F}$. Let $\mathscr{F}_\infty$ be the smallest σ-algebra which contains all the $\mathscr{F}_n$'s. In what follows, there is no essential loss of generality if we assume that $\mathscr{F} = \mathscr{F}_\infty$; accordingly we assume so. Let $\{X_n\}_{n\geq 1}$ be a sequence of integrable random variables on $(\Omega, \mathscr{F}, P)$ forming a martingale with respect to $\{\mathscr{F}_n\}_{n\geq 1}$ (in the sequel, we will abbreviate this by "$\{X_n, \mathscr{F}_n\}_{n\geq 1}$ is a martingale"). For each n, define a measure μ_n on $\mathscr{F}_n$ by

$$\mu_n(A) = \int_A X_n \, dP, \qquad A \in \mathscr{F}_n. \tag{1.1}$$

In [1], Luis Baez-Duarte introduced the following definitions.

DEFINITION 1.1. The martingale $\{X_n, \mathscr{F}_n\}_{n\geq 1}$ is *measure dominated* if there is a finite measure μ on $\mathscr{F}$ whose restriction to each $\mathscr{F}_n$ coincides with the measure μ_n defined in (1.1). In such a case μ is said to *dominate* the martingale.

DEFINITION 1.2. Let $\{X_n, \mathscr{F}_n\}_{n\geq 1}$ be a measure dominated martingale. It is said to be a *singular martingale* if the dominating measure μ is singular with respect to P.

Let for each n, S_n stand for the extended real line $[-\infty, \infty]$ and $\mathscr{B}_n$ for the σ-algebra of its Borel subsets. Denote by S the Cartesian product $\prod_{n=1}^\infty S_n$ and by Y_n the projection from S to $(S_n, \mathscr{B}_n)$. Let the σ-algebras generated by $\{Y_1, Y_2, \ldots, Y_n\}$ and $\{Y_n, n \geq 1\}$ be denoted, respectively, by $\mathscr{B}^n$ and $\mathscr{B}$. In the remainder of this note, the symbols S, S_n, $\mathscr{B}^n$, $\mathscr{B}$, and Y_n have the same meaning as given in this paragraph.

Given any sequence $\{X_n\}_{n\geq 1}$ of integrable random variables on $(\Omega, \mathscr{F}, P)$

by *canonical mapping* we mean the measurable mapping T from $(\Omega, \mathscr{F})$ to $(S, \mathscr{B})$ given by

$$T(\omega) = (X_1(\omega), X_2(\omega), \ldots, X_n(\omega), \ldots), \qquad \forall \omega \in \Omega.$$

It is easy to see that if $\{X_n, \mathscr{F}_n\}_{n \geq 1}$ is a martingale on $(\Omega, \mathscr{F}, P)$ then $\{Y_n, \mathscr{B}^n\}_{n \geq 1}$ is a martingale on $(S, B, P \circ T^{-1})$ (where T is, of course, the canonical mapping) with the property that $\{X_n\}_{n \geq 1}$ and $\{Y_n\}_{n \geq 1}$ are equivalent. We call $\{Y_n, \mathscr{B}^n\}_{n \geq 1}$ the *canonical martingale associated with* $\{X_n, \mathscr{F}_n\}_{n \geq 1}$. Note that for various $\{X_n\}$'s what varies is the measure $P \circ T^{-1}$ and not $\{Y_n\}_{n \geq 1}$.

It is not difficult to prove (see, for instance, Theorem 3.1 of [1]) that, if $\{X_n, \mathscr{F}_n\}_{n \geq 1}$ is a measure dominated martingale then the canonical martingale associated with it is also measure dominated. In [1], Baez-Duarte conjectured that if $\{X_n, F_n\}_{n \geq 1}$ is a singular martingale on $(\Omega, \mathscr{F}, P)$ then the corresponding dominating measure for the canonical martingale is concentrated on the boundary of $T\Omega$. We disprove this with the help of an example.

2. An Example

Before proceeding with the construction of the example we would like to observe the following.

Let $\Omega = S$, $\mathscr{F} = \mathscr{B}$ and $X_n = Y_n$ for all n. Then the canonical mapping from $(\Omega, \mathscr{F})$ to $(S, \mathscr{B})$ is just the the identity mapping. So $T\Omega = S$ and hence the boundary of $T\Omega$ is empty. Therefore if we could find a measure Q on (S, B) such that $\{Y_n, \mathscr{B}^n\}_{n \geq 1}$ becomes a singular martingale on $(S, \mathscr{B}, Q)$ we would have disproved the conjecture.

We achieve this in two stages.

(i) Let $\Omega = \{1, 2, 3, \ldots, n, \ldots, \infty\}$. Let $\mathscr{F}_1 = \{\Phi, \Omega\}$ and, for each $n \geq 2$, $\mathscr{F}_n$ be the σ-algebra generated by the partition $\{1\}, \{2\}, \ldots, \{n-1\}, \{n, n+1, \ldots, \infty\}$. As in Section 1, the σ-algebra generated by $\bigcup_n \mathscr{F}_n$ is denoted by $\mathscr{F}$. Clearly, $\mathscr{F}$ is just the class of all subsets of Ω. Let P be the probability measure defined on $\mathscr{F}$ by $P(\{n\}) = 1/2^n$, $n \geq 1$ and $P(\{\infty\}) = 0$. Define a sequence $\{X_n\}_{n \geq 1}$ of integrable random variables on $(\Omega, \mathscr{F}, P)$ by setting,

$$\begin{aligned} X_n(\omega) &= \frac{1}{P[n, \infty]} \qquad \text{if} \quad \omega \geq n \\ &= 0 \qquad\qquad\quad \text{if} \quad \omega < n. \end{aligned}$$

The fact that $\{X_n, \mathscr{F}_n\}_{n \geq 1}$ is a martingale is easily verified. The measures

μ_n's given by the relation (1.1) are nothing but the restriction to $\mathscr{F}_n$ of the probability measure ϵ_∞, concentrated at $\{\infty\}$. Hence $\{X_n, \mathscr{F}_n\}_{n\geq 1}$ is a singular martingale.

(ii) Consider the canonical martingale associated with the singular martingale constructed in (i). We show that it (the canonical martingale) is singular.

The following two facts are easily observed.

(a) Since the set Ω, defined in (i), is countable, $T(A)$ is $\mathscr{B}$ measurable for every subset A of Ω.

(b) Since

$$T(\omega) = (X_1(\omega), X_2(\omega),..., X_n(\omega),...)$$
$$= \left(1, \frac{1}{P[2,\infty]}, \frac{1}{P[3,\infty]}, ..., \frac{1}{P[\omega,\infty]}, 0, 0, ...\right),$$

T is one to one.

Now, denoting the measures corresponding to the canonical martingale defined in (1.1) by $\{\mu_n'\}_{n\geq 1}$, we have

$$\mu_n'(A) = \int_A Y_n \, dP \circ T^{-1} = \int_{T^{-1}(A)} X_n \, dP = \mu_n(T^{-1}(A)), \qquad \forall A \in \mathscr{B}^n.$$

That is, $\mu_n' = \mu_n \circ T^{-1}$. Let μ' be the probability measure concentrated at the point $T(\infty)$.

For any $A \in \mathscr{B}^n$,

$$\begin{aligned} \mu'(A) = 1 &\Leftrightarrow T(\infty) \in A \\ &\Leftrightarrow \mu_n(T^{-1}(A)) = 1 \\ &\Leftrightarrow \mu_n'(A) = 1. \end{aligned}$$

So, for every n, the restriction of μ' to $\mathscr{B}^n$ coincides with μ_n'. That is, μ' is the dominating measure for the canonical martingale. Since,

$$P \circ T^{-1}(T(\Omega - \{\infty\})) = P(\Omega - \{\infty\}) = 1 \qquad \text{and} \qquad \mu'(\{T(\infty)\}) = 1,$$

the measures $P \circ T^{-1}$ and μ' are singular.

Thus $\{Y_n, \mathscr{B}^n\}_{n\geq 1}$ is a singular martingale on $(S, \mathscr{B}, P \circ T^{-1})$. Therefore, in view of the observation made at the beginning of this section, the conjecture made by Baez-Duarte is false.

3. Remark

A slight modification of the above example will show that even if the boundary of $T\Omega$ is *nonempty*, the conjecture is not true. We indicate below the modification to be effected without giving detailed proofs.

Let Ω and P be as in (i) and $s \in S - T(\Omega)$. Define $S' = S - \{s\}$ and take $(\mathscr{B}^n)'$ and $\mathscr{B}'$, respectively, to be the trace σ algebras of $\mathscr{B}^n$ and $\mathscr{B}$, with respect to S'. Let, for each n, Y_n' be the restriction of Y_n to S'. $\{Y_n', (\mathscr{B}^n)'\}_{n\geq 1}$ is a singular martingale on $(S', B', P \circ T^{-1})$ with μ' (defined in (ii)) as the dominating measure. Now, since $T(S') = S - \{s\}$, the boundary of $T(S') = s$. But, the dominating measure corresponding to the canonical martingale associated with $\{Y_n', (\mathscr{B}^n)'\}_{n\geq 1}$ is concentrated at the point

$$\left(1, \frac{1}{P[2, \infty]}, \dots, \frac{1}{P[n, \infty]}, \dots\right);$$

and this point is obviously different from s.

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Reference

1. Luis Baez-Duarte, Another look at martingale theorem, *J. Math. Anal. Appl.* 23 (1968), 551–557.