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ON A ROBUSTNESS PROPERTY OF PBIBD

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Abstract: In this paper we study a robustness property of partially balanced imcomplete block designs based on association schemes with m classes (PBIBD(m)) against the unavailability of data in the sense that, when any t (a positive integer) observations are unavailable the design remains connected w.r.t. treatment. We characterize the robustness property of PBIBD(m) completely for $m=2$ and partially for $m=3$.

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Key words: Association graphs; Partially balanced imcomplete block design; Robust designs; Unavailable data.

1. Introduction

The robustness of BIBD against the unavailability of data and w.r.t. the estimability of parameters was studied in Ghosh (1982). In this paper we consider the robustness of PBIBD(m). Cheng (1978, 1981) investigated various optimum properties of PBIBD(2). The robustness of connected balanced block designs against the loss of one treatment and w.r.t. the large value of the ratio of a lower bound of the efficiency of the resulting design to the efficiency of the original design, was considered in Kageyama (1980). The robustness in the work of Kageyama is entirely different from the robustness in this paper. This paper therefore presents a further property of PBIBD.

PBIB designs were first studied in Bose and Nair (1939), association schemes in Bose and Shimamoto (1952), association matrices and graphs in Bose and Mesner (1959). The concept of connectedness in a block design which plays an important role in this paper was introduced in Bose (1939). It seems particularly appropriate

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to present this work in honor of Professor R.C. Bose on the occasion of half a century of his contributions in teaching and research.

2. The problem

Consider a block design (BD) with v treatments in b blocks. We can associate with it a multigraph G in the following manner: the vertices are treatments; the vertices i_1 and i_2 are joined by $\lambda_{i_1 i_2}$ edges if they occur together in exactly $\lambda_{i_1 i_2}$ blocks. It can be seen that a BD is connected w.r.t. treatment if and only if the corresponding G is connected.

In this paper we consider a special kind of block designs, namely PBIBD(m); the parameters are v, b, r, k, λ_i, n_i and p^i_{jl} $(i, j, l = 1, \ldots, m)$. We are interested only in a completely connected PBIBD(m), i.e. connected w.r.t. treatment and block. If some observations in a PBIBD are unavailable the resulting design may not be another PBIBD and, moreover, may or may not even be connected w.r.t. treatment.

Definition 1. A completely connected PBIBD(m) is said to be **robust against** the unavailability of any t, a positive integer, observations if the design obtained by omitting any t observations remains connected w.r.t. treatment.

Clearly, $t \leqslant r-1$. It is easy to see that the edge-connectivity of G (i.e. the minimum number of edges whose removal results in a disconnected graph) throws light on the value of t.

If any $r-1$ observations are unavailable, then the maximum number of edges to be removed from G is $(r-1)(k-1)$. It is clear that if the edge-connectivity of G is more than $(r-1)(k-1)$, then the resulting graph and hence the design remain connected. Indeed, the robustness problem as stated earlier is not identical with finding the edge-connectivity of the graph G. The robustness problem will, however, be completely solved if the edge-connectivity of G is always greater than $(r-1)(k-1)$. But unfortunately the edge-connectivity may even be less than or equal to $(r-1)(k-1)$.

Given an association scheme with v treatments and m classes, we define association graphs, $G_1, G_2, \ldots, G_m$ as follows: the vertices are treatments; in G_i, two vertices are joined if the corresponding treatments are ith associates. Note that G_i is a regular graph of degree n_i. We now state a conjecture.

Conjecture. Let each of the graphs G_i, $i = 1, \ldots, m$, for a given association scheme with v treatments and m classes be connected. Then, the edge-connectivity of each G_i is n_i.

In Section 3, we prove the conjecture for $m = 2$ and 3. In Section 4, we show that the edge-connectivity of G corresponding to PBIBD(m) is $r(k-1)$ when $m = 2, 3$ and each of G_l, $l=1,\dots,m$, is connected. We also give complete characterization of robustness for $m=2$.

3. Edge-connectivity of association graphs

Consider an m-class association scheme with parameters v, n_l, p^l_{jl} $(i,j,l=1,\dots,m)$. We recall the relation

$$n_i p^i_{jl} = n_j p^j_{il} = n_l p^l_{ij}. \tag{1}$$

Definition 2. For any ordered pair (i,j), $1\leqslant i,\ j\leqslant m$, $i\neq j$,

$$t^i_j = \sum_{l\neq j} p^i_{jl} \tag{2}$$

where $m\geqslant 3$.

Definition 3. For any i, $1\leqslant i\leqslant m$,

$$t^i = \sum_{(j,l)} p^i_{jl}, \tag{3}$$

where the summation is taken over all (j,l), $j\neq i$, $l\neq i$, $1\leqslant j,l<m$.

Definition 4. For any partition U,V of the treatment containing v treatments, $T_i(U,V)$ denotes the number of ordered pairs (x,y) such that $x\in U$, $y\in V$ and x,y are ith associates. When there is no confusion, we write T_i in place of $T_i(U,V)$. Let $\alpha=\text{Min}\{|U|,|V|\}$. Clearly $\alpha\leqslant\frac{1}{2}v$. Furthermore,

$$\sum_{i=1}^{m} T_i = \alpha(v-\alpha). \tag{4}$$

Lemma 1. *For any (i,j), $1\leqslant i,\ j\leqslant m$, $i\neq j$, and the partition (U,V) of the treatment set, we have*

$$2p^j_{ji}T_j \geqslant p^i_{jj}T_i. \tag{5}$$

Proof. Fix i,j, $i\neq j$, (U,V) and let

$$S=\{(x,y,z)\mid x\in U,\ y\in V,\ (x,y) \text{ are } i\text{th associates and both } (x,z),\ (y,z) \text{ are } j\text{th associates}\}.$$

Since $i\neq j$,

$$|S| = p^i_{jj}T_i. \tag{6}$$

Now for any pair (x,z) such that $x\in U$, $z\in V$ and (x,z) are jth associates, there are clearly $p^j_{ji}+p^j_{ij}=2p^j_{ji}$ vertices y such that elements of one pair of (x,y) and (y,z)

are ith associates and the other are jth associates, where y may or may not be in V. Therefore

$$|S| \leqslant 2p^j_{ji}T_j, \tag{7}$$

and (5) follows from (6) and (7).

Lemma 2. *For any i, $1 \leqslant i \leqslant m$, $m \geqslant 3$, and the partition (U, V) of the treatment set we have*

$$t^i T_i \geqslant 2 \sum_{j \neq i} t^j_i T_j. \tag{8}$$

Proof. Fix i, $1 \leqslant i \leqslant m$, (U, V) and let

$$S = \{(x, y, z) \mid x \in U,\ y \in V,\ (x, y) \text{ are } i\text{th associates, } (x, z)\ j\text{th associates, } (z, y)\ l\text{th associates, } j \neq i,\ l \neq i\}.$$

Here j may be equal to l. Clearly,

$$|S| = t^i T_i. \tag{9}$$

Now given any pair (x, z) such that $x \in U$, $z \in V$ and (x, z) are jth associates, $j \neq i$, the number of y such that one of (x, y) and (y, z) are ith associates and the other are lth associates, where $l \neq i$, is equal to $2t^j_i$, where y may or may not be in V. Therefore

$$|S| \leqslant 2 \sum_{j(j \neq i)} t^j_i T_j, \tag{10}$$

and (8) follows from (9) and (10).

Theorem 1. *For $m = 2$ and 3, the edge-connectivity of G_i is n_i, $i = 1, \ldots, m$, i.e., $T_i \geqslant n_i$ for all i and for any U, V.*

Proof. We consider the cases $m = 2$ and 3 separately.

Case 1. $m = 2$. Since G_1 and G_2 are connected, we have $p^2_{11} \cdot p^1_{22} \neq 0$, i.e. $p^1_{12} \cdot p^2_{12} \neq 0$. Let (U, V) be any partition of the treatment set.

(a) If $\alpha = |U| = 1$, then clearly $T_i \geqslant n_i$, $i = 1, 2$.

(b) If $\alpha = |U| = 2$, then $n_i > 1$ and $T_i \geqslant 2(n_i - 1) \geqslant n_i$.

(c) Suppose $\alpha \geqslant 3$. We have $v \geqslant 6$. If now $T_i \leqslant 2n_i - 1$ for $i = 1, 2$, then

$$3(v-3) \leqslant \alpha(v - \alpha) = T_1 + T_2 \leqslant 2n_1 + 2n_2 - 2 = 2(v-2).$$

Thus $v \leqslant 5$, a contradiction. Therefore, for at least one i, $T_i \geqslant 2n_i$. Let without loss of generality (WLG) $T_2 \geqslant 2n_2$. Then by Lemma 1

$$2p^1_{12}T_1 \geqslant p^2_{11}T_2 \geqslant 2n_2 p^2_{11} = 2n_1 p^1_{12}.$$

Since $p^1_{12} \neq 0$, we have $T_1 \geqslant n_1$.

Case 2. $m = 3$.

(a) Assume first that $\alpha \geqslant 8$. If for each i, $i = 1, 2, 3$, $T_i \leqslant 4n_i - 1$, then by (4) we have

$$8(v-8)\leqslant \alpha(v-\alpha)=\sum_{i=1}^{3} T_i\leqslant \sum_{i=1}^{3}(4n_i-1)=4(v-1)-3,$$

which implies that $4v\leqslant 57$, which is not true since $v\geqslant 2\alpha\geqslant 16$. Therefore there exists an i such that $T_i\geqslant 4n_i$. Let WLG $T_3\geqslant 4n_3$. Now by (1) and Lemma 2 we have

$$\begin{aligned}2(p^1_{31}+p^1_{32})T_1+2(p^2_{31}+p^2_{32})T_2&\geqslant (p^3_{11}+p^3_{22}+p^3_{12}+p^3_{21})T_3\\ &\geqslant 4(p^3_{11}+p^3_{22}+p^3_{12}+p^3_{21})n_3=4[(p^1_{31}+p^1_{32})n_1+(p^2_{31}+p^2_{32})n_2]. \qquad (11)\end{aligned}$$

Let, if possible, $t^3=p^3_{11}+p^3_{22}+p^3_{12}+p^3_{21}=0$. In G_1, consider two vertices a,b which are 3rd associates. Since G_1 is connected, there exists a chain $a=x_0,x_1,\dots,x_n=b$ such that x_j, x_{j+1} are 1st associates $0\leqslant j\leqslant n-1$. It then follows inductively on j that x_0, x_j are 1st or 2nd associates for every j, $1\leqslant j\leqslant n$, contradicting x_0, x_n are 3rd associates. Therefore $t^3\neq 0$ and $T_i\geqslant 2n_i$ for at least one i, $i=1,2$. Let WLG $T_2\geqslant 2n_2$. Again by connectivity the situation $p^2_{11}=0$ and $p^3_{11}=0$ can not occur. If $p^2_{11}\neq 0$, then by Lemma 1, we have

$$2p^2_{12}T_1\geqslant p^2_{11}T_2\geqslant 2p^2_{11}n_2=2p^1_{12}n_1.$$

Since $p^1_{12}\neq 0$, we have $T_1\geqslant n_1$. If $p^3_{11}\neq 0$, similarly $T_1\geqslant n_1$.

(b) Assume $\alpha\leqslant 7$. Fix any i and (U,V) with $|U|=\alpha\leqslant |V|$.

(b.1) If $n_i\geqslant \alpha$ then each vertex in U has at least $(n_i-\alpha+1)$ ith associates in V and hence

$$T_i\geqslant \alpha(n_i-\alpha+1)=(\alpha-1)(n_i-\alpha)+n_i\geqslant n_i.$$

(b.2) Consider the case $n_i<\alpha$. We know $p^i_{ii}\leqslant n_i-1$. If $p^i_{ii}=n_i-1$ then $p^i_{ij}=0$ for $j=1,\dots,m$, $j\neq i$ and we have a subset with n_i+1 vertices such that any two vertices are ith associates. Thus G_i is disconnected. Therefore $p^i_{ii}\leqslant n_i-2$.

(b.2.1) If there is at most one vertex in U such that all its ith associates are in U then clearly $T_i\geqslant \alpha-1\geqslant n_i$.

(b.2.2) Consider the situation where there are at least two vertices $x,y\in U$ such that all their ith associates are in U. If there are no such x and y which are in addition ith associates then there are n_i ith associates of x in U and each of these n_i vertices has at least one ith associate in V. Therefore $T_i\geqslant n_i$. Let x and y be ith associates; $z_1,\dots,z_\beta$, where $\beta=p^i_{ii}$, be the vertices which are joined to both x and y, and $z_{\beta+1},\dots,z_{n_i-1}$ (respectively $z_{n_i},z_{n_i+1},\dots,z_{2n_i-\beta-2}$) be the vertices adjacent in G_i to x (respectively y) but not to y (respectively x). We have

$$7\geqslant \alpha=|U|=2n_i-p^i_{ii}. \qquad (12)$$

From (12) and $p^i_{ii}\leqslant n_i-2$ we get $2\leqslant n_i\leqslant 5$ and $p^i_{ii}\geqslant 2n_i-7$.

Since $p^i_{ii}\geqslant 0$, we have $2p^i_{ii}\geqslant 2n_i-7$, i.e., $p^i_{ii}>n_i-4$. Therefore

$$p^i_{ii}=n_i-3 \text{ or } n_i-2. \qquad (13)$$

We consider these two cases separately.

(b.2.2.1) Let $p^i_{ii}=n_i-2$.

(i) If $n_i=2$ then $p_{ii}^i=0$ and G_i, by connectedness, is a single cycle and hence $T_i \geqslant 2$.

(ii) If $n_i=3$ then $p_{ii}^i=1$. But $p_{ii}^i=1$ implies n_i is even. Thus $n_i=1$ is even. Thus $n_i \neq 3$.

(iii) In case $n_i=4$, we have $p_{ii}^i=2$. In G_i, x is adjacent to y, z_1, z_2 and z_3, and y is adjacent to x, z_1, z_2 and z_4. Considering (x, z_j) and (y, z_j), $j=1,2,3$, $n_i=4$, $p_{ii}^i=2$, and taking $U=\{x, y, z_1, z_2, z_3, z_4\}$, it can be seen that the vertices in U are disconnected from the other vertices in G_i and hence G_i is disconnected; a contradiction. Thus $n_i \neq 4$.

(iv) If $n_i=5$ then $p_{ii}^i=3$ and, in G_i, z_4 and z_5 are ith associates of z_1, z_2 and z_3. One of z_2 and z_3 must be an ith associate of z_1. Assume WLG that z_1 and z_2 are ith associates. But then x, y, z_4 and z_5 are ith associates of z_1 and z_2, i.e., $p_{ii}^i=4$; a contradiction. Thus $n_i \neq 5$.

(b.2.2.2) Let $p_{ii}^i=n_i-3$. Here, $n_i=3$ or 4.

(i) If $n_i=4$ then $p_{ii}^i=1$ and $\alpha=7$. Let if possible z_1 and z_2 are ith associates. Then y and z_2 are ith associates of x and z_1, i.e., $p_{ii}^i=2$; a contradiction. Similarly it can be shown z_1 and z_j, $j=2,3,4,5$ are not ith associates. Taking (x, z_3) and (y, z_4), it can be seen that (z_2, z_3) and (z_4, z_5) are ith associates. Moreover z_2 (respectively z_3) is joined in G_i to at most one of z_4, z_5. These imply that $T_i \geqslant 6 \geqslant n_i$.

(ii) If $n_i=3$ then $p_{ii}^i=0$ and we get, from (12), $6 \leqslant \alpha \leqslant 7$. If $T_i=2$ then, since the sum of the degrees in any graph is even, the degree sequence of the subgraph $G_i(U)$ induced by G_i on the vertices of U with $\alpha=6$ is $(3,3,3,3,3,1)$ or $(3,3,3,3,2,2)$. Since $p_{ii}^i=0$, the former $G_i(U)$ is not possible. Consider $G_i(U)$ with the degree sequence $(3,3,3,3,2,2)$. Since no triangle is possible in $G_i(U)$, we may assume x is adjacent to z_1, z_2 and y; z_3 is adjacent to y, z_2 and z_1; z_4 is adjacent to y and z_2. Suppose (z_2, y) are jth associates, (z_1, z_2) kth associates, (z_1, z_4) lth associates with $j, k, l \neq i$. Then $p_{ii}^j=3$, $p_{jj}^k=2$ and $p_{ii}^l \leqslant 1$. Since $m=3$, we have a contradiction. If $T_i=1$ then the degree sequence of $G_i(U)$ with $\alpha=7$ is $(3,3,3,3,3,3,2)$. Now x and y can be any two of the 6 vertices with degree 3. We may assume in $G_i(U)$, x is adjacent to z_1, z_2 and y; z_3 is adjacent to y, z_2 and z_1; z_4 is adjacent to y, z_2 and z_5; z_1 and z_5 are adjacent. Let (z_1, z_2) be jth associates, (z_2, y) kth associates and (x, z_5) lth associates with $j, k, l \neq i$. Then $p_{ii}^j=2$, $p_{ii}^k=3$ and $p_{ii}^l=1$. We thus have a contradiction. Therefore, for $6 \leqslant \alpha \leqslant 7$ and $n_i=3$, $T_i=1,2$ and hence $T_i \geqslant n_i$. This completes the proof of Theorem 1.

Hence our conjecture in Section 2 is true when $m=2$ and 3.

4. Robustness of PBIBD

Cosider a PBIBD(m) with parameters $u, v, r, k, \lambda_i, n_i, p_{jl}^i$ $(i, j, l=1, \ldots, m)$. We have the relation

$$n_1\lambda_1 + n_2\lambda_2 + \cdots + n_m\lambda_m = r(k-1). \tag{14}$$

For any partition U, V of the treatment set containing v treatments we denote by $\alpha(U,V)$, the number of lines joining U with V. Clearly,

$$\alpha(U,V)=\lambda_1T_1+\lambda_2T_2+\cdots+\lambda_mT_m, \tag{15}$$

$$T_1+T_2+\cdots+T_m=\alpha(v-\alpha)\geqslant v-1=n_1+n_2+\cdots+n_m. \tag{16}$$

Lemma 3. *If $T_i\geqslant n_i$, $i=1,\ldots,m$, then the edge-connectivity of the multigraph G corresponding to* PBIBD(m) *is $r(k-1)$.*

Proof. It follows from (14) and (15), for any U, V with $T_i\geqslant n_i$ for all i, $\alpha(U,V)\geqslant r(k-1)$. This completes the proof.

It follows from the results in Section 3, in case G_i, $i=1,\ldots,m$, are connected, the edge connectivity of G corresponding to PBIBD(m), $m=2,3$, is $r(k-1)$.

Lemma 4. *$T_i\geqslant n_i$ for at least one i.*

Proof. If $T_i<n_i$ for all i, then $T_1+\cdots+T_m<n_1+\cdots+n_m$, a contradiction to (16). We write (15) as

$$\alpha(U,V)=\alpha(v-\alpha)\lambda_i+\sum_{\substack{j=1\\ j\neq i}}^{m}(\lambda_j-\lambda_i)T_j. \tag{17}$$

Therefore, we have

$$\alpha(U,V)\geqslant (v-1)\lambda_i+\sum_{\substack{j=1\\ j\neq i}}^{m}(\lambda_j-\lambda_i)T_j. \tag{18}$$

We now solve the robustness problem completely for $m=2$.

Lemma 5. *Consider $m=2$. If $T_j\geqslant n_j$ and $\lambda_j\geqslant\lambda_i$, $i,j=1,2$, $i\neq j$, then $\alpha(U,V)\geqslant r(k-1)$.*

Proof.
$$\begin{aligned}\alpha(U,V)&\geqslant(v-1)\lambda_i+(\lambda_j-\lambda_i)T_j\\ &\geqslant(v-1)\lambda_i+(\lambda_j-\lambda_i)n_j=r(k-1),\end{aligned}$$

by (14) and (18).

Lemma 6. *If $m=2$ and $T_i\neq 0$, $i=1,2$, then $\alpha(U,V)\geqslant r(k-1)$.*

Proof. If $p^1_{22}\neq 0$, $p^2_{11}\neq 0$, then we know $\alpha(U,V)\geqslant r(k-1)$. If $p^1_{22}=0$, $p^2_{11}=0$, then it follows that $n_1=n_2=0$. Therefore this case is impossible. In case $p^1_{22}\neq 0$, $p^2_{11}=0$, we have $p^1_{11}+p^1_{12}=n_1-1$, i.e., $p^1_{11}=n_1-1$ and $p^1_{22}=n_2$. Thus $T_1\geqslant 1+p^1_{11}=n_1$ and $T_2\geqslant p^1_{22}=n_2$. If $p^1_{22}=0$, $p^2_{11}\neq 0$, then $p^2_{11}=n_1$ and $p^2_{22}=n_2-1$. We get $T_1\geqslant p^2_{11}=n_1$,

$T_2 \geqslant 1 + p_{22}^2 = n_2$. Therefore, $\alpha(U, V) \geqslant r(k-1)$. This completes the proof.

Lemma 7. *Consider the case $m=2$ and $T_i=0$. Then, $\alpha(U,V) \geqslant (r-1)(k-1)+1$ if and only if $n_i n_j \lambda_j \geqslant n_i \lambda_i - (k-2)$, $j \neq i$, $i,j=1,2$.*

Proof. If $T_1=0$ then $p_{11}^2=0$. In this case $v=m_1(n_1+1)$. The treatments can be divided into m_1 groups of n_1+1 treatments such that any two treatments in a group are first associates. We have $\alpha=c(n_1+1)$, c is an integer with $1 \leqslant c \leqslant \frac{1}{2}m_1$. Now,

$$\alpha(U,V)=\alpha(v-\alpha)\lambda_2 \geqslant (r-1)(k-1)+1 \quad \text{for all } \alpha=c(n_1+1),\ 1 \leqslant c \leqslant \tfrac{1}{2}m_1$$

iff

$$\min_{\alpha} \alpha(v-\alpha)\lambda_2 \geqslant (r-1)(k-1)+1 = r(k-1)-(k-2),$$

i.e.

$$(n_1+1)n_2\lambda_2 \geqslant n_1\lambda_1 + n_2\lambda_2 - (k-2),$$

i.e.

$$n_1 n_2 \lambda_2 \geqslant n_1\lambda_1 - (k-2).$$

Similar is the case $T_2=0$. This completes the proof.

One can easily find an example of group divisible (GD) design not satisfying the condition in Lemma 7. It is to be noted that the size of experiments for such GD designs will be very large. It follows from Lemmas 3-7, the robustness against the unavailability of any $r-1$ observations of all connected PBIBD(2) except for GD designs not satisfying the condition in Lemma 7.

Remarks. In case $m=1$ it follows from (15) and (16), $\alpha(U,V) \geqslant r(k-1)$. Thus the edge-connectivity of the multigraph G corresponding to a BIBD is $r(k-1)$. Hence a BIBD is robust against the unavailability of any $r-1$ observations. However, the robustness of BIBD against the unavailability of all observations in any $r-1$ blocks does not follow from the results in this paper. This is done in Ghosh (1982).

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