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ON THE PROBLEM OF AUGMENTED FRACTIONAL FACTORIAL DESIGNS

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ABSTRACT

Let D be a saturated fractional factorial design of the general $k_1 \times k_2 \times \ldots \times k_t$ factorial such that it consists of m distinct treatment combinations and it is capable of providing an unbiased estimator of a subvector of m factorial parameters under the assumption that the remaining k-m, $(k = \prod_{i=1}^{t} k_i)$, factorial parameters are negligible. Such a design will not provide an unbiased estimator of the variance σ^2. Suppose that D is an optimal design with respect to some optimality criterion (e.g. d-optimality, a-optimality or e-optimality) and it is desirable to augment D with c treatment combinations with the aim to estimate σ^2 unbiasedly. The problem then is how to select the c treatment combinations such that the augmented design D^* retains its optimality property. This problem, in all its generality is extremely complex. The objective of this paper is to provide some insight in the problem by providing a partial answer in the case of the 2^t factorial, using the d-optimality criterion.

1. INTRODUCTION

It is well known that the theory of fractional factorial designs presents many interesting and difficult algebraic, combinatorial and geometric problems. For a comprehensive introduction into the subject see the book by Raktoe, Hedayat and Federer (1981) and the numerous references mentioned in it.

The topic of augmented designs has been treated by several authors in different contexts, e.g. Federer(1956,1961), Banerjee and Federer(1963,1964), Gaylor and Merril(1968), Dykstra(1971), Federer and Raghavarao(1975), Pesotan and Raghavarao(1975), Raghavarao and Pesotan(1977) and Pesotan, Raghavarao and Raktoe(1977).

Our motivation in tackling the augmented fractional factorial design problem is different from the above authors, in the sense that we start with an optimal saturated design D for a subvector of factorial effects and ask for a design D^*, which is obtained by augmenting D with c treatment combinations, such that σ^2 can be estimated unbiasedly and D^* retains its optimality property.

In Section 2, we provide the general setting for the formulation of the augmentation problem and then we specialize to the 2^t factorial where the design matrices are simply (-1,1)-matrices. Section 3 gives a solution of this problem for the 2^t factorial under the added assumption that the initial design D is orthogonal and hence optimal in the sense of d-,a-, and e-optimality (see Kiefer(1960)). In Section 4 we consider a balanced initial design D of the 2^t factorial and solve the problem when optimal augmentation is limited to one extra treatment combination. Finally, in Section 5 we provide a discussion of further work in this area.

2. FORMULATION OF THE PROBLEM OF AUGMENTED FRACTIONAL FACTORIAL DESIGNS

Consider the general $k_1 \times k_2 \times \ldots \times k_t$ factorial, $k_i \geq 2$, where the i-th factor has k_i levels from the set $K_i = \{0,1,2,\ldots,$

$k_i-1\}$. Let $K = K_1 \times K_2 \times \dots \times K_t$ be the Cartesian product of the sets K_i. With a treatment combination $(i_1 i_2 \dots i_t)$ in K associate an observation $y(i_1 i_2 \dots i_t)$. A factorial effect will be denoted by $A_1^{i_1} A_2^{i_2} \dots A_t^{i_t}$ with at least one of the $i_j \neq 0$ and $i_j \in K_j$, $j=1,2,\dots,t$, and the mean will be indicated by $A_1^0 A_2^0 \dots A_t^0$.

Let Y_K be the set of all observations associated with the full replicate K and let P_K be the set of all effects including the mean. Let $X_K = X_1 \otimes X_2 \otimes \dots \otimes X_t$ be the Kronecker product of real colunwise orthogonal matrices X_i of order k_i with each first column entry of X_i equal to 1. Then X_K is a real columnwise orthogonal matrix of order $k = \prod_{i=1}^{t} k_i$ with the sum of the entries of each column of X_i and of X besides the first is equal to zero. Associate with the observation vector X_K and the column vector P_K of parameters the well known linear model:

$$\begin{cases} E\,|Y_K| = X_K\,P_K\,, \\ \mathrm{Cov}\,|Y_K| = \sigma^2 I_k\,. \end{cases} \quad (2.1)$$

From the experimenter's viewpoint the complete parametric vector P_K can be partitioned as

$$P_K' = (P_1' \,\vdots\, P_2' \,\vdots\, P_3'), \quad (2.2)$$

where P_1 is a $N_1 \times 1$ vector of parameters to be estimated, P_2 is a $N_3 \times 1$ vector of parameters not of interest and not assumed to be known, and P_3 is a $N_3 \times 1$ vector of parameters assumed to be known (which without loss of generality, can be taken to be zero), such that $1 \le N_1 \le k$, $0 \le N_2 \le k-1$, and $0 \le N_3 \le k-N_1-N_2 \le k-1$. Explicitly the following four cases occur:

(i) $N_1 = k$, $N_2 = N_3 = 0$;

(ii) $N_2 = 0$, $N_3 \neq 0$;

(iii) $N_2 \neq 0$, $N_3 \neq 0$;

and

(iv) $N_2 \neq 0$, $N_3 = 0$.

By the <u>degree</u> of a factorial effect $A_1^{i_1} A_2^{i_2} \dots A_t^{i_t}$ we mean the number of nonzero exponents among $(i_1 i_2 \dots i_t)$. A <u>fractional</u>

factorial design, or simply, design is a collection of treatment combinations of K. (Note that repetitions of treatment combinations are allowed). A design is said to be of resolution R if all factorial effects up to degree ν are estimable, where ν is the greatest integer less than R/2, under the assumption that all factorial effects of degree R-ν and higher are zero. The designs of resolution R have been divided into two types in the literature, namely:

(a) R = 2r, known as designs of even resolution, and

(b) R = 2r+1, known as designs of odd resolution.

It follows that a design of even resolution is a special case of (iii) and an odd resolution design is a special case of (ii).

In the formulation below we will limit ourselves to case(ii) since a similar formulation can be done for case (iii). The observation vector of a design D consisting of m treatment combinations will be denoted by Y_D and the model for Y_D and a given subvector P_1 with p_1 parameters is read off from the full model (2.1), which in case (ii) gives rise to:

$$\begin{cases} E\left[Y_D\right] = X_{D1}P. \\ Cov\left[Y_D\right] = \sigma^2 I_m . \end{cases} \tag{2.3}$$

If the design D is such that the rank of X_{D1} is equal to p_1 then:

$$\begin{cases} \hat{P}_1 = \left[X'_{D1}X_{D1}\right]^{-1} X_{D1}Y_D = M_D^{-1}X'_{D1}\ Y_D, \\ Cov\left[\hat{P}_1\right] = M_D^{-1}\sigma^2. \end{cases} \tag{2.4}$$

We are now ready to formulate the problem of augmented fractional factorial designs. Let D consist of m distinct treatment combinations and let P_1 in (2.3) consist of p_1 = m parameters. Such a design to estimate P_1 is called a saturated factorial design. Assume that D is an optimal design relative to a given optimality criterion (see Kiefer(1960)). Since D is saturated the variance σ^2 in (2.3) cannot be estimated unbiasedly. The problem of augmented fractional factorial designs is to augment

D with $c > 0$ treatment combinations, resulting in the design D^*, such that D^* retains its optimality property.

Obviously, this problem in all its generality is difficult to resolve. Below, we specialize to the 2^t factorial, where $X_X = X_2 \otimes X_2 \otimes \ldots \otimes X_2$, and $X_2 = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$. The design matrix X_{D1} of a saturated design D relative to a given vector P_1 is then a(-1,1)-matrix. We will also restrict the development using the criterion of d-optimality.

3. AUGMENTED OPTIMAL FRACTIONAL 2^t FACTORIAL DESIGNS

Let P_1 be a fixed parametric vector in the 2^t factorial under case (ii) of Section 2. Let D be a given saturated d-optimal design relative to P_1, i.e. det (M_D) is maximum in the class $\mathcal{D}_{p_1}$ of all designs with p_1 distinct treatment combinations. The problem to be considered now is how to augment D with c extra treatment combinations so that the resulting design D^* of cardinality $p_1 + c$ is d-optimal in the class $\mathcal{D}_{p_1+c}$. Write the design D^* as $D^* = D \cup D_A$, where D_A consists of c additional distinct treatment combinations, and let X_D, X_{D_A} and X_{D*} be the design matrices of D, D_A and D^* respectively. The problem before us is to select D_A such that the corresponding $\det(M_{D*}) = \det(X'_{D*}X_{D*})$ is maximum in $\mathcal{D}_{p_1+c}$

Now,

$$X_{D*} = \begin{bmatrix} X_D \\ \hline X_{D_A} \end{bmatrix} \tag{3.1}$$

Since det $(X'_D X_D) \neq 0$, it follows from a well known expansion that

$$\det(X'_{D*}X_{D*}) = \det(X'_D X_D).\det\left[I_c + X_{D_A}(X'_D X_D)^{-1}X'_{D_A}\right]. \tag{3.2}$$

Thus the maximization problem reduces to maximizing

$$\det\left[I_c + X_{D_A}(X'_D X_D)^{-1}X'_{D_A}\right]. \tag{3.3}$$

For the case c=1, X_{D_A} is a row vector $\underset{\sim}{x}'$ and hence the above expansion becomes

$$\det(X'_{D*}X_{D*}) = \det(X'_D X_D)\cdot\left[1 + \chi'(X'_D X_D)^{-1}\chi\right]. \qquad (3.4)$$

Since D is fixed, the maximization problem for $c = 1$ reduces to maximizing the scalar $\chi'(X'_D X_D)^{-1}\chi$ and this has been considered and illustrated by Dykstra (1971) in a regression context.

To make some headway in the general problem we further assume that D is an orthogonal design relative to P_1, i.e. $X'_D X_D = \alpha I_{p_1}$, where α is a fixed constant. Notice that this assumption is equivalent to assuming that X_D is a Hadamard matrix of order p_1 which in turn implies that D is d-, a-, and e-optimal. It now follows that equation (3.3) under this additional assumption can be re-written as:

$$\det(I_c + \alpha^{-1}X_{D_A}X'_{D_A}) = \det(B_{D_A}), \text{ say}, \qquad (3.5)$$

and the maximization problem in this setting reduces to maximizing $\det(B_{D_A})$ given in (3.5).

If $\lambda_1, \lambda_2, \ldots, \lambda_r$ are the nonzero characteristic roots of $X_{D_A}X'_{D_A}$ then the c characteristic roots of B_{D_A} are given by $\alpha^{-1}\lambda_1 + 1$, $\alpha^{-1}\lambda_2 + 1\ldots, \alpha^{-1}\lambda_r + 1, 1, 1, \ldots, 1$, where $r = \text{rank}(X_{D_A}) \leq \min(c, p_1)$.

Therefore, to maximize $\det(B_{D_A})$, we must maximize the product

$$\prod_{i=1}^{r}(\alpha^{-1}\lambda_i + 1) \qquad (3.6)$$

for all r, $1 \leq r \leq \min(c, p_1)$. Noting that for all choices of D_A, trace $(X'_{D_A}X_{D_A}) = p_1 c$, the maximization of the quantity given in equation (3.6) is subject to the restriction $\sum_{i=1}^{r}\lambda_i = p_1 c$.

Now, for a fixed r

$$\max_{\sum_{i=1}^{r}\lambda_i = p_1 c}\left\{\prod_{i=1}^{r}(\alpha^{-1}\lambda_i + 1)\right\} = \left(\alpha^{-1}\frac{p_1 c}{r} + 1\right)^r, \qquad (3.7)$$

which is equal to $\prod_{i=1}^{r}(\alpha^{-1}\lambda_i + 1)$ with $\lambda_1 = \lambda_2 = \ldots = \lambda_r = \frac{p_1 c}{r}$. We have now to maximize the quantity given in (3.7) for all r, where $1 \leq r \leq \min(c, p_1)$.

The following lemma will be useful in the sequel.

Lemma 3.1. Let x be any non-negative real number and u and v be any two non-negative integers such that $u \neq 0$, and $u \le v$. Then

$$(1 + \frac{xv}{u})^u \le (1+x)^v . \tag{3.8}$$

Proof. Follows immediately from a term by term comparison of the binomial expansions of the two sides of (3.8).

We therefore obtain the following inequality:

$$\left[1 + \alpha^{-1}\frac{p_1 c}{r}\right]^r = \left[1 + \left(\frac{\alpha^{-1} p_1 c}{\min(c,p_1)}\right)\left(\frac{\min(c,p_1)}{r}\right)\right]^r \le \left[1 + \frac{\alpha^{-1} p_1 c}{\min(c,p_1)}\right]^{\min(c,p_1)} \tag{3.9}$$

Thus the quantity (3.7) is maximized when $r = \min(c,p_1)$. Hence we have established the following theorem.

Theorem 3.1. If D is a saturated orthogonal 2^t fractional factorial design relative to P_1 in the class $\mathcal{D}_{p_1}$, then an optimal design D^* with an additional set of c treatment combinations relative to P_1 in the class $\mathcal{D}_{p_1+c}$ will be obtained if $r=\min(c,p_1)$, where $r = \text{rank}(X_{D_A})$, and the nonzero characteristic roots of $X_{D_A}'X_{D_A}$ are all equal.

The above theorem provides only a sufficient condition for the existence of an optimal augmentation. In some cases the conditions of the theorem are attainable and in some cases they are not. We shall show that when $c \le p_1$ an optimal augmentation always exists. Also, when $c > p_1$ and c is a Hadamard number (i.e. a Hadamard matrix of order c exists) then an optimal augmentation exists.

Theorem 3.2. There exists an optimal augmentation for all $c \le p_1$.

Proof. Since $c \le p_1$ we have $r = c$. Clearly $\det(X_{D^*}'X_{D^*})$ will be maximum when all the c positive characteristic roots of $X_{D_A}'X_{D_A}$ are the same and equal to $\frac{p_1 c}{r} = p_1$. Since the positive characteristic roots of $X_{D_A}'X_{D_A}$ and $X_{D_A}X_{D_A}'$ are the same, it follows that $\det(X'_{D^*}X_{D^*})$ will be maximized when $X_{D_A}X'_{D_A} = p_1 I_c$, i.e. when the rows of X_{D_A} are mutually orthogonal. Since p_1 is a Hadamard number we can choose any c rows of H_{p_1}, a Hadamard matrix of order p_1, as our X_{D_A}. The corresponding treatment combinations D_A form the desired augmentation D_A.

Theorem 3.3. If c is a Hadamard number (i.e. a Hadamard matrix of order c exists) and $c > p_1$ then an optimal augmentation exists.

Proof. In this case the maximum given in (3.9) is obtained when $X'_{D_A}X_{D_A} = c\, I_{p_1}$. Thus since both c and p_1 are Hadamard numbers we can choose any p_1 columns of H_c as our design matrix X_{D_A} and the corresponding design D_A will provide the optimal augmentation.

Note that the case c=1 is interesting. Since $X_{D_A}X'_{D_A} = p_1 I_1$, it is clear that the addition of any treatment combination to D will give the same value of $\det(X'_{D^*}X_{D^*})$ whenever D is an orthogonal design relative to P_1 in the 2^t factorial.

Example 3.1. Consider the 2^3 factorial and let D be the orthogonal saturated resolution III design

$$D = \{(000), (110), (101), (011)\}.$$

Note that in this case P_1 consists of the mean and the three main effects. The design matrix X_D relative to P_1 is the Hadamard matrix

$$X_D = \begin{bmatrix} 1 & -1 & -1 & -1 \\ 1 & 1 & 1 & -1 \\ 1 & 1 & -1 & 1 \\ 1 & -1 & 1 & 1 \end{bmatrix}$$

Suppose the problem is to augment D with c = 3 treatment combinations such that the resulting design D^*, consisting of 7 treatment combinations, is d-optimal. Then, since $c < p_1$ in this case, according to Theorem 3.2 an optimal design D^* can be read off from the augmented design matrix X_{D^*} and it consists of D and the treatment combinations corresponding to any three rows of X_D, e.g. picking the first three rows of X_D results in:

$D^* = D \cup D_A$, where $D_A = \{(000),(110),(101)\}$. With $c = 8$, then according to Theorem 3.3, an optimal augmented design D^* can be read off from the augmented design matrix X_{D^*}, which consists of X_D and the first column and any 3 other columns of a semi-normalized Hadamard matrix, viz. taking the first 4 columns of $H_8 = H_2 \otimes H_2 \otimes H_2$, with $H_2 = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$, we have:

$$X_{D^*} = \begin{bmatrix} 1 & -1 & -1 & -1 \\ 1 & 1 & 1 & -1 \\ 1 & 1 & -1 & 1 \\ 1 & -1 & 1 & -1 \\ \hline 1 & -1 & -1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & 1 & -1 \\ 1 & -1 & -1 & 1 \\ 1 & 1 & 1 & -1 \\ 1 & 1 & -1 & 1 \\ 1 & -1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix} = \begin{bmatrix} X_D \\ \hline X_{D_A} \end{bmatrix}$$

This results in:

$D^* = D \cup D_A$, where D_A is the complete replicate of the 2^3 factorial, i.e. $D_A = \{(000),(100),(010),(001),(110),(101),(011),(111)\}$.

Remarks: (i) It should be intuitively clear that the augmentation should be done in such a way that the additional treatment combinations D_A lead to independent rows in the design matrix X_{D_A}. If any two treatments lead to dependency of the corresponding rows in X_{D_A} then there is no gain in information relative to estimation of P_1 by including both of them in D_A.

(ii) We could have formulated the augmentation problem by starting with an unsaturated design D. However, the motivation vis-a-vis the most economical design (in terms of minimal number of treatment combinations), which is incorporated in the saturated case, would be lost. Note that the same argument, given for the saturated case above, would follow through for an unsaturated design whose design matrix is column-wise orthogonal in the 2^t setting.

(iii) One may generalize the above results to the general mixed factorial setting as long as the initial design D, under an appropriate linear model, leads to $X_D'X_D = \alpha I$ for a real constant α.

4. OPTIMAL AUGMENTED BALANCED 2^t FRACTIONAL FACTORIAL DESIGNS

Consider D to be a saturated balanced design in the 2^t factorial and let us restrict augmentation by c = 1 treatment combination, i.e. D_A consists of a single treatment combination. Further assume that the mean is always present in P_1 and is its first element. The problem now is to select D_A in such a way that $\det(X_{D^*}'X_{D^*})$ is maximum, where $D^* = D \cup D_A$.

It is clear from equation (3.2) that we are seeking a (-1,1) - column vector $\underset{\sim}{x}$ which will be such that

$$\max_{x_i = \pm 1}.(\underset{\sim}{x}'A_D^{-1}\underset{\sim}{x}) \tag{4.1}$$

will be achieved, where A_D is the $p_1 \times p_1$ information matrix. Since D is a balanced design it follows from Raktoe and Federer (1973) that:

$$A_D^{-1} = A^{-1} = \left[\begin{array}{c|c} a & b\underset{\sim}{1}' \\ \hline b\underset{\sim}{1} & dI+eJ \end{array}\right] \tag{4.2}$$

where a, b, d, e are constants, $\underset{\sim}{1}$ is a (p_1-1)-column vector whose entries are all 1's, I is the identity matrix of order (p_1-1), and J is the matrix of order (p_1-1) with entries all equal to 1.

For any (-1,1)-row vector $\underset{\sim}{x}' = (x_1, x_2, \ldots, x_{p_1})$, we have by direct calculation

$$\max_{x_i=\pm 1}.\underset{\sim}{x}'A^{-1}\underset{\sim}{x} = \max_{x_i=\pm 1}.[f+2bx_1(\sum_{j=2}^{p_1}x_j)+e(\sum_{j=2}^{p_1}x_j)^2], \tag{4.3}$$

where $f = a + d(p_1-1)$ is a positive constant. This leads to the consideration of the following cases.

Case(1): e = 0. Under this case we have the following possibilities:

(i) b = 0. Clearly from (4.3) in this case any treatment combination selected for the augmentation will do to obtain the maximization.

(ii) $b > 0$. In this case we must select $\underset{\sim}{x}$ in such a way that x_1 and $\sum_{j=2}^{p_1} x_j$ have the same sign and their product is maximized. This means that the treatment combination selected must be either $0' = (00\ldots0)$ or its foldover$(11\ldots1)$ in order to obtain an optimal augmentation.

(iii) $b < 0$. In this case we must select $\underset{\sim}{x}$ in such a way that x_1 and $\sum_{j=2}^{p_1} x_j$ have opposite signs and their product is maximized. This means that the treatment combination selected must be either $(100\ldots0)$ or its foldover $(011\ldots1)$ in order to obtain an optimal augmentation.

Case(2): $e \neq 0$. Since $f = a + d(p_1 - 1)$ is a constant the maximization depends merely on maximizing the other terms in equation (4.3). Since

$$\max_{x_1=\pm1}.[e(\sum_{j=2}^{p_1} x_j)^2 + 2bx_1(\sum_{j=2}^{p_1} x_j)]$$

$$= \max_{x_1=\pm1}.[e(\frac{bx_1}{e} + \sum_{j=2}^{p_1} x_j)^2 - \frac{b^2}{e} x_1^2], \tag{4.4}$$

we are led to the following possibilities:

(i) $e > 0$. We may choose x_1 to be +1 or -1 arbitrarily. Then if $b \neq 0$, we take $\sum_{j=2}^{p_1} x_j$ equal to respectively $(p_1 - 1)$ or $-(p_1 - 1)$ according as $(\frac{bx_1}{e})$ is positive or negative. If $b = 0$, any choice of a $(-1,1)$-vector $\underset{\sim}{x}$ such that $(\sum_{j=2}^{p_1} x_j)^2$ is maximum will do. This means that in either case the additional treatment combination needed to obtain an optimal augmentation is anyone of $\underset{\sim}{0}' = (00\ldots0)$, $(10\ldots0)$, $(01\ldots1)$, or $(11\ldots1)$.

(ii) $e < 0$. In this case the maximum of the expression given in equation (4.4) will be obtained by calculating the minimum value, $\min_{x_1=\pm1}.(\frac{bx_1}{e} + \sum_{j=2}^{p_1} x_j)^2$. Choose x_1 to be +1 or -1 such that $(\frac{bx_1}{e}) \geq 0$. Write $\frac{bx_1}{e} = h + g$, where $h \geq 0$ is the integer part of $\frac{bx_1}{e}$ and $0 \leq g < 1$. Then select $x_2, x_3, \ldots x_{p_1}$ such that $\sum_{j=2}^{p_1} x_j$ is respectively equal to $-(p_1-1)$ if $h \geq p_1-1$; $-h$ if $0 \leq h < p_1-1$ and p_1-h-1 is even; $-(h+1)$ if $0 \leq h < p_1-1$ and p_1-h-1 is odd. Such choices for the x_i clearly minimize the desired expression. This means that the selection of any treatment combination $(t_1 t_2 \ldots t_{p_1-1})$ such that

$$
\begin{aligned}
& t_1 = 1, \text{ if } \frac{b}{e} > 0, \\
& \quad = -1, \text{ if } \frac{b}{e} < 0, \\
& \quad = 1 \text{ or } -1, \text{ if } \frac{b}{e} = 0, \\
& t_i = 0, \text{ for all } i \geq 2 \text{ if } h \geq p_1-1, \text{ and} \\
\text{if} \quad & h < p_1-1 \text{ then take} \\
& t_i = 0, \text{ for any } (p_1+h-1)/2 \text{ subscripts } i \geq 2, \\
& \quad = 1, \text{ for the remaining } i, \text{ if } (p_1-h-1) \text{ is even,} \\
\text{or,} \quad & \\
& t_i = 0, \text{ for any } (p_1+h)/2 \text{ subscripts } i \geq 2, \\
& \quad = 1, \text{ for the remaining } i, \text{ if } (p_1-h-1) \text{ is odd,}
\end{aligned}
\tag{4.5}
$$

will lead to an optimal augmentation. Similarly, if x_1 is selected as +1 or -1 such that $(\frac{bx_1}{e}) \leq 0$, then the selection of the foldover of any treatment combination described in equation (4.5) (i.e. the treatment combination obtained by interchanging 0's for 1's and 1's for 0's) for the corresponding case will also lead to an optimal augmentation.

We have thus established the following theorem.

Theorem 4.1. Let D be a balanced saturated design relative to P_1 in the 2^t factorial, where the mean is the first element in P_1. Then an optimal augmentation D to D* with one additional treatment combination is achieved in the following ways: (i) when $e = 0$ and $b = 0$, any choice of a treatment combination will lead to an optimal augmentation; (ii) when $e = 0$, $b > 0$, then the selection of either (00...0) or its foldover (11...1) will lead to an optimal augmentation; (iii) when $e = 0$ and $b < 0$, then the selection of either (10...0) or its foldover (01...1) will lead to an optimal augmentation; (iv) if $e > 0$, then the choice of anyone of the treatment combinations (00...0), (01...1), (10...0) or (11...1) will lead to an optimal augmentation; (v) if $e < 0$, then the choice of any treatment combination satisfying (4.5) or their foldovers will lead to an optimal augmentation.

We now illustrate the results in Theorem 4.1 with an example.

Example 4.1. In the 2^4 factorial let D be the saturated main effect plan given by:

$$D = \{(0111), (1011), (1101), (1110), (1111)\}.$$

Then under the usual model it can be easily verified that

$$A_D = \left[\begin{array}{c|cccc} 5 & 3 & 3 & 3 & 3 \\ \hline 3 & 5 & 1 & 1 & 1 \\ 3 & 1 & 5 & 1 & 1 \\ 3 & 1 & 1 & 5 & 1 \\ 3 & 1 & 1 & 1 & 5 \end{array}\right] = \left[\begin{array}{c|c} 5 & 3\,\underset{\sim}{1}' \\ \hline 3\,\underset{\sim}{1} & 4\,I_{4\times4} + J_{4\times4} \end{array}\right]$$

and

$$A_D^{-1} = \left[\begin{array}{c|c} 2 & -\frac{3}{4}\underset{\sim}{1}' \\ \hline -\frac{3}{4}\underset{\sim}{1} & \frac{1}{4}(I_{4\times4} + J_{4\times4}) \end{array}\right]$$

Hence in the notation of the theorem we have:

$a = 2$, $b = -\frac{3}{4}$, $d = \frac{1}{4}$, and $e = \frac{1}{4}$. Since $e > 0$ we are in case (iv) of Theorem 4.1 and hence an optimal augmentation will be achieved by selecting anyone of the treatment combinations (0000), (0111), (1000), or (1111).

5. DISCUSSION

The results in Section 4 for the case $c = 1$ are a generalization of those in Section 3 in the sense that orthogonality is implied in equation (4.2) when $b = 0$ and $e = 0$ so that $a = d$. Further, one may generalize the results obtained in Section 4 to the unsaturated case with the only restriction being that the mean is the first element of P_1. As earlier, the motivation of starting with the most economic design (i.e. a D with a minimal number of treatment combinations) would be lost. The development of the theory for the general mixed factorial, even for the case $c = 1$, appears complicated since the design matrices will not be simply (-1,1)-matrices. All the results obtained in this paper can be further generalized by not only augmenting treatment combinations but also augmenting parameters. The work of Pesotan et al (1975,1977) and Raghavarao et al (1977) has made a beginning in this direction by considering the 2^t series. See as well Section 17C in Raghavarao (1971), where ideas analagous to those in this paper concerning the augmentation of singular weighing designs are discussed. It is clear that considerable further work needs to be done to resolve many of the problems of augmented fractional factorial designs.

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