LINEAR ALGEBRA AND ITS APPLICATIONS 9, 155–167 (1974)

Projections under Seminorms and Generalized Moore Penrose Inverses

Sujit Kumar Mitra* and C. Radhakrishna Rao*

Indiana University, Bloomington, Indiana 47401
and Indian Statistical Institute, Calcutta 35, India

Recommended by Gene Golub

ABSTRACT

The definition of a projector under a seminorm is given. Such a projector is not unique. Operators projecting into a given linear subspace under a seminorm form an affine linear subalgebra of the linear associative algebra of square matrices. The authors have introduced elsewhere the concept of a minimum seminorm semileast squares inverse of a complex matrix. It is shown here that the same concept could also be defined in terms of projectors under seminorms. This extends a similar definition for the Moore Penrose inverse given in terms of orthogonal projectors under the usual Euclidean norms. Various properties of a projector under a seminorm and also of a minimum seminorm semileast squares inverse are obtained including representations giving general solutions for both.

1. INTRODUCTION

The concept of projection under a seminorm was introduced in Rao and Mitra [6, 7]. In the present publication, we propose to study this operator in greater detail while correcting some errors in earlier results.

The study of projection under a seminorm arose out of its importance in the statistical problem of estimation of parameters in linear models when the random variables have a singular covariance matrix. We hope it will find applications in other areas too.

We denote by $\mathbf{E}^p$ the vector space of all complex p-tuples. Let A be a

*Research supported by National Science Foundation Grant No. GP32822.

complex matrix of order $m\times n$ and M a nonnegative definite matrix of order $m\times m$. Let $\mathfrak{M}(A)$ denote the vector space spanned by the columns of A.

DEFINITION A matrix $P_{A(M)}$ is called a projector into $\mathfrak{M}(A)$ with respect to the seminorm defined by $\|x\|_M=(x^*Mx)^{1/2}$, $x\in \mathbf{E}^m$ if and only if the following two conditions hold

$$P_A y\in \mathfrak{M}(A) \qquad \forall y\in \mathbf{E}^m \tag{1.1a}$$

$$\| y-P_A y\|_M \leq \| y-Ax\|_M \qquad \forall x\in \mathbf{E}^n,\ y\in \mathbf{E}^m. \tag{1.1b}$$

We denote $P_{A(M)}$ simply by P_A when the seminorm involved is understood with reference to context. The class of such matrices which are projectors into $\mathfrak{M}(A)$ is denoted by $\{P_{A(M)}\}$ or simply by $\{P_A\}$.

In Rao and Mitra [6,7] is shown the existence of a matrix G, called M-semileast squares inverse of A, such that $\hat{x}=Gy$ is a M-semileast squares solution of $Ax=y$, that is, one which minimizes $(y-Ax)^*M(y-Ax)$. Then $A\hat{x}=AGy$ and one choice of P_A is AG. This establishes the connection between projection operators and generalized inverses.

2. PROJECTION OPERATOR UNDER SEMINORMS

We establish the following lemmas and theorems concerning projection operators under seminorms.

LEMMA 2.1. *The matrix P of order $m\times m$ is a projector onto $\mathfrak{M}(P)$ iff*

$$P^*MP=MP. \tag{2.1}$$

Proof. With (1.1a) trivially satisfied, one need only consider (1.1b). Observe that $(I-P)^*MP\neq 0 \Rightarrow y^*(I-P)^*MPz\neq 0$ for some $y,z\in \mathcal{E}^m \Rightarrow (y-Py)^*M(y-Py)>(y-Px)^*M(y-Px)$ where $x=y+cz$ and $\bar{c}=y^*(I-P)^*MPz/z^*P^*MPz$, which contradicts (1.1b). Thus the necessity of (2.1) is established. To prove sufficiency, check that $(2.1)\Rightarrow (y-Px)^*M(y-Px)=(y-Py)^*M(y-Py)+(y-x)^*P^*MP(y-x)$ which shows that for each x we have indeed the strict inequality in (1.1b) unless

$$MPx=MPy. \tag{2.2}$$

As a simple consequence of Lemma 2.1, we have Lemma 2.2.

LEMMA 2.2. *Condition* (2.1) *is equivalent to*

$$(MP)^* = MP, \tag{2.3a}$$

$$MP^2 = MP. \tag{2.3b}$$

THEOREM 2.1. *P is a projector into $\mathcal{M}(A)$ iff*

$$\text{(i)} \quad \mathcal{M}(P) \subset \mathcal{M}(A), \tag{2.4a}$$

$$\text{(ii)} \quad P^*MP = MP \ \ (\text{or } MP = P^*M), \tag{2.4b}$$

$$\text{(iii)} \quad MPA = MA. \tag{2.4c}$$

The proof of Theorem 2.1 is similar to that of Lemma 2.1.

THEOREM 2.2. *The conditions of Theorem* 2.1 *are equivalent to the following*

$$\text{(i)} \quad \mathcal{M}(P) \subset \mathcal{M}(A), \tag{2.5a}$$

$$\text{(ii)} \quad P^*MP = MP, \tag{2.5b}$$

$$\text{(iii)} \quad \text{Rank}(MP) = \text{Rank}(MA). \tag{2.5c}$$

Proof. (2.4a)$\Rightarrow P = AK$ for some matrix K. Then $MP = MAK \Rightarrow \text{Rank}(MP) < \text{Rank}(MA)$. But (2.4c)$\Rightarrow \text{Rank}(MP) > \text{Rank}(MA)$. Thus $\text{Rank}(MP) = \text{Rank}(MA)$ so that (2.5 a,b,c) follow from (2.4 a,b,c). (2.5a) and (2.5c) $\Rightarrow MA = MPR$ for some matrix R. Using (2.5b) $MA = MPR = P^*MPR = P^*MA = MPA$ which establishes the reverse equivalence.

Note 1. (2.5b) and (2.5c) or (2.4c)$\Rightarrow$(1.1b). This, however, is no guarantee that P would project into the right subspace for which we need (1.1a) or equivalently (2.5a). When M is positive definite, (2.5c) and (2.4c) are respectively equivalent to $\text{Rank}(P) = \text{Rank}(A)$ and $PA = A$ and these two together imply $\mathcal{M}(P) = \mathcal{M}(A)$ and hence (2.5a).

Note 2. If P is a projector into $\mathcal{M}(A)$ and $\mathcal{M}(P) \subset \mathcal{M}(B) \subset \mathcal{M}(A)$, then P is also a projector into $\mathcal{M}(B)$. Let $\mathbf{C}$ denote the collection of all such linear subspaces of $\mathbf{E}^m$ with the property that if the subspace $\mathcal{S}$ is a member of $\mathbf{C}$ then P is a projector into $\mathcal{S}$. $\mathbf{C}$ is thus a partially ordered set with respect to the set inclusion relation. The minimal and maximal elements in $\mathbf{C}$ are given by $\mathcal{M}(P)$ and $\mathcal{M}(P) + \mathcal{N}(M)$, respectively, where $\mathcal{N}(M)$ is the nullspace of M. The linear transformation M maps each linear space in C onto the same image space, which we denote by $\mathcal{S}(P)$. Observe that $\mathcal{S}(P)=\mathcal{M}(MP)$ is a characteristic of the projector P.

THEOREM 2.3. *For $i=1,2$ let P_i be a projector into $\mathcal{M}(P_i)$. Then*

(a) *P_1+P_2 is a projector if and only if $MP_1P_2=MP_2P_1=0$.*
(b) *P_1-P_2 is a projector if and only if $MP_1P_2=MP_2P_1=MP_2$.*
(c) *P_1P_2 is a projector if and only if $MP_1P_2=MP_2P_1$.*

Proof. Similar propositions for the usual projectors (orthogonal or oblique) are well known (see, for example, Chapter 5 of Rao and Mitra 7) and Theorem 2.3 could be proved on the same lines. We shall however prove the necessity parts of Theorem 2.3(a) and (c) to show the changes that may be needed in the proof.

$$M(P_1+P_2)^2=M(P_1+P_2), MP_i^2=MP_i \quad \text{and} \quad (MP_i)^*=MP_i$$

$$\Rightarrow M(P_1P_2+P_2P_1)=0$$

$$\Rightarrow MP_1P_2^2+MP_2P_1P_2=P_1^*MP_2^2+MP_2P_1P_2$$

$$=P_1^*MP_2+MP_2P_1P_2=MP_1P_2+MP_2P_1P_2=0$$

$$\Rightarrow P_2^*MP_1P_2+P_2^*MP_2P_1P_2=MP_2P_1P_2+MP_2^2P_1P_2$$

$$=2MP_2P_1P_2=0.$$

Back substitution in the preceding steps show $MP_1P_2=MP_2P_1=0$, thus establishing the necessity part of Theorem 2.3(a).

$$(MP_i)^*=MP_i, \qquad (MP_1P_2)^*=MP_1P_2 \Rightarrow$$

$$MP_1P_2=P_1^*MP_2=P_1^*P_2^*M=(P_1^*P_2^*M)^*=MP_2P_1.$$

This establishes the necessity part of Theorem 2.3 (c).

When M is positive semi-definite, a projection into $\mathcal{M}(A)$ is not unique. We have, however, the following result.

THEOREM 2.4. *If P and $\bar{P}$ are two choices of a projector into $\mathcal{M}(A)$, then so are*

$$P\bar{P} \quad \text{and} \quad \lambda P+(1-\lambda)\bar{P} \tag{2.6a}$$

for any complex number λ, *and*

$$MP^2 = M\bar{P}^2 = MP\bar{P} = M\bar{P}P = MP = M\bar{P}. \tag{2.6b}$$

Proof. From (2.4a) and (2.4c) it follows that

$$M\bar{P}P = MP^2 = MP. \tag{2.7}$$

Using (2.4b) and (2.7) we have

$$(P-\bar{P})^*M(P-\bar{P}) = M(P-\bar{P})^2 = 0.$$

Hence $MP = M\bar{P}$ and (2.6b) is established.

To show that $\tilde{P} = \lambda P + (1-\lambda)\bar{P}$ is indeed a projector into $\mathfrak{M}(A)$, check that $\tilde{P}$ trivially satisfies (2.4a) and (2.4c). Also

$$\tilde{P}^*M\tilde{P} = M\tilde{P}^2 = M\left[\lambda^2P^2 + \lambda(1-\lambda)P\bar{P} + (1-\lambda)\lambda\bar{P}P + (1-\lambda)^2\bar{P}^2\right]$$

$$= \left[\lambda^2 + 2\lambda(1-\lambda) + (1-\lambda)^2\right]MP = MP$$

$$= M\left[\lambda P + (1-\lambda)\bar{P}\right] = M\tilde{P}.$$

Thus $\tilde{P}$ satisfies (2.4b) as well as being thus a projector into $\mathfrak{M}(A)$.

Theorem 2.4 shows that the class of projectors into $\mathfrak{M}(A)$, under a seminorm, is a affine sublinear space of the linear space of matrices of order $m \times m$. Further, it is also closed under multiplication. In this sense we have the following result:

THEOREM 2.5. *The projectors into* $\mathfrak{M}(A)$ *under a seminorm span an affine linear subalgebra of the linear associative algebra of* $m \times m$ *matrices.*

The following two theorems lead to an explicit algebraic representation of a projector under a seminorm by showing first how a projector with respect to a seminorm is related to an orthogonal projector with respect to the usual Euclidean norm.

THEOREM 2.6. *Let* $M = C^*DC$ *where D is nonnegative definite. Then*

(a) $\{P_{CA(D)}C\} \subset \{CP_{A(M)}\}$,

(b) $\{P_{CA(D)}C\} = \{CP_{A(M)}\}$ *if and only if either*

$$\mathfrak{M}(C^*) \subset \mathfrak{M}(M) + \mathfrak{N}(A^*), \tag{2.8a}$$

and/or

$$C^*C \quad \text{is positive definite.} \tag{2.8b}$$

Proof. (a) is a simple consequence of the fact that

$$\| y - Ax\|_M = \|Cy - CAx\|_D.$$

For (b) the sufficiency of (2.8a) follows from Theorem 2.6(a) and Theorem 2.4 since, when $\mathfrak{M}(C^*) \subset \mathfrak{M}(M) + \mathfrak{N}(A^*)$, the uniqueness of $MP_{A(M)}$ as implied by (2.6b) implies that $CP_{A(M)}$ be unique irrespective of the choice of the projector $P_{A(M)}$. To prove the sufficiency of (2.8b) choose and fix matrices $P \in \{P_{A(M)}\}$ and $Q \in \{P_{CA(D)}\}$ such that $CP = QC$. Existence of such a pair P, Q follows from Theorem 2.6(a). Also let the matrix K be such that $\mathfrak{M}(K) = \mathfrak{M}(A) \cap \mathfrak{N}(M)$. From (2.6b) and Theorem 2.1 it is seen that a general solution to $P_{A(M)}$ is given by $P_{A(M)} = P + KU$, where U is such that KU is a matrix of order $m \times m$ and is otherwise arbitrary. Observe that $Q + CKU(C^*C)^{-1}C^*$ is indeed one choice of $P_{CA(D)}$ and

$$C(P + KU) = \left(Q + CKU(C^*C)^{-1}C^*\right)C.$$

To prove the "only if" part of Theorem 2.6(b), assume that both (2.8a) and (2.8b) are untrue. Let P and Q be determined as in above. Let u be a vector in $\mathbf{E}^m$ which is not in $\mathfrak{M}(C^*)$ and v be a vector in $\mathfrak{M}(K)$ which is not in $\mathfrak{N}(C)$. It is easily seen that

$$P + vu^* \in \{P_{A(M)}\}$$

$$C(P + vu^*) \notin \{P_{CA(D)}C\}.$$

THEOREM 2.7. (a) *Let* $M = C^*C$. *Then*

$$CP_{A(M)} = P_{CA(I)}C. \tag{2.9a}$$

(b) A general solution to $P_{A(M)}$ *is*

$$P_{A(M)} = A(A^*MA)^- A^*M + A[I - (A^*MA)^- A^*MA]U, \tag{2.9b}$$

where U *is arbitrary and* $(A^*MA)^-$ *called a generalized inverse (or a* *g-inverse) of A^*MA represents a matrix such that*

$$A^*MA(A^*MA)^- A^*MA = A^*MA.$$

Proof. (2.9a) follows from Theorem 2.6(b). The uniqueness of $CP_{A(M)}$ under (2.8a) was noted in the proof of this Theorem. Using the explicit representation of the orthogonal projector given for example in Rao and Mitra [7, p. 111] we have

$$P_{CA(I)}C = CA(A^*C^*CA)^- A^*C^*C = CA(A^*MA)^- A^*M.$$

$A(A^*MA)^- A^*M$ is thus seen to be one choice of $P_{A(M)}$. (2.9b) follows from the fact that

$$\mathfrak{M}\left[A\{I-(A^*MA)^- A^*MA\}\right] = \mathfrak{M}(A) \cap \mathfrak{N}(M)$$

[Mitra and Rao [3]].

3. MINIMUM SEMINORM SEMILEAST SQUARES INVERSE

Let the seminorms of $x \in \mathbf{E}^n$ and $y \in \mathbf{E}^m$ be defined by

$$\|x\|_N = (x^*Nx)^{1/2} \qquad \|y\|_M = (y^*My)^{1/2},$$

where M and N are nonnegative definite matrices. As in Rao and Mitra (6, 7), we define the following:

(a) G is a g-inverse of A if $x = Gy$ is a solution of the consistent equation $Ax = y, \forall y \in \mathfrak{M}(A)$. We represent such an inverse by A^-, the entire class by $\{A^-\}$, and the subclass satisfying $(A^-)^- = A$ by $\{A_r^-\}$.

(b) G is a minimum N-seminorm, M-semileast squares inverse of A if and only if $x = Gy$ has minimum N-seminorm among the semileast squares solutions of $Ax = y$ which is possibly inconsistent. We denote the class of such matrices by $\{A_{MN}\}$. The subclass $\{A_{MN}^+\}$ consists of such inverses A_{MN} which are also in $\{A_r^-\}$

It may be noted that A_{MN} is the Moore-Penrose inverse when M and N are positive definite and is unique. The unique $A_{MN} \in \{A_r^-\}$ and is therefore of the type A_{MN}^+. In general, when M and N are not positive definite, A_{MN} is not unique and need not be a g-inverse in the sense defined in Rao and

Mitra (6). We investigate the properties of A_{MN} and the conditions under which A_{MN} is unique, A_{MN}^{+} exists, and related problems.

THEOREM 3.1. *For G to be A_{MN} it is necessary and sufficient that*

$$\text{(a)}\quad MAGA=MA,\quad (AG)^{*}M=MAG \tag{3.1a}$$

$$\text{(b)}\quad \mathfrak{M}(NG)\subset\mathfrak{M}(A^{*}MA). \tag{3.1b}$$

Proof. For $\hat{x}=Gy$ to satisfy the requirement $\|Ax-y\|_M \geq \|A\hat{x}-y\|_M$ it is necessary and sufficient that $\hat{x}$ is a solution of the equation

$$A^{*}MAx=A^{*}My \tag{3.2a}$$

$$\Leftrightarrow A^{*}MAG=A^{*}M \tag{3.2b}$$

$$\Leftrightarrow (3.1\text{a}).$$

A general solution to (3.2a) is given by

$$x=\hat{x}+[I-(A^{*}MA)^{-}A^{*}MA]z, \tag{3.3}$$

where $z\in\mathbf{E}^n$ is arbitrary. For $\hat{x}$ to have minimum N seminorm in this class, it is necessary and sufficient that

$$(\hat{x})^{*}N[I-(A^{*}MA)^{-}A^{*}MA]z=0\quad \forall y\in\mathbf{E}^m, z\in\mathbf{E}^n \tag{3.4}$$

$$\Leftrightarrow (3.1\text{b}).$$

THEOREM 3.2. *For G to be A_{MN} it is necessary that*

$$\text{(a)}\quad MAGA=MA,\quad (AG)^{*}M=MAG, \tag{3.5a}$$

$$\text{(b')}\quad NGAG=NG,\quad (GA)^{*}N=NGA. \tag{3.5b}$$

The conditions (a) *and* (b′) *are respectively equivalent to*

$$AG\in\{P_A\}\quad\text{and}\quad GA\in\{P_G\}, \tag{3.5c}$$

where P_A is a projector under the M seminorm and P_G is a projector under the N seminorm, as defined in Section 2 of this paper.

Proof. $A^{*}MAG=A^{*}M\Rightarrow A^{*}MA(I-GA)=0\ \forall z\in\mathbf{E}^n$. If $\hat{x}=Gy$, for arbitrary $z \in \mathbf{E}^n$

$$x = \hat{x} + (I - GA)z \tag{3.6}$$

is a solution to (3.2a), though not necessarily the general solution. For $\hat{x}$ to have a minimum N seminorm in this class, it is necessary and sufficient that

$$(\hat{x})^* N(I - GA)z = 0 \quad \forall y \in \mathbf{E}^m, z \in \mathbf{E}^n \tag{3.7}$$

$$\Leftrightarrow G^* N(I - GA) = 0$$

$$\Leftrightarrow \text{condition (b}') \text{ of Theorem 3.2.}$$

Using Theorems 3.1 and 3.2, we have the following result.

THEOREM 3.3. *For G to be A_{MN} it is necessary and sufficient that*

$$AG \in \{P_A\}, \quad GA \in \{P_G\}, \tag{3.8a}$$

and

$$\mathcal{S}(GA) \subset \mathfrak{M}(A^* MA), \tag{3.8b}$$

where $\mathcal{S}(\ )$ is the characteristic of a projector as introduced in Note 2 following Theorem 2.2.

Proof. Necessity of (3.8a) was proved in Theorem 3.2. When (3.8a) or equivalently (3.5a) and (3.5b) holds, G satisfies the equation

$$NGAG = NG.$$

Hence $\mathcal{S}(GA) = \mathfrak{M}(NP_G) = \mathfrak{M}(NGA) = \mathfrak{M}(NG)$.

Necessity of (3.8b) and its sufficiency in conjunction with (3.8a) therefore follows from Theorem 3.1.

THEOREM 3.4. *If* Rank$(A^* MA)$ = Rank(A), *then conditions* (i) *or* (ii) *is necessary and sufficient for G to be A_{MN}.*

$$\text{(i)} \quad AG \in \{P_A\}, \quad GA \in \{P_G\}. \tag{3.9a}$$

$$\text{(ii)} \quad MAGA = MA, (AG)^* M = MAG, NGAG = NG, \text{ and } (GA)^* N = NGA. \tag{3.9b}$$

Proof. The "necessity" part was established in Theorem 3.2. The proof of the "sufficiency" part consists in showing that when the matrices A^*MA and A are of the same rank, x as determined in (3.6) is indeed a general solution to (3.2a), so that the conditions which are shown to be necessary in the proof of Theorem 3.2 are also seen to be sufficient.

Let G be a matrix satisfying (3.2b). Observe that

$$\operatorname{Rank}(A^*MA)=\operatorname{Rank}A \Rightarrow \mathfrak{M}(A^*MA)=\mathfrak{M}(A^*) \Rightarrow GA=JA^*MA$$

for some matrix J. Also

$$A^*MAG=A^*M \Rightarrow A^*MAGA=A^*MA \Rightarrow A^*MAJA^*MA=A^*MA$$

$$\Rightarrow J\in\{(A^*MA)^-\} \Rightarrow GA=(A^*MA)^-A^*MA$$

for a suitable choice of the generalized inverse $(A^*MA)^-$ This shows that here (3.6) and (3.3) determine an identical class of solutions and the proof of Theorem 3.4 is complete.

THEOREM 3.5. *The following statements are true*:

(i) $NG_1=NG_2$ *if* G_1 *and* G_2 *are two choices of* A_{MN}.
(ii) $\{A_{MN}\}=\{A_{MN_0}\}$ *if* $N_0=N+A^*MA$.
(iii) $G_0=N_0^-A^*MA(A^*MAN_0^-A^*MA)^-A^*M$ *is one choice of* A_{MN}.
(iv) $G=G_0+(I-N_0^-N_0)U$ *is a general solution to* A_{MN}.
(v) A_{MN} *is unique if and only if* $N+A^*MA$ *is positive definite*.
(vi) *If* $G\in\{A_{MN}\}$, *then* $\mathfrak{M}[N\{I-(A^*MA)^-A^*MA\}]=\mathfrak{M}\{N(I-GA)\}$.

Proof. By Theorem 3.1, $NG_i=A^*MAK_i$ for some K_i $(i=1,2)$. Hence $(G_1^*-G_2^*)N(G_1-G_2)=(G_1^*-G_2^*)A^*MA(K_1-K_2)=(MA-MA)(K_1-K_2)=0$. Since N is nonnegative definite, this implies (i).

To establish (ii) choose and fix a particular solution $x=u$ of Eq. (3.2a). Then a general solution to (3.2a) is given by $x=u+[I-(A^*MA)^-A^*MA]z$ where $z\in\mathbf{E}^n$ is arbitrary. Observe that for every x so determined

$$\|x\|_{N_0}^2=\|x\|_N^2+\|x\|_{A^*MA}^2=\|x\|_N^2+\|u\|_{A^*MA}^2.$$

Hence if $\hat{x}=Gy$ minimizes the N seminorm of x in this class the same choice would also minimize its N_0 seminorm and vice versa. This establishes (ii). In fact, the proof shows that the statement would remain true if $N_0=N+A^*\Lambda A$ where Λ is a nonnegative definite matrix such that $\mathfrak{M}(A^*\Lambda A)\subset\mathfrak{M}(A^*MA)$, Λ could be arbitrary otherwise.

To establish (iii), one has to verify that G_0 satisfies conditions (a) and (b) of Theorem 3.1. (a) is straightforward. For (b), check that $NN_0^{-}A^*MA = N(N+A^*MA)^{-}A^*MA = N \pm A^*MA$, the parallel sum of the nonnegative definite matrices N and A^*MA as defined by Anderson and Duffin (1). (b) follows from the following property of the parallel sum established by these authors (see also Section 10.1.6 in Rao and Mitra (7) in this connection):

$$\mathfrak{M}(N \pm A^*MA) = \mathfrak{M}(N) \cap \mathfrak{M}(A^*MA).$$

(iv) In view of (i) and (iii), conditions (a) and (b) of Theorem 3.1 could be replaced by the following equivalent condition

$$\begin{pmatrix} A^*MA \\ N \end{pmatrix} G = \begin{pmatrix} A^*M \\ NG_0 \end{pmatrix}.$$

A general solution to this equation is given by

$$G = G_0 + Z,$$

where Z is a general solution to the corresponding homogeneous equation

$$\begin{pmatrix} A^*MA \\ N \end{pmatrix} Z = 0 \Leftrightarrow (N + A^*MA)Z = N_0 Z = 0.$$

This establishes (iv).

(v) follows from (iv).

(vi) To prove (vi), observe that in view of (i)

$$N(I - GA) = N(I - G_0A),$$

and that $G_0A = (A^*MA)^{-}A^*MA$ for a particular choice $N_0^{-}A^*MA(A^*MAN_0^{-}A^*MA)^{-}$ of $(A^*MA)^{-}$. Note that $\mathfrak{M}[N\{I-(A^*MA)^{-}A^*MA\}]$ is independent of the choice of $(A^*MA)^{-}$. This concludes the proof of Theorem 3.5.

It is seen from Theorem 3.1 that A_{MN} is not necessarily a g-inverse of A, that is, the relation $AA_{MN}A = A$ may not be true for every member of $\{A_{MN}\}$. Theorem 3.6 gives the conditions under which the subclass $\{A^+_{MN}\}$ is not empty.

THEOREM 3.6. *$\{A^+{}_{MN}\}$ is not empty if and only if*

$$\mathfrak{M}(N) \cap \mathfrak{M}(A^*) \subset \mathfrak{M}(A^*M). \tag{3.10}$$

Proof. We consider the general solution A_{MN} as given in (iv), Theorem 3.5 with

$$G_0 = N_0^- A^* MA (A^* MAN_0^- A^* MA)^- A^* M,$$

as in (iii), Theorem 3.5. If the solution is a g-inverse, then

$$A[G_0 + (I - N_0^- N_0) U] A = A$$

$$\Leftrightarrow A(I - G_0 A) = A(I - N_0^- N_0) UA. \quad (3.11)$$

Observe that $\text{Rank}[A(I - G_0 A)] = \text{Rank}[A^* : A^* MA] - \text{Rank}(A^* MA)$ by Lemma 7.1.2 of Rao and Mitra [7]. Also,

$$\text{Rank}[A(I - N_0^- N_0)] = \text{Rank}[A^* : N_0] - \text{Rank}(N_0).$$

Since $\mathcal{M}(A^* MA) \subset \mathcal{M}(N_0)$, we have

$$\text{Rank}[A(I - G_0 A)] \geq \text{Rank}[A(I - N_0^- N_0)]. \quad (3.12)$$

This shows that eq. (3.11) in U is inconsistent unless strict equality holds in (3.12). The equivalence of this condition with (3.10) follows from the representation of the intersection of linear spaces given in Lemma 2 of Mitra and Rao [3]. Also since $(I - N_0^- N_0) = (I - G_0 A)K$ for some K, if (3.10) holds and the inequality in (3.12) can be replaced by equality, then

$$\mathcal{M}[A(I - N_0^- N_0)] = \mathcal{M}[A(I - G_0 A)],$$

in which case given A^- there exists U, such that

$$A(I - N_0^- N_0) U = A(I - G_0 A) A^- \quad (3.13)$$

Such a U clearly satisfies (3.11), thus establishing the consistency of (3.11) and the sufficiency of (3.10).

Under the condition (3.10), we have established the existence of $G_1 \in \{A_{MN}\}$ such that $AG_1 A = A$. Now we choose

$$G_2 = G_1 A G_1. \quad (3.14)$$

It is easy to see that $G_2 \in \{A_{MN}\}$ and $AG_2 A = A$ and $G_2 A G_2 = G_2$, so that $G_2 \in \{A_{MN}^+\}$ as defined.

COROLLARY 3.6.1. *If* (3.10) *holds, then the necessary conditions in Theorem* 3.2 *are also sufficient.*

Proof. If (3.10) holds, then by Theorem 3.6 there exists a $G_2 \in \{A^+\}$.

Using (vi), Theorem 3.5,

$$\mathfrak{M}\{N(I-(A^*MA)^-A^*MA)\}=\mathfrak{M}\{N(I-G_2A)\}. \tag{3.15}$$

For any G satisfying (3.5 a, b) or (3.5 c)

$$\begin{aligned} G^*N(I-G_2A) &= G^*NGA(I-G_2A)=0 \\ &\Rightarrow G^*N\{I-(A^*MA)^-A^*MA\}=0 \\ &\Rightarrow \mathfrak{M}(NG)\subset\mathfrak{M}(A^*MA). \end{aligned}$$

By Theorem 3.1, $G\in\{A_{MN}\}$.

In Rao and Mitra [6, 7] *it was wrongly stated that the conditions in Lemma* 3.2 *are necessary and sufficient for G to be* A_{MN}. *This is however true only with an additional condition as in Theorem* 3.3. *The authors are indebted to Dr. M. Sibuya and Dr. K. Tanabe for a remark which led to the detection of this error.*

The authors also wish to thank the referee for his comments which led to Theorem 2.6. *Theorem* 2.7 *appears in a form which was essentially communicated to the authors by the referee. His comments in general were helpful in improving the presentation of this paper.*

REFERENCES

1 W. N. Anderson, Jr. and R. J. Duffin, Series and parallel addition of matrices, *J. Math. Anal. Appl.* **26**, 576–594 (1969).
2 S. K. Mitra, Unified least squares approach to linear estimation in a general Gauss-Markoff model. *SIAM J. Appl. Math.* **25**, 671–680 (1973).
3 S. K. Mitra and C. R. Rao, Some results in estimation and tests of linear hypotheses under the Gauss-Markoff model, *Sankhyā*, Ser A **30**, 281–290 (1968).
4 C. R. Rao, Unified theory of linear estimation, *Sankhyā*, Ser A **338** 371–394 (1971).
5 C. R. Rao, Unified theory of least squares, *Communications in Statistics,l 1, 1–8 (1973).*
6 C. R. Rao and S. K. Mitra, Further contributions to the theory of generalized inverse of matrices and its applications, *Sankhyā*, Ser A **33**, 298–300 (1971a).
7 C. R. Rao and S. K. Mitra, *Generalized Inverse of Matrices and its Applications*, Wiley, New York (1971b).
8 M. Sibuya, Subclasses of generalized inverse of matrices, *Ann. Inst. Statist. Math.* **22**, 543–556 (1970).
9 G. Zyskind and F. B. Martin, On best linear estimation and a general Gauss-Markov theorem in linear models with arbitrary nonnegative covariance structure, *SIAM J. Appl. Math.* **17**, 1190–1202 (1969).

Received March 2, 1973; revised December 17, 1973