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A New Method for Constructing Factorisable Representations for Current Groups and Current Algebras

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Abstract. Let $C_c^\infty(R^n, G)$ denote the group of infinitely differentiable maps from n-dimensional Euclidean space into a simply connected and connected Lie group, which have compact support. This paper introduces a class of factorisable unitary representations of $C_c^\infty(R^n, G)$ with the property that the unitary operator U_f corresponding to a function f in $C_c^\infty(R^n, G)$ depends not only on f, but also on the derivatives of f up to a certain order. In particular these representations can not be extended to the group of all continuous functions from R^n to G with compact support.

§ 1. Introduction

Let G be a simply connected and connected Lie group and let $\mathscr{G}$ be its Lie algebra. Let $\exp:\mathscr{G}\to G$ denote the exponential map. We denote by $C_c^\infty(R, G)$ the class of all C^∞ maps from R into G with compact support. A map $\varphi: R\to G$ is said to have compact support if takes the value e, i.e., the identity element of G outside a compact set. Let $C_0^\infty(R, \mathscr{G})$ denote the class of all infinitely differentiable maps from R into the vector space $\mathscr{G}$ with compact support. For any $f\in C_0^\infty(R, \mathscr{G})$, we define $\operatorname{Exp} f\in C_c^\infty(R, G)$ by writing $(\operatorname{Exp} f)(x)=\exp f(x)$, for all $x\in R$. $C_c^\infty(R, G)$ is a group (under pointwise multiplication) and $C_0^\infty(R, \mathscr{G})$ is a Lie algebra (under pointwise addition, scalar multiplication and Lie brackets). These may respectively be called as current group and current algebra over R. We give $C_0^\infty(R, \mathscr{G})$ the usual Schwarz topology. A homomorphism $\varphi\to U_\varphi$ of the group $C_c^\infty(R, G)$ into the group of unitary operators on a Hilbert space H is said to be a *unitary representation* or simply a representation if $U_{\operatorname{Exp} f_n}$ converges weakly to $U_{\operatorname{Exp} f}$ whenever $f_n\to f$ as $n\to\infty$ in the topology of $C_0^\infty(R, \mathscr{G})$.

For any compact set $K\subset R$, let $C(K, G)\subset C_0^\infty(R, G)$ be the subgroup of all those maps with support contained in K. If K_1, K_2 are two disjoint compact subsets of R, $C(K_1\cup K_2, G)$ can be identified in a natural manner with the cartesian product $C(K_1, G)\times C(K_2, G)$. Indeed, for any $\varphi\in C(K_1\cup K_2, G)$, define

$$\begin{aligned}\varphi_i(x)&=\varphi(x) \quad \text{if} \quad x\in K_i\\ &=e \quad \text{if} \quad x\notin K_i, \quad i=1,2.\end{aligned}$$

Then $\varphi=\varphi_1\varphi_2$. The map $\varphi\to(\varphi_1,\varphi_2)$ gives the required identification. For any representation U of $C_c^\infty(R,G)$, we define the representation U^K of the subgroup $C(K,G)$ by

$$U_\varphi^K=U_\varphi,\ \varphi\in C(K,G).$$

We say that a representation U of $C_c^\infty(R,G)$ is *factorisable* if, for any two disjoint compact sets $K_1, K_2\subset R$, the representation $U^{K_1\cup K_2}$ is unitarily equivalent to the tensor product $U^{K_1}\otimes U^{K_2}$. This unitary equivalence will of course depend on K_1 and K_2. Examples of such factorisable representations based on the unitary representations of G and their first cohomologies were first constructed by Streater [6] and Araki [1]. Further development of these ideas may be found in the works of Parthasarathy and Schmidt [4, 3], Vershik, Gelfand and Graev [7], and Guichardet [2]. However, most of these examples have the degenerate property that they factorise completely. These representations extend to borel maps from R into G and the factorisability property extends to pairs of disjoint borel sets. This is mainly because the representations constructed in these papers do not involve the derivatives of smooth maps in a certain sense. One may compare this with the following situation in the classical theory of distributions. To evaluate the Dirac δ at a testing function φ one need not know the derivations of φ. However to evaluate the distributions $\delta', \delta'', \ldots$ one requires a knowledge of $\varphi', \varphi'', \ldots$. The main aim of this paper is to construct factorisable representations U which for their evaluation at $\mathrm{Exp} f, f\in C_0^\infty(R,\mathscr{G})$ requires a knowledge of $f, f', f'', \ldots$. A beginning in this direction was already made by Schmidt [5] in the case when G is the Heisenberg group, whose representations lead to canonical commutation relations.

§ 2. The Leibnitz Extension of a Lie Algebra

In order to outline the method of constructing factorisable representations we need to construct an extension of the Lie algebra $\mathscr{G}$. To this end consider the space $\mathscr{G}_n$ which is the $n+1$-fold Cartesian product of $\mathscr{G}$. Any element X of $\mathscr{G}_n$ can be written as

$$X=(X_0, X_1, \ldots, X_n),\ X_i\in\mathscr{G}\quad \text{for each } i.$$

Between two elements X and X' in $\mathscr{G}_n$ define the bracket operation by

$$[X,X']=X'',$$

where

$$\begin{aligned} X_0''&=[X_0,X_0'],\\ X_j''&=\sum_{r=0}^{j}\binom{j}{r}[X_r,X_{j-r}']. \end{aligned} \tag{2.1}$$

An easy computation shows that for $X, Y, Z\in\mathscr{G}_n$,

$$[[X,Y]Z]=T$$

where

$$T_r = \sum_{k_1+k_2+k_3=r} (r!/k_1!k_2!k_3!)[[X_{k_1}, Y_{k_2}], Z_{k_3}].$$

This shows that

$$[[X, Y], Z] + [[Y, Z], X] + [[Z, X], Y] = 0.$$

In other words $\mathscr{G}_n$ becomes a Lie algebra. We shall call $\mathscr{G}_n$ the n^{th} *Leibnitz extension* of the Lie algebra $\mathscr{G}$. The mapping $X \to (X, 0, 0, \ldots, 0)$ is an isomorphism of $\mathscr{G}$ into $\mathscr{G}_n$. All elements of the form $(0, X_1, X_2, \ldots, X_n)$, $X_i \in \mathscr{G}$, $i = 1, 2 \ldots n$ constitute a nilpotent Lie subalgebra $\mathscr{N}^{(n)}$ of $\mathscr{G}_n$. Further

$$[(X, 0, 0, \ldots, 0), (0, X_1, X_2, \ldots, X_n)]$$
$$= (0, [X, X_1], [X, X_2], \ldots, [X, X_n]).$$

Thus $\mathscr{G}$ acts as a Lie algebra of derivations of the nilpotent Lie algebra $\mathscr{N}^{(n)}$. In other words $\mathscr{G}_n$ is a semi-direct sum of $\mathscr{G}$ and $\mathscr{N}^{(n)}$.

Remark 2.1. Since any Lie algebra $\mathscr{G}$ can be represented as a Lie algebra of matrices, we shall assume that $\mathscr{G}$ is a Lie algebra of real matrices in all our computations hereafter. Let the order of the matrices in $\mathscr{G}$ be $k \times k$.

Lemma 2.2. *The map*

$$A : (0, X_1, X_2, \ldots, X_n) \to A(X_1, X_2, \ldots, X_n),\ X_i \in \mathscr{G},\ i = 1, 2 \ldots n$$

where

$$A(X_1, X_2, \ldots, X_n) = \begin{pmatrix} 0 & X_1/1! & X_2/2! & \ldots & & X_n/n! \\ 0 & 0 & X_1/1! & X_2/2! & \ldots & X_{n-1}/n-1! \\ 0 & 0 & 0 & X_1/1! & \ldots & X_{n-2}/n-2! \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \ldots & \ldots & 0 \end{pmatrix}$$

is an isomorphism of the Lie algebra $\mathscr{N}^{(n)}$ into the Lie algebra of all matrices of order $k(n+1) \times k(n+1)$.

Proof. This follows from a routine computation and is left to the reader.

Lemma 2.3. *Let A be the map defined in the preceding lemma. Then the matrix* $\exp A(X_1, X_2, \ldots, X_n)$ *is of the form*

$$\begin{pmatrix} I & A_1 & A_2 & \ldots & \ldots & A_n \\ 0 & I & A_1 & A_2 & \ldots & A_{n-1} \\ 0 & 0 & I & A_1 & \ldots & A_{n-2} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & 0 & \ldots & I \end{pmatrix}$$

where

$$A_j = \sum_{p=1}^{j} 1/p! \sum_{\substack{m_1+\ldots+m_p=j \\ 1\leq m_i\leq j}} m_1!^{-1} X_{m_1} m_2!^{-1} X_{m_2} \ldots m_p!^{-1} X_{m_p} .$$

Proof. It is left to the reader.

Remark 2.4. Let H be the group generated (algebraically) by all matrices of the form $\exp A(X_1, X_2, \ldots, X_n)$, $X_i \in \mathscr{G}$, $i = 1, 2 \ldots n$. Its Lie algebra is isomorphic with $\mathscr{k}^{(n)}$. Let G be the simply connected group for which the Lie algebra is $\mathscr{G}$. Then for any $X_0 \in \mathscr{G}$, the element $\exp X_0$ of G acts as an automorphism of H as follows:

$$\exp X_0 : \exp A(X_1, X_2, \ldots, X_n)$$
$$\rightarrow \exp A(e^{\mathrm{ad}\, X_0}(X_1), e^{\mathrm{ad}\, X_0}(X_2), \ldots, e^{\mathrm{ad}\, X_0}(X_n)) .$$

Hence we can form the semi-direct product $G \odot H$ of the two groups G and H. $G \odot H$ consists of all pairs (g, h), $g \in G$, $h \in H$. The multiplication operation is defined by

$$(g, h) \cdot (g', h') = (gg', h \cdot g(h')) ,$$

where $h' \rightarrow g(h')$ is the automorphism of H induced by g. The Lie algebra of the group $G \odot H$ is then isomorphic to the Lie algebra $\mathscr{G}_n$. In particular $\mathscr{G}_1$ is the Lie algebra of the semidirect product of G and the additive group $\mathscr{G}$, where G acts as the adjoint representation in $\mathscr{G}$.

Lemma 2.4. *For any* $X = (X_0, X_1, \ldots, X_n) \in \mathscr{G}_n$, *the exponential map from* $\mathscr{G}_n$ *into* $G \odot H$ *is defined as follows: let*

$$A_j(t) = \sum_{p=1}^{j} \sum_{\substack{m_1+\ldots+m_p=j \\ 1\leq m_i\leq j}} \int_{0<t_1<t_2<\ldots<t_p<t} \left(\prod_{k=1}^{p} e^{t_k \mathrm{ad}\, X_0}(m_k!^{-1} X_{m_k}) \right) dt_1 \ldots dt_p \quad (2.2)$$

for $j = 1, 2 \ldots n$. *Let*

$$A(t) = \begin{pmatrix} I & A_1(t) & A_2(t) & \ldots A_n(t) \\ 0 & I & A_1(t) & \ldots A_{n-1}(t) \\ 0 & 0 & I & \ldots A_{n-2}(t) \\ \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & \ldots\ldots 0 & I \end{pmatrix}$$

Then

$$\exp tX = (\exp tX_0, A(t)) \quad \textit{for all} \quad t \in R .$$

Proof. Indeed, differentiating (2.2) at $t = 0$, we get

$$dA_j/dt|_{t=0} = j!^{-1} X_j .$$

Thus

$$dA(t)/dt|_{t=0}=\begin{pmatrix} 0 & X_1/1! & \dots & X_n/n! & \\ 0 & 0 & X_1/1! & \dots & X_{n-1}/(n-1)! \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Further

$$\begin{aligned}&(\exp tX_0, A(t))\cdot(\exp sX_0, A(s))\\ &=(\exp(t+s)X_0, A(t)\cdot\exp tX_0(A(s)),\end{aligned}$$

where

$$\exp tX_0(A(s))=\begin{pmatrix} I & B_1 & B_2 & \dots & B_n \\ 0 & I & B_1 & \dots & B_{n-1} \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & I \end{pmatrix}$$

and

$$\begin{aligned}B_j&=B_j(t,s)=e^{tX_0}A_j(s)e^{-tX_0}\\ &=\sum_{p=1}^{j}\sum_{\substack{m_1+\dots+m_p=j\\ m_i\geqq 1 \text{ for all } i}}\int_{0<t_1<t_2<\dots<t_p<s}\prod_{k=1}^{p}e^{(t_k+t)\operatorname{ad}X_0}(k!^{-1}X_{m_k})dt_1\dots dt_p\\ &=\sum_{p=1}^{j}\sum_{\substack{m_1+\dots+m_p=j\\ m_i\geqq 1 \text{ for all } i}}\int_{0<t_1<t_2<\dots<t_p<t+s}\prod_{k=1}^{p}e^{t_k\operatorname{ad}X_0}(k!^{-1}X_{m_k})dt_1\dots dt_p. \end{aligned}\tag{2.3}$$

A straightforward matrix multiplication shows that

$$A(t)\cdot\exp tX_0(A(s))=\begin{pmatrix} I & C_1 & C_2 & \dots & C_n \\ 0 & I & C_1 & \dots & C_{n-1} \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & I \end{pmatrix},$$

where

$$C_j=\sum_{r=0}^{j}A_r(t)B_{j-r}(t,s),$$

$$A_0(t)=B_0(t,s)=I,$$

and where A_r and B_r is defined by (2.2) and (2.3) respectively. Now an easy computation gives $C_j=A_j(t+s)$. This shows that $(\exp tX_0, A(t))$ is a one parameter group with the generator $(X_0, X_1, X_2, \dots, X_n)$. The proof is complete.

Corollary 2.5. *When $n=1$ and $G \odot H$ is identified with the semidirect product of G and the additive group $\mathcal{G}$, where G acts as adjoint representation in $\mathcal{G}$, we have*

$$\exp t(X_0, X_1) = \left(\exp tX_0, \frac{e^{t\,\mathrm{ad}\,X_0}-1}{t\,\mathrm{ad}\,X_0}(X_1)\right)$$

for all $t \in R$.

Proof. This follows from the preceding lemma by noting that

$$\int_0^t e^{t_1\,\mathrm{ad}\,X_0}(X_1)dt_1 = \frac{e^{t\,\mathrm{ad}\,X_0}-1}{t\,\mathrm{ad}\,X_0}(X_1).$$

§ 3. Representation of Current Algebras and Current Groups

In the preceding section we gave a complete description of the group associated with the n-th Leibnitz extension $\mathcal{G}_n$ of a Lie algebra $\mathcal{G}$. The following lemma yields the required embedding of $C_0^\infty(R, \mathcal{G})$ into $C_0^\infty(R, \mathcal{G}_n)$ for writing down our representations.

Lemma 3.1. *Let Π_n be the map from $C_0^\infty(R, \mathcal{G})$ into $C_0^\infty(R, \mathcal{G}_n)$ defined by*

$$\Pi_n(f)(x) = (f(x), f'(x), f''(x), \ldots, f^{(n)}(x))$$

for all $x \in R$, $f \in C_0^\infty(R, \mathcal{G})$.

Then Π_n is a Lie algebra isomorphism of $C_0^\infty(R, \mathcal{G})$ into $C_0^\infty(R, \mathcal{G}_n)$.

Proof. This follows immediately from the fact that

$$d^j[f, g]/dx^j = \sum_{r=0}^{j} \binom{j}{r} [f^{(r)}(x), g^{(j-r)}(x)]$$

and the commutation rules in $\mathcal{G}_n$ are defined by (2.1).

As mentioned in § 1, we define for any $f \in C_0^\infty(R, \mathcal{G})$, $\mathrm{Exp} f$ as the element in $C_e^\infty(R, G)$ with the property

$$(\mathrm{Exp} f)(x) = \exp f(x), \quad x \in R.$$

Consider the group $G \odot H$ described in Remark 2.4. We shall call it the n-th Leibnitz extension of the group G. For any $f \in C_0^\infty(R, \mathcal{G})$, we define $\mathrm{Exp}_n f$ as the element in $C_e^\infty(R, G \odot H)$ with the property

$$(\mathrm{Exp}_n f)(x) = (\exp f(x), A^f(x)),$$

where

$$A^f(x) = \begin{pmatrix} I & A_1^f(x) & A_2^f(x) & \ldots & A_n^f(x) \\ 0 & I & A_1^f(x) & \ldots & A_{n-1}^f(x) \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \ldots & I \end{pmatrix}$$

$$A_j^f(x) = \sum_{p=1}^{j} \sum_{\substack{m_1+\ldots+m_p=j \\ m_i \geq 1 \text{ for all } i}} \int_{0<t_1<t_2<\ldots<t_p<1} \prod_{k=1}^{p} e^{t_k \operatorname{ad} f(x)}$$
$$\cdot m_k!^{-1} f^{(m_k)}(x) dt_1 dt_2 \ldots dt_p \,. \tag{3.1}$$

for $j = 1, 2 \ldots n$. With this notation we have the following corollary to Lemma 3.1.

Theorem 3.2. *Let G be a connected and simply connected Lie group whose n-th Leibnitz extension is G_n. Suppose $\varphi \to U_\varphi$ is a factorisable representation of the current group $C_c^\infty(R, G_n)$. Then the map*

$$U^{(n)}: \operatorname{Exp} f \to U_{\operatorname{Exp}_n f}, f \in C_0^\infty(R, \mathscr{G})$$

determines a factorisable representation of the current group $C_c^\infty(R, G)$. In particular this determines a factorisable representation of the current algebra $C_0^\infty(R, \mathscr{G})$.

Remark 3.3. To construct a factorisable representation U of the current group $C_c^\infty(R, G_n)$ one may start with a projection valued measure on the Borel subsets of R, a unitary represention V of the group G_n commuting with the projection valued measure and a first order cocycle for the representation V, and adopt the procedure outlined in [4]. Since G is a subgroup of G_n it follows that $C_c^\infty(R, G)$ is a subgroup of $C_c^\infty(R, G_n)$. Hence the restriction of U to $C_c^\infty(R, G)$ yields a representation $U^{(0)}$ of $C_c^\infty(R, G)$. The representation $U^{(n)}$ of Theorem 3.1 obtained from U may be considered as the n-th derivative of the representation $U^{(0)}$

Example 3.4. We shall now illustrate the procedure outlined in the preceding remark in a special case. Let G be a compact, connected, simply connected and semi-simple Lie group with Lie algebra $\mathscr{G}$ and Cartan Killing form $B(X, Y)$, $X, Y \in \mathscr{G}$. Let $g \to \operatorname{Ad} g$ be the adjoint representation of G acting in $\mathscr{G}$. Let G_1 denote the first Leibnitz extension of G. Then G_1 is the semi direct product of G and the additive group $\mathscr{G}$ in which G acts as a group of automorphisms through the adjoint representation. Any element of G_1 can be expressed as a pair (g, X), $g \in G$, $X \in \mathscr{G}$. Then $(g, X) \to \operatorname{Ad} g$ is an irreducible unitary representation U of G_1 acting in the vector space $\mathscr{G}$ with the positive definite inner product $-B$. Define the map $\delta: G_1 \to \mathscr{G}$ by

$$\delta(g, X) = X \,.$$

Then δ is a first order cocycle for the representation U. Hence the function

$$\Phi(g, X) = \exp \tfrac{1}{2} B(X, X)$$

is an infinitely divisible positive definite function on the group G_1.

Let now $\varphi: R \to \mathscr{G}$ be a C_0^∞ map from R into $\mathscr{G}$. Then the map $t \to (\varphi(t), \varphi'(t))$ is a C_0^∞ map from R into $\mathscr{G}_1$ the Lie algebra of G_1. Let

$$\psi(t) = \frac{e^{\operatorname{ad} \varphi(t)} - 1}{\operatorname{ad} \varphi(t)} (\varphi'(t)) \,,$$

and let

$$K(\operatorname{Exp} \varphi) = \exp \tfrac{1}{2} \int B(\psi(t), \psi(t)) dt \,. \tag{3.2}$$

Then K is an infinitely divisible positive definite functional on $C_c^\infty(R, G)$ which extends to $C_c^1(R, G)$, the group of all C^1 maps from R into G with compact support. This positive definite functional defines a factorisable representation of $C_c^1(R, G)$ which cannot be extended to all bounded borel maps from R into G with compact support.

Since the factorisable representation corresponding to (3.2) is in a sense a continuous tensor product of copies of the irreducible adjoint representation of G one is tempted to conjecture that (3.2) yields an irreducible factorisable representation of $C_c^1(R, G)$.

Remark 3.5. The theory outlined above extends in a natural manner when R is replaced by R^m and one considers current groups $C_c^\infty(R^m, G)$. To describe this extension we adopt the following conventions. Let, for any positive integer N, F_N denote the set of all ordered m-tuples $\underline{j}=(j_1, j_2, \ldots, j_m)$ of non-negative integers such that $j_1+j_2+\ldots+j_m<N$. For any $\underline{j}\in F_N$, let $\underline{j}!=j_1!j_2!\ldots j_m!$, where $0!=1$. A general point of R^m will be denoted by $x=(x_1, x_2, \ldots, x_m)$. Let $|\underline{j}|=j_1+j_2+\ldots+j_m$. For any C^∞ map f from R^m into the Lie algebra $\mathscr{G}$, let

$$f^{(\underline{j})}=\partial^{|\underline{j}|}f/\partial x_1^{j_1}\partial x_2^{j_2}\ldots\partial x_m^{j_m}.$$

We now define the N-th Leibnitz extension $\mathscr{G}_N$ of $\mathscr{G}$ as the set of all maps X from F_N into $\mathscr{G}$ with Lie bracket $[X, Y]$ defined by

$$[X, Y](\underline{j})=\sum(\underline{j}!/\underline{r}!(\underline{j}-\underline{r})!)[X(\underline{r}), Y(\underline{j}-\underline{r})]$$

where the summation is over all $\underline{0}\leqq\underline{r}\leqq\underline{j}$. Here $\underline{r}\leqq\underline{j}$ means that $r_i\leqq j_i$ for all $i=1, 2, \ldots, m$. Then $\mathscr{G}_N$ is a Lie algebra. As before $\mathscr{G}$ may be embedded in $\mathscr{G}_N$ by mapping any $X\in\mathscr{G}$ to the element X with $X(\underline{0})=X$, $X(\underline{i})=0$ for $\underline{i}\neq\underline{0}$. Let us say that $\underline{i}<\underline{j}$ if $\underline{i}\neq\underline{j}$ but $\underline{i}\leqq\underline{j}$. As before all elements X such that $X(\underline{0})=0$ constitute a nilpotent Lie subalgebra $\mathscr{k}^{(N)}$ of $\mathscr{G}_N$. $\mathscr{G}_N$ is a semidirect sum of $\mathscr{G}$ and $\mathscr{k}^{(N)}$. For $X\in\mathscr{k}^{(N)}$, we define the matrix $A(X)$ whose $(\underline{i}, \underline{j})^{\text{th}}$ element is $X(\underline{i}+\underline{j})$ if $\underline{j}>\underline{i}$ and 0 otherwise. The order of the matrix is $ck\times ck$ where c is the cardinality of F_N and k is the order of the matrices which constitute the Lie algebra $\mathscr{G}$. Lemma 2.3 now holds with the convention

$$A_{\underline{j}}=\sum_{p=1}^{|\underline{j}|}p!^{-1}\sum_{\underline{m}_1+\ldots+\underline{m}_p=\underline{j}}\underline{m}_1!^{-1}X(\underline{m}_1)\ldots\underline{m}_p!^{-1}X(\underline{m}_p)$$

Lemma 2.4 holds with the condition

$$A_{\underline{j}}(t)=\sum_{p=1}^{|\underline{j}|}\sum_{\underline{m}_1+\ldots+\underline{m}_p=\underline{j}}\int_{0<t_1<t_2\ldots<t_p<t}\prod_{l=1}^{p}e^{t_l\,\mathrm{ad}\,X(0)}\cdot(\underline{m}_l!^{-1}X(\underline{m}_l))\,dt_1\ldots dt_p.$$

Then Theorem 3.2 holds with the condition that in defining the map $f \to \mathrm{Exp}_n f$ we change (3.1) to

$$A_{\underline{l}}^{f} = \sum_{p=1}^{|\underline{l}|} \sum_{\underline{m}_1 + \ldots + \underline{m}_p = \underline{l}} \int_{0 < t_1 < t_2 \ldots < t_p < 1} \prod_{i=1}^{p} e^{t_i \operatorname{ad} f(x)} (\underline{m}_i!^{-1} f^{(\underline{m}_i)}(x)) dt_1 \ldots dt_p .$$

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