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FUZZY PREFERENCES AND SOCIAL CHOICE*

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The traditional Arrowian framework assumes that the aggregation rule maps n-tuples of *exact* individual preferences into an exact social preferences relation. In this paper, both individual and social preferences are permitted to be *fuzzy*. Under a relatively strong transitivity condition, the fuzzy counterparts of Arrow's conditions result in oligarchic and not dictatorial aggregation rules. The disctatorship result is restored if these conditions are supplemented by Positive Responsiveness. Under a weaker transitivity condition, these impossibilities disappear. However, if the social preference relation is constrained to generate exact social choice, new impossibilities may emerge.

Key words: fuzzy preference, Arrowian framework, aggregation rules, exact social choice.

1. Introduction

The traditional literature on the theory of social choice explores the problems of aggregating individual preference relations so as to arrive at a social preference relation, the latter being interpreted as embodying judgements about social welfare. In this framework, it is assumed that individual and social preferences are *exact*, i.e., individuals as well as the society 'know for sure' whether an alternative x is at least as good as y or vice versa. However, it is not unusual for our values and preferences to be characterised by different degrees of imprecision. The answer to 'is x better than y?' is often 'may be' rather than 'yes' or 'no'. This imprecision or vagueness in individual preferences or value judgements is more appropriately modelled through fuzzy rather than exact binary relations[1].

In an important paper, Barrett et al. (1985) (henceforth BPS), analyse the properties of rules which aggregate fuzzy individual preferences into a fuzzy social

[1] An alternative method for relaxing the traditional framework is by permitting the social ordering to be stochastic. This has been explored by Barbera and Sonnenschein (1978), Heiner and Pattanaik (1983) and McLennan (1980) amongst others.

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preference relation. Assuming that the domain of the aggregation rules consists of n-tuples of fuzzy *strict* preference relations, BPS essentially proves results which are the counterparts of the well known conclusions of Arrow (1963) and Gibbard (1969) in the more traditional, exact framework.

In this paper, I present an alternative framework in which the aggregation rule maps n-tuples of fuzzy *weak* orderings into fuzzy *weak* orderings. Thus, in contrast to BPS, the (fuzzy) weak preference relation 'at least as good as' is taken to be the primitive concept. In addition, the derived strict preference relation is assumed to satisfy a strong *antisymmetry* property. These seemingly mild modifications lead to qualitatively different results. The main reason for the different conclusions is that the structure of fuzzy binary relations is different from that of exact binary relations. For instance, let R stand for an *exact* weak preference relation, with P and I the corresponding strict and indifference relations. It is known that (see Sen, 1970) transitivity of R implies (i) PP-transitivity, i.e., transitivity of P, (ii) PI-transitivity, i.e., for all $x, x, z: xPy$ and yIz implies xPz, (iii) IP-transitivity, i.e., for all $x, y, z: xIy$ and yPz implies xPz, (iv) II-transitivity, i.e., transitivity of I. Moreover, these implications play an important role in precipitating the Arrow-type impossibility results. There is no clearly unambiguous way of deriving the P and I relations from R, when the latter is a fuzzy binary relation. In section 2, I offer some arguments in support of the construction used in this paper. It is then shown that transitivity of R does not necessarily imply PP, PI or IP transitivity in the case of fuzzy binary relations.

Of course, if individual (and social) preferences are assumed to be strict, then the notions of II, PI and IP transitivity become vacuous, while PP transitivity is identical to transitivity itself. In order to render these notions nonvacuous, it is necessary to have a framework in which individual (and social) preferences are permitted to be weak. The arguments in the preceding paragraph would then provide some hope of being able to avoid the Arrow-type impossibility results. And surely one of the main motivations for relaxing the traditional framework is to see whether the impossibility results can be converted into possibilities.

The concept of transitivity itself can be defined in several alternative ways in the context of fuzzy binary relations. Using a relatively strong version of transitivity, I show that aggregation rules which satisfy fuzzy counterparts of Arrow's Independence and Pareto conditions and whose range consists of transitive fuzzy binary relations are characterised by oligarchies, and not necessary by dictatorships In order to get the Arrow-type dictatorships, it is necessary to impose a further condition – the fuzzy counterpart of Positive Responsiveness. However, if we are content with a weaker notion of transitivity, then we can even avoid oligarchies.

These results seem to indicate that the greater flexibility permitted within the fuzzy framework offers clear advantages. However, a note of caution is in order. The *fuzzy social preferences* relation has to generate *exact social choice*. I show that the necessity to generate *rationalisable* social choice may under certain circumstances lead to a dictatorship result. However, further work along these lines has to be done before any definite conclusion can be reached in this context.

The plan of the paper is as follows. In section 2, I discuss certain structural features of fuzzy binary relations. These are used in the proofs of the results on aggregation in section 3. I conclude in section 4.

2. Structure of fuzzy binary relations

Let X be a set of alternatives $(3 \leq X < \infty)$, and let $\underline{X} = 2^X - \{\emptyset\}$. The elements of $\underline{X}$ are *exact subsets* of X. A fuzzy *subset* of X is a function $A : X \to (0, 1)$. Clearly, an exact subset of X can be considered to be a function $A : X \to (0, 1)$ such that $A(X) \subseteq \{0, 1\}$. If A is an exact subset of X, then an element $x \in X$ either belongs to A, so that $A(x) = 1$, or does not belong to A, in which case $A(x) = 0$. Fuzzy subsets generalise this notion by allowing for different degrees of membership.

A fuzzy binary weak preference relation[2] (FWPR) on X is a function $R : X^2 \to (0, 1)$, while an *exact binary weak preference relation* (EWPR) on X is an FWPR R such that $R(X^2) \subseteq \{0, 1\}$.

Definition 2.1. An FWPR R is *reflexive* iff for all $x \in X$, $R(x, x) = 1$, and it is *connected* iff for all distinct $x, y \in X$, $R(x, y) + R(y, x) \geq 1$.

Note that when R is an EWPR, then these definitions correspond to the traditional definitions.

The most general notion of transitivity of an FWPR is defined by means of a binary operation $*$ on $(0, 1)$ as follows:

$$\text{for all } x, y, z \in X,\ R(x, z) \geq R(x, y) * R(y, z).$$

This is known as max-star transitivity. To obtain useful structural properties of max-star transitive relations, certain restrictions have to be imposed. Ovchinnikov (1984) suggests that the $*$ operator be a *triangular norm*, i.e., a function

$$T : (0, 1)^2 \to (0, 1)$$

satisfying

(A.1)	$T(x, 1) = x$	Boundary Condition
(A.2)	$T(x, y) \leq T(u, v)$ if $x \leq u, y \leq v$	Monotonicity
(A.3)	$T(x, y) = T(y, x)$	Commutativity
(A.4)	$(T(x, y), z) = T(x, T(y, z))$	Associativity.

In this paper, I will consider two specific $*$ operators satisfying (A.1)–(A.4).

[2] For a discussion of fuzzy binary relations, see Kaufman (1975) or Ovchinnikov (1981).

Definition 2.2. An FWPR satisfies:
(2.2.1) *Type* 1 *transitivity* (T_1) iff for all $x, y, z \in X$, $R(x,z) \geq \min(R(x,y), R(y,z))$.
(2.2.2) *Type* 2 *transitivity* (T_2) iff for all $x, y, z \in X$, $R(x,z) \geq R(x,y) + R(y,z) - 1$.

Proposition 2.3. *If an FWPR is T_1-transitive, then it is max-star transitive under any triangular norm.*

Proof. Suppose R is T_1-transitive. We need to show that for all $x, y, z \in X$, $R(x,z) \geq R(x,y) * R(y,z)$ for all $*$ operators satisfying (A.1)–(A.4).

It is sufficient to show that for all $\alpha, \beta \in (0,1)$, $\alpha * \beta \leq \min(\alpha, \beta)$. Choose $\alpha, \beta \in (0,1)$ such that $\alpha \geq \beta$. From the Boundary condition, $1 * \beta = \beta$. From the Monotonicity condition, $\alpha * \beta \leq 1 * \beta$. Hence, $\alpha * \beta \leq \min(\alpha, \beta)$.

Hence, T_1-transitivity implies max-star transitivity under any $*$ operator satisfying a triangular norm. Of course, these notions of transitivity coincide with the usual definition of transitivity when R is an EWPR. □

Let A and B be two fuzzy subsets. The *union* of A and B is the set $A \cup B$: for all $x \in X$, $A \cup B(x) = \max(A(x), B(x))$. The *intersection* of A and B is the set $A \cap B$: for all $x \in X$, $A \cap B(x) = \min(A(x), B(x))$. If $\min(A(x), B(x)) = 0$ for all $x \in X$, then $A \cap B = \emptyset$.

Given an FWPR R, there is no clearcut way of deriving the indifference and strict preference relations from R. One could try to proceed by way of analogy with the case of EWPRs. For instance, if R is an EWPR, and if $(x,y) \in R$, then $(x,y) \in R$, or $(x,y) \in I$; so that R is the union of P and I. Moreover, P and I are antisymmetric and symmetric relations respectively. Also, no pair (x,y) can belong to both P and I, so that $P \cap I = \emptyset$.

Unfortunately, the next proposition shows that there is no mileage to be gained from proceeding with this analogy.

Proposition 2.4. *Let R be a connected FWPR satisfying*

(i) $R = P \cup I$.
(ii) *I is symmetric, i.e., for all $x, y \in X$, $I(x,y) = I(y,x)$.*
(iii) *P is antisymmetric, i.e., for all $x, y \in X$, $P(x,y) > 0 \Rightarrow P(y,x) = 0$.*
(iv) $P \cap I = \emptyset$.

Then, either R is an EWPR, or for all $x, y \in X$, $R(x,y) = R(y,x) = I(x,y) = I(x,y)$.

Proof. From (i), for all $x, y \in X$

$$R(x,y) = \max(P(x,y),\ I(x,y))$$

$$R(y,x) = \max(P(y,x),\ I(y,x)).$$

Suppose $P(x,y) > I(x,y)$. Then, $R(x,y) = P(x,y)$. So, $I(x,y) = 0$ by (iv). Since I is symmetric, $I(y,x) = 0$. Similarly, $P(y,x) = 0$ from antisymmetry of P. Hence,

$R(y,x)=0$. But, by connectedness of R, $R(x,y)+R(y,x)\geq 1$. Hence, $R(x,y)=1$, $R(y,x)=0$, so that R is an EWPR.

Suppose $I(x,y)>P(x,y)$. By (iv), $P(x,y)=0$. Also, $I(y,x)=I(x,y)>0$. Hence, by (iv) again, $P(y,x)=0$. Hence, $R(x,y)=R(y,x)=I(x,y)=I(y,x)$. Finally, if $I(x,y)=P(x,y)$, then $R(x,y)=I(x,y)=P(x,y)=0$ by (i) and (iv). From connectedness of R, we must have $R(y,x)=P(y,x)=1$, so that R is an EWPR.

This completes the proof of the proposition. □

If one or more of the conditions in proposition 2.4 have to be given up, then it seems to me that the condition $P\cap I=\emptyset$ is a natural candidate. As I have argued earlier, it is not unnatural for strict preference and indifference to coexist over a pair of alternatives when preferences are fuzzy. The other conditions, together with an additional mild restriction, then uniquely specify the P and I relations given any FWPR.

Proposition 2.5. *Let R be a connected FWPR satisfying*

(i) $R=P\cup I$

(ii) *I is symmetric*

(iii) *P is antisymmetric*

(iv) *for all $x,y\in X$, $R(x,y)=R(y,x)\Rightarrow P(x,y)=P(y,x)$.*

$$\text{Then, for all } x,y\in X,\ P(x,y)=\begin{cases}R(x,y) & \text{if } R(x,y)>R(y,x)\\ 0, & \text{otherwise}\end{cases} \tag{1}$$

$$I(x,y)=\min(R(x,y),\ R(y,x)). \tag{2}$$

Proof. Suppose $R(x,y)>R(y,x)$, but $P(x,y)\neq R(x,y)$. From (i), $R(x,y)=\max(P(x,y),\ I(x,y))$. Hence, $P(x,y)\neq R(x,y)$ implies that $R(x,y)=I(x,y)$. By symmetry, $I(y,x)=I(x,y)$. Hence, $R(y,x)\geq I(y,x)=R(x,y)$. This contradicts $R(x,y)>R(y,x)$.

Hence, $P(x,y)=R(x,y)$ if $R(x,y)>R(y,x)$. Antisymmetry of P implies $P(y,x)=0$.

Also, $R(y,x)=I(y,x)$. By symmetry of I, $I(x,y)=I(y,x)$. Hence, (2) is also satisfied. Finally, note that if $R(x,y)=R(y,x)$, then (iii) and (iv) imply that $P(x,y)=0$ and $R(x,y)=I(x,y)=\min(R(x,y),\ R(y,x))$.

This completes the proof of the proposition. □

Remark 2.6. This derivation of P and I was suggested by Ovchinnikov (1981).

Of course, there have been other suggestions in the literature about the derivation of the P and I relations from an FWPR. One possibility is the following:

$$\begin{aligned}\text{for all } x,y\in X,\ P(x,y)&=\max(0,\ R(x,y)-R(y,x))\\ I(x,y)&=\min(R(x,y),\ R(y,x)).\end{aligned} \tag{3}$$

Another possibility is:

$$\text{for all } x, y \in X: P(x, y) = 1 - R(y, x)$$
$$I(x, y) = \min(R(x, y),\ R(y, x)). \tag{4}$$

As the reader can easily check, both (3) and (4) fail to satisfy the requirements of $R = P \cup I$ and $P \cap I = \emptyset$. In view of Propositions 2.4 and 2.5, *throughout this paper, we will stick to the derivation of P and I outlined by* (1) *and* (2) *in Proposition* 2.5.

The relationship between the various transitivity properties of FWPRs is considerably more complicated than that of EWPRs. Some of these are presented below.

Definition 2.7. Let R be an FWPR, with P and I its antisymmetric and symmetric components. R satisfies
(2.7.1) *PP-transitivity* iff for all $x, y, z \in X$, $P(x, z) \geq P(x, y) * P(y, z)$.
(2.7.2) *IP-transitivity* iff for all $x, y, z \in X$, $P(x, z) \geq I(x, y) * P(y, z)$
(2.7.3) *PI-transitivity* iff for all $x, y, z \in X$, $P(x, z) \geq P(x, y) * I(y, z)$
(2.7.4) *II-transitivity* iff for all $x, y, z \in X$, $I(x, z) \geq I(x, y) * I(y, z)$.

Remark 2.8. The * operator in Definition 2.7 is to be interpreted as a binary operator on (0, 1) as in the max-star transitivity of R.

Proposition 2.9. *Let R be a FWPR, with P and I its antisymmetric and symmetric components.*
(2.9.1) *Max-star transitivity of R does not imply PI or IP transitivity.*
(2.9.2) *T_1-transitivity of R implies PP-transitivity*[3].
(2.9.3) *T_2-transitivity of R does not imply PP-transitivity.*

Proof. (2.9.1) Ovchinnikov (1981) shows that PI as well as IP implies $P \cap I = \emptyset$. Since $P \cap I \neq \emptyset$ in our construction, the proposition follows.
(2.9.2) Suppose R satisfies T_1-transitivity, but not PP-transitivity. Then, for some $x, y, z \in X$, we must have:

$$R(x, z) \geq \min(R(x, y),\ R(y, z)) \tag{5}$$

and

$$P(x, z) < \min(P(x, y),\ P(y, z)). \tag{6}$$

Then,

$$0 < P(x, y) = R(x, y) > R(y, x) \tag{7}$$

$$0 < P(y, z) = R(y, z) > R(z, y). \tag{8}$$

(5) and (6) imply

[3] Ovchinnikov (1981) states this result without proof.

$$0 = P(x, z) \Rightarrow R(x, z) \leq R(z, x). \tag{9}$$

Case 1. Suppose

$$\min(R(x, y),\ R(y, z)) = R(x, y). \tag{10}$$

By (5),

$$R(x, z) \geq R(x, y) \tag{11}$$

T_1-transitivity of R implies

$$R(y, x) \geq \min(R(y, z),\ R(z, x)). \tag{12}$$

But, (10) and $R(y, x) \geq R(y, z) \Rightarrow R(y, x) \geq R(x, y)$ contradicting (7). Similarly, (9), (11) and $R(y, x) \geq R(z, x) \Rightarrow R(y, x) \geq R(x, y)$. Hence, $\min(R(y, z),\ R(z, x)) > R(y, x)$ contradicting (12), so that case 1 is not possible.

Case 2. Suppose

$$\min(R(x, y),\ R(y, z)) = R(y, z) \tag{13}$$

Hence, from (5),

$$R(x, z) \geq R(y, z). \tag{14}$$

By T_1-transitivity of R;

$$R(z, y) \geq R(z, x) \tag{15}$$

or

$$R(z, y) \geq R(x, y). \tag{16}$$

(15), (9) and (14) imply $R(z, y) \geq R(y, z)$.

Similarly, (16) and (13) imply $R(z, y) \geq R(y, z)$. In either case, (8) is violated.

(2.9.3) A counter example is sufficient.

Let $R(x, x) = R(y, y) = R(z, z) = 1$; $R(x, y) = R(y, z) = R(z, x) = \frac{2}{3}$; $R(x, z) = R(y, x) = R(z, y) = \frac{1}{3}$. Then, the corresponding P values are: $P(x, x) = P(y, y) = P(z, z) = 0$; $P(x, y) = P(y, z) = P(z, x) = \frac{2}{3}$; $P(x, z) = P(y, x) = P(z, y) = 0$.

The reader can check that R is T_2-transitive over (x, y, z). However, $P(x, y) + P(y, z) - 1 = \frac{1}{3} > P(x, z) = 0$. Hence, PP-transitivity is violated.

This proposition, in particular (2.9.1) and (2.9.3), will play an important role in the possibility theorems of the next section.

3. Possibility theorems for aggregation rules

In common with the traditional framework of social choice, let $N = \{1, \ldots, n\}$; $n \geq 2$, be a given finite set of individuals. X is the set of feasible alternatives. Let

H_1 (respectively H_2) denote the set of reflexive and connected FWPRs over X satisfying T_1-transitivity (respectively T_2-transitivity). H_E denotes the set of reflexive, connected, transitive EWPRs over X. Clearly, $H_E \subsetneq H_1 \subsetneq H_2$. Elements of H_1 or H_2 will be denoted by R_i, R, etc., R_i standing for the preference ranking of individual i.

Definition 3.1. *A fuzzy aggregation rule* (FAR) is a function $f: T^n \to T$ where $\emptyset \neq T$ is a set of fuzzy binary relations over X.

Remark 3.2. In what follows, T will be identified with either H_1 or H_2. Intuitively, an FAR specifies a fuzzy social preference relation given an n-tuple of fuzzy individual preference relations, one for each individual.

Notation 3.3. *The elements of T^n are indicated by $(R_1, \ldots, R)$, $(R'_1, \ldots, R'_n)$, etc. When f is an FAR, we will write $R = f(R_1, \ldots, R_n)$, $R' = f(R'_1, \ldots, R'_n)$ and so on.*

We now introduce certain properties of FARs.

Definition 3.4. Let $f: T^n \to T$ be an FAR. f satisfies (3.4.1) *Independence of Irrelevant Alternative* (IIA) if for all $(R_1, \ldots, R_n)$, $(R'_1, \ldots, R'_n) \in T^n$, and for all distinct $x, y \in X$, $(R_i(x, y) = R'_i(x, y)$ and $R_i(y, x) = R'_i(y, x)$ for all $i \in N)$ implies $(R(x, y) = R'(x, y)$ and $R(y, x) = R'(y, x))$.
(3.4.2) *Pareto Criterion* (PC) if for all $(R_1, \ldots, R_n) \in T^n$, all distinct $x, y \in X$, $P(x, y) \geq \min_{i \in N} P_i(x, y)$.
(3.4.3) *Positive Responsiveness* (PR) if for all $(R_1, \ldots, R_n)$, $(R'_1, \ldots, R'_n) \in T^n$, for all distinct $x, y \in X$, $(R_i = R'_i$ for all $i \neq j)$, $(R(x, y) = R(y, x))$ and $((P_j(x, y) = 0$ and $P'_j(x, y) > 0$, or $(P_j(y, x) > 0$ and $P'_j(y, x) = 0))$ implies $(P'(x, y) > 0)$.

These conditions are the fuzzy counterparts of well known conditions in the exact framework, and need no further elaboration.

Notation 3.5. *Non-empty subsets of N are called coalitions and are denoted by C, C', etc.*

Definition 3.6. Let f be an FAR.
(3.6.1) A coalition C is an *oligarchy* if for all distinct $x, y \in X$ and all $(R_1, \ldots, R_n) \in T^n$, (i) $(P_i(x, y) > 0$ for all $i \in C) \Rightarrow (P(x, y) > 0)$, and (ii) $(P_j(x, y) > 0$ for some $j \in C) \Rightarrow (P(y, x) = 0)$.
(3.6.2) An individual $j \in N$ is a *dictator* if for all distinct $x, y \in X$ and all $(R_1, \ldots, R_n) \in T^n$, $(P_j(x, y) > 0) \Rightarrow (P(x, y) > 0)$.
(3.6.3) An individual $j \in N$ is a *vetoer* if for all distinct $x, y \in X$ and all $(R_1, \ldots, R_n) \in T^n$, $(P_j(x, y) > 0) \Rightarrow (P(y, x) = 0)$.

The fuzzy counterpart of Gibbard's oligarchy result follows.

Proposition 3.7. *Let $f: H_1^n \to H_1$ be an FAR satisfying IIA and PC. Then, there exists a unique oligarchy C.*

Proof. The proof is exactly analogous to the proof of the corresponding result in BPS (Theorem 3.5), and is also similar to the proof of Gibbard's theorem in the exact framework. (For a sketch of the proof of the latter, see for instance Sen, 1985.)

Remark 3.8. Proposition 3.7 assumes that individual and social preferences are T_1-transitive. As shown in BPS, the proposition is true under any transitivity condition on FWPRs which ensures that the corresponding strict preference relations satisfy:

$$\text{for all } x, y, z \in X,\ P(x, y) > 0$$

and

$$P(y, z) > 0 \Rightarrow P(x, z) > 0. \tag{17}$$

Note that from Proposition 2.9.2, it follows that T_1-transitivity ensures that (17) is satisfied.

Proposition 3.9. *There exists a nondictatorial FAR $f: H_1^n \to H_1$ satisfying IIA and PC.*

Proof. Consider the following FAR $\bar{f}$.

For all $x \in X$, for all $(R_1, \ldots, R_n) \in H_1^n$, $R(x, x) = 1$.

For all distinct $x, y \in X$, for all $(R_1, \ldots, R_n) \in H_1^n$,

$$\bar{R}(x, y) = \begin{cases} 1 & \text{if } \forall i \in N: R_i(x, y) > R_i(y, x) \\ \beta & \text{otherwise,} \end{cases}$$

where $\beta \in (\frac{1}{2}, 1)$. $\bar{R}$ is connected since $\beta \geq \frac{1}{2}$. $\bar{f}$ obviously satisfies IIA. $\bar{f}$ is also symmetric across individuals (anonymous) and hence nondictatorial. To check that $\bar{f}$ satisfies PC, note that if $\min_{i \in N} P_i(x, y) > 0$, then $\bar{P}(x, y) = 1$ for all $x, y \in X$. So, we only need to show that $\bar{R}$ is T_1-transitive.

Consider any $x, y, z \in X$. We need to show that $\bar{R}(x, z) \geq \min(\bar{R}(x, y), \bar{R}(y, z))$. Now, $\bar{R}(x, z) = 1$ or $\bar{R}(x, z) = \beta$. Similarly, $\min(\bar{R}(x, y), \bar{R}(y, z)) = \beta$ or 1. if $\min(\bar{R}(x, y), \bar{R}(y, z)) = \beta$, then the required inequality is satisfied.

Suppose $\min(\bar{R}(x, y), \bar{R}(y, z)) = 1$, i.e., $\bar{R}(x, y) = \bar{R}(y, z) = 1$. From the definition of $\bar{f}$, this means that $\forall i \in N: P_i(x, y) > 0$ and $P_i(y, z) > 0$. By proposition 2.9.2, T_1-transitivity of R_i implies PP-transitivity. Hence, $P_i(x, z) > 0$ for all $i \in N$. This means that $\bar{R}(x, z) = 1$. Hence, $\bar{R}(x, z) \geq \min(\bar{R}(x, y), \bar{R}(y, z))$.

Remark 3.10. In view of Proposition 2.3, the above proposition is true when the *range* of $\bar{f}$ is expanded to include FWPRs which are max-star transitive under any triangular norm. Note, however, that a corresponding expansion of the *domain* of

$\bar{f}$ may not be permissible. This is because the proof requires PP-transitivity of R_i, and this is not necessarily valid under every notion of max-star transitivity - see, for example, Proposition 2.9.3.

Remark 3.11. If preferences are assumed to be exact, then the conditions imposed here lead to Arrow's (1963) result. The fuzzy framework allows for the construction of nondictatorial FARs. Of course, $\bar{f}$ is oligarchic, N being the unique oligarchy.

It is interesting to analyse why one is able to avoid the Arrow Impossibility Theorem in the fuzzy framework. As a general remark, it is pertinent to note that Arrow's result uses the fact that transitivity of R implies PI and IP transitivity. However, as Proposition 2.9.1 shows, PI and IP transitivity are *independent* properties in the fuzzy framework, and do not follow as corollaries of transitivity of the FWPR.

I come now to the particular construction of $\bar{f}$. In an intuitive sense, strict preference under $\bar{f}$ coincides with the Pareto dominance relation - this becomes clear if one assumes that individual preferences are *exact*, while allowing social preferences to be fuzzy. The problem with the Pareto dominance relation is that it is not *connected*. Suppose that for some $x, y \in X$, and for some $x, y \in X$, and for some n-tuple of individual preferences, neither x nor y Pareto dominates the other. Then, $\bar{R}(x, y) = \bar{R}(y, x) = \beta \geq \frac{1}{2}$, and R is connected. The 'exact counterpart' of $\bar{R}$, would not be connected since $R(x, y) = R(y, x) = 0$. Again, in an intuitive sense, the greater flexibility permitted by allowing for intermediate values between 0 and 1 in the fuzzy framework explains the difference in the nature of the results in the two alternative frameworks.

However, there is no nondictatorial FAR satisfying Positive Responsiveness in addition to the conditions imposed in Proposition 3.9.

Proposition 3.12. *Let $n \geq 3$, and $f: H_1^N \to H_1$ be an FAR satisfying IIA, PC and PR. Then f is dictatorial.*

Proof. By Proposition 3.7, there is a unique oligarchy C. If C consists of a single individual, then that individual is a dictator. So, let $i, j \in C$.

Let x, y be distinct alternatives in X. Consider $(R_i, \ldots, R_n) \in H_1^n$ such that: $R_i(x, y) > R_i(y, x)$ *and* $R_j(y, x) > R_j(x, y)$. Since i and j are *vetoers*, $P(x, y) = P(y, x) = 0$. Hence, $R(x, y) = R(y, x)$. Let $k \in N - \{i, j\}$. Such k exists since $n \geq 3$. f cannot satisfy PR because $R(x, y) = R(y, x)$ *irrespective* of the preferences of individual k. Hence, C contains only a single individual, who is a dictator. □

Remark 3.13. This proposition is the fuzzy counterpart of a result in Mas-Collel and Sonnenschein (1972).

The results proved so far have assumed that individual and social preferences are T_1-transitive. With T_2-transitivity, all the impossibilities disappear.

Proposition 3.14. *There exists an $f: H_2^n \to H_2$ satisfying IIA, PC, PR.*

Proof. Construct the following $\hat{f}$.
For all $x, y \in X$, for all $(R_1, \ldots, R_n) \in H_2^n$,

$$\hat{R}(x, y) = \frac{1}{n} \sum_{i \in N} R_i(x, y).$$

That $\hat{f}$ satisfies IIA is obvious. I first show that $\hat{f}$ satisfies PC. Consider any $(R_1, \ldots, R_n)$ and $x, y \in X$. If

$$\min_{i \in N} P_i(x, y) = P_j(x, y) = 0,$$

then $\hat{P}(x, y) \geq P_j(x, y)$. Suppose $P_j(x, y) > 0$. Then, for all $i \in N$, $R_i(x, y) > R_i(y, x)$. Hence,

$$\hat{R}(x, y) = \frac{1}{n} \sum_{i \in N} R_i(x, y) > \frac{1}{n} \sum_{i \in N} R_i(y, x) = \hat{R}(y, x).$$

Since $\hat{R}(x, y) > \hat{R}(y, x)$, $\hat{P}(x, y) = \hat{R}(x, y)$. Since $\hat{R}$ is a convex combination of

$$(R_1, \ldots, R_n), \quad \hat{R}(x, y) \geq \min_{i \in N} R_i(x, y) = R_j(x, y).$$

Hence $\hat{f}$ satisfies PC.
To check that $\hat{R}$ is connected, note that

$$\hat{R}(x, y) + \hat{R}(y, x) = \frac{1}{n} \sum_{i \in N} R_i(x, y) + \frac{1}{n} \sum_{i \in N} R_i(y, x)$$
$$= \frac{1}{n} \sum_{i \in N} (R_i(x, y) + R_i(y, x)) \geq 1$$

since each individual R_i is connected.
Finally, consider any $x, y, z \in X$. $\forall i \in N$, $R_i(x, z) \geq R_i(x, y) + R_i(y, z) - 1$. Summing over i and dividing each side by n, we get $\hat{R}(x, z) \geq \hat{R}(x, y) + \hat{R}(y, z) - 1$. Hence, $\hat{R}$ is T_2-transitive.

Remark 3.15. $\hat{f}$ is also symmetric across both individuals and alternatives.

Remark 3.16. The Gibbard oligarchy result is avoided in this case because T_2-transitivity does not ensure that (17) is satisfied. (See Remark 3.8.)

The previous results show that if we permit judgements about social welfare to be fuzzy, the force of the Arrow-type impossibility theorems is much weaker. However, it can be argued that even if social preferences are permitted to be fuzzy, the final social *choice* must perforce be exact. So, before one can pass any judgement about the benefits or advantages of fuzzy subset theory in social choice, one must also discuss the implications of these results for exact social choice.

This hinges on the relationship postulated between choice and the fuzzy social preferences. Dutta et al. (1986) discuss several alternative notions of *rationalisability* of exact choice by fuzzy preferences in the context of revealed preference theory[4]. I introduce below one of the more plausible notions discussed in their paper, and present a result using that notion. The purpose of this result is merely to point out that the optimism generated by the previous results in this section *may be* misplaced. But, a more detailed analysis has to be made before one can arrive at anything resembling a definite answer to the problem referred to above.

Definition 3.17. An exact choice function (ECF) is a function $C: \underline{X} \to \underline{X}$ such that for all $A \in \underline{X}$, $\emptyset \neq C(A) \subset A$.

Notation 3.18. Let R be any FWPR, and $A \in \underline{X}$. The *R-greatest set* in A is $G(A, R): \underline{X} \to (0, 1)$ such that for all $x \in X - A$, $G(A, R)(x) = 0$ and for all $x \in A$, $G(A, R)(x) = \min_{y \in A} R(x, y)$. Let $B(A, R) = \{x \in A / G(A, R)(x) \geq G(A, R)(y)$ for all $y \in A\}$.

Definition 3.19. An ECF C is *H-rationalisable in terms of an FWPR R* iff for all $A \in X$, $C(A) = B(A, R)$.

Under H-rationalisability, the agent chooses those elements in A which score the highest with the function $G(A, R)$[5].

Definition 3.20. Let $f: T^n \to T$ be an FAR, and C an ECF. f *H-generates* C iff for all $(R_1, \ldots, R_n) \in T^n$, C is H-rationalisable in terms of $f(R_1, \ldots, R_n)$.

The following condition, first proposed by Bordes (1976), is a well-known rationality condition in social choice.

Definition 3.21. An ECF C satisfies *Property* $(\beta+)$ iff for all $x, y \in X$, and all $A, B \in X$, $(x, y \in A \subseteq B$ and $y \in C(A)$ and $x \in C(B)) \to (y \in C(B))$.

The following lemma, proved in Dutta et al. (1986), shows the importance of Property $(\beta+)$ in the present context.

Lemma 3.22. *If C is an ECF which is H-rationalisable in terms of some* $R \in H_1$ *then C satisfies Property* $(\beta+)$.

[4] See also Basu (1984).

[5] The intuitive plausibility of H-rationalisability is discussed in Dutta et al. (1986).

Proposition 3.23. *Let $F: H_1^n \to H_1$ be a nondictatorial FAR satisfying IIA and PC, and C an ECF which is H-generated by f. Then, there is $A \in X$, and $(R_1, \ldots, R_n) \in H_1^n$ such that* (i) *$x, y \in A$ and $P_i(x, y) = 1$ for all $i \in N$, and* (ii) *$y \in C(A)$.*

Proof. Let $A = \{x, y, z\}$, and f and C satisfy the hypothesis of the proposition. From Proposition 3.7, there is a unique oligarchy C. Since f is nondictatorial, $|C| \geq 1$.

Without loss of generality, assume that $\{1, 2\} \subseteq C$. Construct $(R_1, R_2, \ldots, R_n)$ as follows:

(i) for all $i \in N$, $R_i(x, y) = 1$ and $R_i(y, x) < 1$.

(ii) $R_1(y, z) > R_1(z, y)$ and $R_1(x, z) > R_1(z, x)$.

(iii) $R_2(z, y) > R_2(y, z)$ and $R_2(z, x) > R_2(x, z)$.

(iv) for all $i \in N - \{1, 2\}$, $R_i(y, z) = R_i(z, y) = R_i(z, x) = R_i(x, z)$.

The reader can check that there exist $(R_1, \ldots, R_n)$ H^n satisfying restrictions (i)-(iv).

Since f satisfies PC, $P(x, y) = 1$, and hence $G(\{x, y\}, R)$ $(x) = 1$, and $G(\{x, y\}, R)$ $(y) = 0$. Since $1 \in C$, (ii) implies that $P(z, y) = P(z, x) = 0$. Similarly, (iii) implies that $P(y, z) = P(x, z) = 0$. Also, $P(z, y) = P(y, z) = 0$ implies $R(y, z) = R(z, y)$, and $P(z, x) = P(x, z) = 0$ implies that $R(z, x) = R(x, z)$. Hence, $G(\{y, z\}, R)(y) = G(\{y, z\}, R)(z)$ and $G(\{x, z\}, R)(x) = G(\{x, z\}, R)(z)$.

Since C is H-generated by R, $G(\{x, y\}, R)(x) > G(\{x, y\}, R)(y) \Rightarrow C(\{x, y\}) = \{x\}$. Also, $C(\{y, z\}, R)(y) = G(\{y, z\}, R)(z) \Rightarrow C(\{y, z\}) = \{y, z\}$ and $G(\{x, z\}, R)(x) = G(\{x, z\}, R)(z) \Rightarrow C(\{x, z\}) = \{x, z\}$.

Suppose $y \notin C(A)$. Since C satisfies Property ($\beta+$), and $y \in C(\{y, z\})$, this implies that $z \notin C(A)$. But, $z \in C(\{x, z\})$. Hence, if $z \notin C(A)$, then $x \notin C(A)$.

Hence, if $y \notin C(A)$, then $C(A) = \emptyset$. So, $y \in C(A)$ and the proposition is proved. □

Proposition 3.23 should be compared with Propositions 3.7 and 3.9. The latter results showed that the class of FARs (with domain H_1^n and range H_1) satisfying IIA and PC must be oligarchic but not necessarily dictatorial. But, if the nondictatorial FARs in this class are further constrained to H-generate exact choice functions, then in some situations, an alternative which is Pareto-dominated will also be chosen. Thus, the necessity to generate *exact* social choice may under certain circumstances lead to strong impossibility results even in a fuzzy framework.

There is another way of interpreting this result. The concept of H-rationalisability in terms of T_1-transitive FWPRs requires the exact choice function to satisfy Property ($\beta+$). Suppose the domain of the aggregation rule is restricted to H_E^n, i.e., individual preferences are assumed to be exact. If IIA and the Pareto condition are rephrased in choice-functional terms, and if the aggregation rule is defined to be a mapping from H_E^n to the set of choice functions, then it is known that property ($\beta+$) leads to the existence of a dictator. So, essentially what this result shows is that a dictator on the domain H_E^n remains a dictator on the expanded domain H_1^n.

Of course, a corresponding result for T_2-transitivity is not necessarily true. I do not know the nature of the restrictions imposed on exact choice functions by the concept of H-rationalisability in terms of T_2-transitive FWPRs. These restrictions could, however, be weaker than known rationality conditions such as Property ($\beta+$). Hence an impossibility result even when social choice is assumed to be *exact* is not inevitable in a fuzzy framework.

4. Conclusion

In contrast to BPS, I have permitted individual and social preferences to be fuzzy weak preference relations. I have argued that differences in the structure of fuzzy binary relations from that of exact binary relations become apparent only if the binary relations are allowed to be weak. And qualitative differences in the nature of the possibility theorems in the two frameworks do follow from these differences in the structure of binary relations.

In particular, with a relatively strong version of the transitivity condition on FWPRs, the counterparts of the conditions assumed by Arrow (1963) lead to an oligarchy in the fuzzy framework. The dictatorship result is restored if Positive Responsiveness is imposed in addition to the other conditions. However, with a weaker form of transitivity, all these conditions can be satisfied without leading to oligarchic rules.

Fuzzy social preferences cannot be an end in themselves – they must ultimately be used to define *exact* social choice. One possibility is to derive exact choice functions which are rationalisable in terms of the fuzzy social preference ordering. For the strong version of the transitivity requirement and using one possible approach to rationalisability, I have shown that the dictatorship result is essentially restored.

However, further work needs to be done on the relationship between rationalisable choice functions and fuzzy preferences. In particular, one open problem is to derive the restrictions imposed on choice functions by rationalisability in terms of general max-star transitive fuzzy binary relations. If these restrictions turn out to be mild, then it may be possible to define reasonably acceptable aggregation rules which map n-tuples of fuzzy individual preference orderings into exact social choices.

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