RESEARCH NOTES

A NOTE ON KOMAMIYA—JOSHI'S BOUND FOR MINIMUM DISTANCE CODES

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Summary

The superiority of Hamming's upper bound for the number of alphabets in a minimum distance code over that of Komamiya—Joshi's bound in all practically important cases is pointed out in this note.

1. Introduction

Let C_n denote the set of all sequences of length n in binary symbols 0 and 1. For any two elements.

$$\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n)$$
$$\text{and} \quad \beta = (\beta_1, \beta_2, \ldots, \beta_n)$$

the Hamming distance (1) $\partial(\alpha, \beta)$ between α and β is defined to be the number of positions in which α and β differ from each other. Defining the norm of α by

$$\| \alpha \| = \sum_{i=1}^{n} \alpha_i$$

it is easy to see that

$$\partial(\alpha, \beta) = \| \alpha + \beta \|$$

where $\alpha + \beta$ is the vector obtained by co-ordinatewise addition modulo 2 of the vectors α and β.

For any given positive integer $d < n$, a subset $M(n, d)$ of C_n such that no two elements of $M(n, d)$ of C_n are at a distance less than d is called a d-minimum distance code. Let $M(n, d)$ denote thenumber of elements in $M(n, d)$.

When d is odd, say $d = 2t + 1$, Hamming gave an upper bound for $M(n, d)$ which is now known as Hamming's shpere-packing bound and is given by the inequality

$$[M(n, d)] < \frac{2^n}{\binom{n}{0} + \binom{n}{1} + \cdots + \binom{n}{t}} \tag{1}$$

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It is known that such a code corrects any combination of t or fewer independent errors when n symbols are transmitted over a noisy channel.

Komamiya (2) gave another bound for $M(n, d)$ and the same was rediscovered by Joshi (3) by an idependent and similper method. They prove that $[M(n, d)] < 2^{n-d+1}$ (2)

We prove in this note that the inequality (1) is sharper than (2) in all practically important cases. More specifically, we prove that

$$\frac{2^n}{\binom{n}{0} + \binom{n}{1} + \dots + \binom{n}{t}} < 2^{n-d+1} \tag{3}$$

for all odd values of $d = 2t + 1$.

and

$$\frac{2^n}{\binom{n}{0} + \binom{n}{1} + \dots + \binom{n}{t-1}} < 2^{n-d+1} \tag{4}$$

for all even values of $d = 2t$, provided

$n \geqslant 2t + 3$ and $t > 1$.

To prove (3) observe that (3) can be written as

$$2^{2t} = 2^{d-1} < \binom{n}{0} + \binom{n}{1} + \dots + \binom{n}{t} \tag{5}$$

Evidently, for a given 't', if the above is true for a particular value of n, then it is true for all higher values of n. Since $n \geqslant d = 2t + 1$ in this case we need only prove that

$$\begin{aligned} 2^{2t} &< \binom{2t+1}{0} + \binom{2t+1}{1} + \dots + \binom{2t+1}{t} \\ &= \tfrac{1}{2}(1+1)^{2t+1} = 2^{2t} \end{aligned}$$

which establishes the inequality (3).

To prove (4), we write it as

$$2^{2t-1} = 2^{d-1} < \binom{n}{0} + \binom{n}{1} + \dots + \binom{n}{t-1} \tag{6}$$

As before if for a given 't' the above holds good for a particular value of n, it holds good for al subsequent values of n.

It can be verified directly that

$$\binom{2t+1}{t} < 2^{2t-1} \tag{7}$$

for $t = 4$.

Let (7) hold good for a particular value t_0 of t. Then

$$\binom{2(t_0+1)+1}{t_0+1} = \binom{2t_0+3}{t_0+1} = \binom{2t_0+1}{t_0} \cdot \frac{(2t_0+2)(2t_0+3)}{(t_0+1)(t_0+2)}$$

$$< 4 \cdot \binom{2t_0+1}{t_0} < 4 \cdot 2^{2t_0-1} = 2^{2(t_0+1)-1}$$

Hence by induction, (7) is true for all $t \geqslant 4$.

Putting $n = 2t + 1$ in (3) and combining with (7), we have

$$2^{2t-1} < \binom{2t+1}{0} + \binom{2t+1}{1} + \dots + \binom{2t+1}{t-1}$$

for $t \geqslant 4$ and $n = 2t + 1$, and hence for all $n \geqslant 2t + 1$.

It can be verified directly that (a) for $t = 3$, (6) holds good for all $n \geqslant 2t + 2$ (b) for $t = 2$, it holds good for all $n \geqslant 2t + 3$.

This completely establishes (4).

To complete our assertion we note that

$$[M(n, 2t)] \leqslant [M(n, 2t-1)] \leqslant \frac{2^n}{\binom{n}{0} + \binom{n}{1} + \dots + \binom{n}{t-1}}$$

by (1), where as

$$[M(n, 2t)] < 2^{n-2t+1}$$

by (2), and by what we have proved above, it follows that Hamming's bound is better than Komamiya—Joshi bound for all odd values of 'd', and for even values $d \neq 2$ it is better for all $n \geqslant d + 3$

We may remark that while n should necessarily be greater than $2t$, in practice it is much larger. Hence the restriction that $n \geqslant 2t + 3$ is not a practical limitation.

References

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