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Absolute Stability of a Stochastic Integro-Differential System

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A stochastic integro-differential system (*) $\dot{X}(t, \omega) = A(\omega) X(t, \omega) + \int_0^t B(t-s, \omega) X(s, \omega)\, ds + b(\omega)\, \Phi(\sigma(t, \omega))$ subject to $\sigma(t, \omega) = \langle c(\omega), X(t, \omega)\rangle$, $t \geqslant 0$ is considered where the integrals are interpreted as Bochner integrals. Existence and absolute stability of random solutions of (*) are studied.

1. INTRODUCTION

The purpose of this paper is to study the existence and absolute stability of a random solution of a stochastic integro-differential system of the form

$$\dot{X}(t, \omega) = A(\omega) X(t, \omega) + \int_0^t B(t-s, \omega) X(s, \omega)\, ds + b(\omega)\, \Phi(\sigma(t, \omega)),$$

$$\sigma(t, \omega) = \langle c(\omega), X(t, \omega)\rangle, \qquad t \in R_+, \tag{S}$$

where

(i) $\omega \in \Omega$, where Ω is the supporting set of the probability measure space$(\Omega, \mathscr{A}, \mu)$,

(ii) $X(t, \omega)$ is the unknown $n \times 1$ random vector for every $t \geqslant 0$,

(iii) $b(\omega)$ and $c(\omega)$ are $n \times 1$ constant random vectors,

(iv) $A(\omega)$ is a constant $n \times n$ random matrix,

(v) $B(t, \omega)$ is an $n \times n$ random matrix kernel, and

(vi) $\Phi(\sigma)$ is a real valued function defined on the real line R.

The integral in (S) is interpreted as a Bochner integral. Further assumptions concerning the functions in (S) will be stated in Section 2.

Absolute stability of deterministic integro-differential systems of type (S) has been studied recently by Corduneanu [1] using some results of Grossman and Miller [4]. Our approach to the problem will be similar to Corduneanu [1]. Stochastic integro-differential systems were studied by Prakasa Rao and Rama Mohana Rao [6] and Tsokos [7].

2. Preliminaries

Let R^n be the Euclidean n space with norm $\| x \| = \sup | x_i |$. If A is an $n \times n$ matrix, define $\| A \| = \sup \| Ax \|/\| x \|$ $(= \sup_i \sum_k | a_{ik} |)$. Let $L(p, n)$ be the Banach space formed from all $X: \Omega \to R^n$ with finite pth moments $(p \geqslant 1)$ and define $| \| x \| |_p = (E(\| x \|^p))^{1/p}$. Here E is the expectation operator. Let $J = [0, \infty)$. $X: J \times \Omega \to R^n$ is a stochastic process. It can also be thought of as a map $X: J \to L(p, n)$ if $| \| X(t, \omega)\| |_p < \infty$. In this case, we say that X is *L_p-differentiable* if the difference quotient

$$(X(t + h, \omega) - X(t, \omega))/h \tag{2.1}$$

converges in norm topology on $L(p, n)$ as $| h | \to 0$. If almost all the sample paths of X are differentiable, then we say that X is *SP-differentiable.* X is called *SP-integrable* if, for almost all its sample paths, $\int_0^\infty \| X(t, \omega)\| \, dt$ exists and is finite. If, considered as a map from J into $L(p, n)$, X is Bochner integrable, we say that X is *L_p-integrable.*

Let $C_c = C_c(J, L(p, n))$ be the space of all continuous functions from J into $L(p, n)$ with the topology of uniform convergence on compact sets. The space C_c is a locally convex space whose topology is defined by the following family of seminorms:

$$| x(t, \omega)|_n = \sup_{0 \leqslant t \leqslant n} | \| x(t, \omega)\| |_p , \qquad n \geqslant 1.$$

By a *random solution* of (S), we shall mean a function $X \in C_c(J, L(p, n))$ satisfying (S) μ a.e. Any random solution of (S) is said to be *absolutely stable* if $\| X(t, \omega)\| \to 0$ as $t \to \infty$ μ a.e.

3. Some Basic Results

We shall now obtain some elementary results which are necessary to obtain the main theorem on the existence and stability behavior of solutions of (S). Suppose the following conditions are satisfied.

$$\begin{aligned} &\text{(i)} \quad \operatorname*{ess\text{-}sup}_{\omega \in \Omega} \| A(\omega)\| = \beta < \infty; \\ &\text{(ii)} \quad \operatorname*{ess\text{-}sup}_{\omega \in \Omega} \| b(\omega)\| = d < \infty; \\ &\text{(iii)} \quad \operatorname*{ess\text{-}sup}_{\omega \in \Omega} \| c(\omega)\| = c < \infty; \end{aligned} \tag{A1}$$

(i) $\| B(t, \omega)\| \in L_1(R_+, L_\infty(\Omega, \mathscr{A}, \mu))$;

(ii) $\underset{\omega \in \Omega}{\text{ess-sup}} \int_0^t \| B(s, \omega)\| \, ds \leqslant m(t)$ where $m \in C[R_+, R]$;

and

(iii) $\sup_{0 \leqslant s \leqslant u} | \| B(s, \omega)\| |_\infty \leqslant l(u)$ where $l \in C[R_+, R]$. (A2)

Define formally

$$\psi(t, \omega) = A(\omega) + \int_0^t B(t - u, \omega) \, du \tag{3.1}$$

and

$$R(t, \omega) = I + \int_0^t R(t - u, \omega) \, \psi(u, \omega) \, du, \tag{3.2}$$

where I is the identity matrix. Under the assumptions (A1) and (A2), it was shown in [6] that

$$\| \psi(t, \omega)\| \in L_\infty(\Omega, \mathscr{A}, \mu), \quad \| R(t, \omega)\| \in L_\infty(\Omega, \mathscr{A}, \mu) \qquad \text{for } t \in R_+, \tag{3.3}$$

and the integrals in (3.1) and (3.2) exist as Bochner integrals.

In fact,

$$\| R(t, \omega)\| \leqslant 1 + \left\{\int_0^t (\beta + m(u)) \, du\right\} \exp \left\{\int_0^t (\beta + m(u)) \, du\right\} = n(t) \quad \text{(say)}. \tag{3.4}$$

Furthermore, $R(t, \omega)$ is L_p-differentiable and

$$\dot{R}(t, \omega) = A(\omega) R(t, \omega) + \int_0^t B(t - s, \omega) R(s, \omega) \, ds. \tag{3.5}$$

We shall now state a probabilistic version of sufficiency part of a theorem of Grossman and Miller [4].

THEOREM 3.1. *Suppose that* (A1)(i) *and* (A2)(i) *hold and there exists a random variable* $\alpha(\omega)$ *such that* $\alpha(\omega) \geqslant 0$ a.e. μ *and*

$$\det(sI - A(\omega) - \tilde{B}(s, \omega)) \neq 0 \qquad \text{a.e. } \mu \text{ for } \operatorname{Re}(s) \geqslant -\alpha(\omega), \tag{A3}$$

where $\tilde{B}(s, \omega)$ *is the Laplace transform of* $B(s, \omega)$ i.e.,

$$\tilde{B}(s, \omega) = \int_0^\infty e^{-st} B(t, \omega) \, dt. \tag{3.6}$$

Then

$$\| R(t, \omega)\| \in L_1(R_+, R) \quad \text{a.s.} \tag{3.7}$$

Proof. Let D be the set of all $\omega \in \Omega$ such that $\| A(\omega)\| < \infty$, $\alpha(\omega) \geqslant 0$ and $\det(sI - A(\omega) - \tilde{B}(s, \omega)) \neq 0$ for $\mathrm{Re}(s) \geqslant -\alpha(\omega)$. $\mu(D) = 1$ by hypothesis. Take any $\omega_0 \in D$. $\| B(t, \omega_0)\| \in L_1(R_+, R)$ and $\det(sI - A(\omega_0) - \tilde{B}(s, \omega_0)) \neq 0$ for $\mathrm{Re}(s) \geqslant 0$ since $-\alpha(\omega_0) \leqslant 0$. Hence, by [4, Theorem 2.8],

$$\| R(t, \omega_0)\| \in L_1(R_+, R)$$

for every $\omega_0 \in D$. In other words,

$$\| R(t, \omega)\| \in L_1(R_+, R) \qquad \text{a.s.} \ \blacksquare$$

As a consequence of (3.3) and (3.7), it follows that

$$\| R(t, \omega)\| \in L_1(R_+, L_\infty(\Omega, A, \mu)). \tag{3.8}$$

Let $f(t, \omega) = b(\omega)\, \Phi(\sigma(t, \omega))$, where Φ satisfies the following condition:

$\Phi(\sigma)$ is a continuous real valued function on R such that $\sigma\Phi(\sigma) > 0$ for $\sigma \neq 0$ and there exists $\varphi > 0$ such that $|\Phi(\sigma)| \leqslant \varphi$ for $\sigma \in R$. (A4)

THEOREM 3.2. *Under the assumptions* (A1), (A2), *and* (A4), *any random solution of*

$$\dot{X}(t, \omega) = A(\omega)\, X(t, \omega) + \int_0^t B(t - s, \omega)\, X(s, \omega)\, ds + f(t, \omega) \tag{3.9}$$

can be represented in the form

$$X(t, \omega) = R(t, \omega)\, X(0, \omega) + \int_0^t R(t - s, \omega)\, f(s, \omega)\, ds, \tag{3.10}$$

where $R(t, \omega)$ is the associated integral resolvent defined by (3.2).

This result is a special case of [6, Theorem 4.1].

4. EXISTENCE AND STABILITY

Since

$$\sigma(t, \omega) = \langle c(\omega), X(t, \omega)\rangle, \tag{4.1}$$

it follows from (3.10) that

$$\sigma(t, \omega) = \langle c(\omega), R(t, \omega)\, X(0, \omega)\rangle + \int_0^t \langle c(\omega), R(t - s, \omega)\, b(\omega)\rangle\, \Phi(\sigma(s, \omega))\, ds. \tag{4.2}$$

Let

$$h(t, \omega) = \langle c(\omega), R(t, \omega)\, X(0, \omega)\rangle, \tag{4.3}$$

$$k(t, \omega) = \langle c(\omega), R(t, \omega)\, b(\omega)\rangle. \tag{4.4}$$

The stochastic integral equation (4.2) can be written in the form

$$\sigma(t, \omega) = h(t, \omega) + \int_0^t k(t - s, \omega)\, \Phi(\sigma(s, \omega))\, ds, \tag{4.5}$$

where the integral on the right-hand side of (4.5) is a Bochner integral. In view of (A1), (3.3), and (4.4), it follows that

$$k(t, \omega) \in L_1(R_+, L_\infty(\Omega, \mathscr{A}, \mu)). \tag{4.6}$$

Let

$$\tilde{R}(s, \omega) = \int_0^\infty e^{-st} R(t, \omega)\, dt, \qquad \mathrm{Re}(s) \geqslant 0. \tag{4.7}$$

Since $\| R(t, \omega)\| \in L_1(R_+, L_\infty(\Omega, \mathscr{A}, \mu))$, (A1) and (A2) hold, and from the fact that the second term on the right-hand side of (3.5) is a convolution, it follows that

$$\| \dot{R}(t, \omega)\| \in L_1(R_+, R) \qquad \text{a.s.} \tag{4.8}$$

Furthermore,

$$\| \dot{R}(t, \omega)\| \leqslant \| A(\omega)\|\, \| R(t, \omega)\| + \int_0^t \| B(t - s, \omega)\|\, \| R(s, \omega)\|\, ds \tag{4.9}$$

and hence

$$\| \dot{R}(t, \omega)\| \in L_\infty(\Omega, \mathscr{A}, \mu) \qquad \text{for every } t \in R_+, \tag{4.10}$$

by (A1), (A2), and (3.3). Therefore,

$$\| \dot{R}(t, \omega)\| \in L_1(R_+, L_\infty(\Omega, \mathscr{A}, \mu)). \tag{4.11}$$

Taking Fourier transforms on both sides of (3.5), one obtains the relation

$$s\tilde{R}(s, \omega) - I = A(\omega)\, \tilde{R}(s, \omega) + \tilde{B}(s, \omega)\, \tilde{R}(s, \omega) \tag{4.12}$$

after some calculations. Hence,

$$\tilde{R}(s, \omega) = [sI - A(\omega) - \tilde{B}(s, \omega)]^{-1}. \tag{4.13}$$

The existence of the inverse follows from (A3).

We shall now state and prove the main theorems of this paper.

Theorem 4.1. *Under the assumptions* (A1)–(A4), *the system* (S) *has at least one random solution for any initial condition* $X(0, \omega) \in L(p, n)$.

Proof. In view of (4.3), (4.4), (3.8), (4.11), and (A1) and the fact that

$X(0, \omega) \in L(p, n)$, it follows that $h(t, \omega)$, $h'(t, \omega)$, $k(t, \omega)$, $k'(t, \omega)$ exist and

$$h(t, \omega), h'(t, \omega) \in L_1(R_+, L_p(\Omega, \mathscr{A}, \mu)), \tag{4.14}$$

$$k(t, \omega), k'(t, \omega) \in L_1(R_+, L_\infty(\Omega, \mathscr{A}, \mu)). \tag{4.15}$$

Since (4.14) holds,

$$h(t, \omega) \in BC(R_+, L_p(\Omega, \mathscr{A}, \mu)), \tag{4.16}$$

where $BC(R_+, L_p(\Omega, \mathscr{A}, \mu))$ is the Banach space of bounded continuous functions from R_+ into $L_p(\Omega, \mathscr{A}, \mu)$.

Define the integral operator T on $C(R_+, L_p(\Omega, \mathscr{A}, \mu))$ by

$$(T\sigma)(t, \omega) = h(t, \omega) + \int_0^t k(t-s, \omega)\, \Phi(\sigma(s, \omega))\, ds, \qquad t \in R_+, \tag{4.17}$$

where the integral is a Bochner integral. In view of (A4), (4.14), and (4.15), $(T\sigma)(t, \omega) \in L_p(\Omega, \mathscr{A}, \mu)$ for each $t \in R_+$. Furthermore,

$$\begin{aligned}
&\|(T\sigma)(t, \omega) - (T\sigma)(t_0; \omega)\|_p \\
&\quad \leqslant \| h(t, \omega) - h(t_0, \omega)\|_p + \varphi \int_0^{t_0} \| k(t-s, \omega) - k(t_0 - s, \omega)\|_p\, ds \\
&\quad \leqslant \| h(t, \omega) - h(t_0, \omega)\|_p + \varphi c d \int_0^{t_0} \| R(t-s, \omega) - R(t_0 - s, \omega)\|_p\, ds \\
&\qquad + \varphi c d \int_{t_0}^{t} \| R(t_0 - s, \omega)\|_p\, ds.
\end{aligned} \tag{4.18}$$

Clearly the first term on the right-hand side of (4.18) tends to zero by (4.16) as $t \to t_0$. Since $\| R(t, \omega)\| \in L_1(R_+, L_\infty(\Omega, \mathscr{A}, \mu))$, the last term approaches zero. From [6, Lemma 3.2], $R(t, \omega)$ is L_p-differentiable. In particular, it follows that the integrand of the second term of (4.18) tends to zero. Further, the integrand is dominated by a continuous function $n(s)$ by (3.4) which is integrable on $[0, t_0]$. Hence, by the Lebesgue dominated convergence theorem, the second term converges to zero. Hence,

$$\|(T\sigma)(t, \omega) - (T\sigma)(t_0, \omega)\|_p \to 0 \qquad \text{as } t \to t_0.$$

This shows that $(T\sigma)(t, \omega) \in C(R_+, L_p(\Omega, \mathscr{A}, \mu))$. In fact, the operator T defined by (4.17) is continuous from $C(R_+, L_p(\Omega, \mathscr{A}, \mu))$ into itself. This can be shown be a method similar to [5, Lemma 2.1]. It is easy to see from (4.17) that

$$\|(T\sigma)(t, \omega)\|_p \leqslant \sup_{t \in R_+} \| h(t, \omega)\|_p + \varphi \int_0^\infty \| k(t, \omega)\|_p\, dt = r \qquad \text{(say)}. \tag{4.19}$$

Consider the set

$$\Sigma = \{\sigma(t, \omega) : \sigma(t, \omega) \in C(R_+, L_p(\Omega, \mathscr{A}, \mu)), \| \sigma(t, \omega) \|_p \leqslant r \text{ for all } t \in R_+\}. \quad (4.20)$$

Equation (4.19) shows that $T\Sigma \subset \Sigma$. Furthermore, $T\Sigma$ is relatively compact in $C(R_+, L_p(\Omega, \mathscr{A}, \mu))$ by the Ascoli–Arzela theorem, since $T\Sigma$ is uniformly bounded by (4.20) and $T\Sigma$ is equicontinuous as the right-hand side of (4.18) tends to zero as $t \to t_0$ and is independent of σ. Hence by the Schauder fixed-point theorem, there exists at least one fixed point σ. Hence there exists at least one random solution $\sigma(t, \omega) \in C[R_+, L_p(\Omega, \mathscr{A}, \mu)]$ satisfying (4.5). Equation (4.1) proves that there exists at least one random solution $X(t, \omega) \in C[R_+, L_p(\Omega, \mathscr{A}, \mu)]$ satisfying the system (S). ∎

THEOREM 4.2. *In addition to assumptions* (A1)–(A4), *suppose the following condition is satisfied.*

(A5) *There exists* $q \geqslant 0$ *such that for every* $\lambda \in R$,

$$\mathrm{Re}\{(1 + i\lambda q)\langle c(\omega), [i\lambda I - A(\omega) - \hat{B}(i\lambda, \omega)]^{-1} b(\omega)\rangle\} \leqslant 0 \qquad \text{a.e. } \mu. \quad (4.21)$$

Then, any random solution $X(t, \omega)$ *of* (S) *with* $X(0, \omega) \in L(p, n)$ *is absolutely stable*; i.e.,

$$\lim_{t \to \infty} \| X(t, \omega) \| = 0 \qquad \text{a.e. } \mu. \quad (4.22)$$

Proof. It is already shown in Theorem 4.1 that a random solution of the system (S) exists almost surely under assumptions (A1)–(A4). Let $X(t, \omega)$ be any such random solution. In view of results in [3, pp. 196–199], there exists a solution $\hat{X}(t, \omega)$ equivalent to $X(t, \omega)$ such that $\hat{X}(t, \omega)$ satisfies (S) almost surely where the integrals are now defined sample pathwise and we shall prove that

$$\lim_{t \to \infty} \| \hat{X}(t, \omega) \| = 0 \qquad \text{a.e. } \mu, \quad (4.23)$$

which in turn proves (4.22).

Note the definitions of $h(t, \omega)$ and $k(t, \omega)$ given in (4.3), (4.4) and their properties from (4.14) and (4.15). Let D be the set of all $\omega \in \Omega$ where essential supremums (with respect to μ) of $h(t, \omega)$, $h'(t, \omega)$, $k(t, \omega)$, $k'(t, \omega)$ are finite and (A5) holds. Fix any $\omega_0 \in D$. Consider the deterministic integral equation

$$\sigma(t, \omega_0) = h(t, \omega_0) + \int_0^t k(t - s, \omega_0) \Phi(\sigma(s, \omega_0)) \, ds, \qquad t \in R_+. \quad (4.24)$$

It follows from [2, Theorem 3.2.2] that any solution $\hat{\sigma}(t, \omega_0)$ of (4.24) satisfies

$$\hat{\sigma}(t, \omega_0) \to 0 \qquad \text{as} \quad t \to \infty. \quad (4.25)$$

Since $\| R(t, \omega_0)\| \in L_1(R_+, R)$ and $\| \dot{R}(t, \omega_0)\| \in L_1(R_+, R)$, it follows that

$$\| R(t, \omega_0)\| \to 0 \quad \text{as} \quad t \to \infty. \tag{4.26}$$

But

$$\dot{X}(t, \omega_0) = R(t, \omega_0) X(0, \omega_0) + \int_0^t R(t - s, \omega_0) b(\omega_0) \Phi(\sigma(s, \omega_0)) ds, \tag{4.27}$$

by (3.10), and the convolution product tends to zero as $t \to \infty$ if at least one of the factors does. Hence,

$$\lim \| \dot{X}(t, \omega_0)\| = 0 \quad \text{as} \quad t \to \infty.$$

Since this holds for every $\omega_0 \in D$ with $\mu(D) = 1$, it follows that (4.23) holds, which in turn proves (4.22). ▮

Remarks. Tsokos [7] studied absolute stability of stochastic differential systems of the form (S) when $B(t, \omega) \equiv 0$. He further assumes that $\sigma(t, \omega) = \langle c(t, \omega), X(t, \omega)\rangle$. His approach is to reduce the study of the system to the study of a nonlinear stochastic integral equation of the form (4.5) and then apply the stability results of (4.5). It seems to us that his results are valid only when $c(t, \omega)$ is independent of t since the kernel for the system (1.0), (1.1), of Tsokos [7] in the reduced stochastic integral equation will not be a convolution kernel when $c(t, \omega)$ depends on t.

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