TRUNCATION IN QUANTAL ASSAY

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Summary

Problems of truncation at known and unknown points in quantal assay are considered. Methods of maximum likelihood and of minimum chi-square for estimating the parameters are discussed. Other quick but rough procedures of estimation, including a graphical iterative procedure, are outlined. Tables of 'Truncated Probits' and of 'Truncated Logits,' which facilitate the graphical procedure are presented. Numerical examples illustrating the use of these tables are worked out.

1. Problems of truncation

Quantal assay as a convenient experimental scheme has attracted the attention of research workers, since this observational set up is particularly useful in situations wherein it is not possible to observe directly a characteristic of an experimental unit, but only the numbers of experimental units, having the characteristics above fixed levels of the characteristic are possible to be observed. Many variations in this kind of observational set up and several kinds of adjustments in the method, to suit the particular experimental conditions have been studied. Adjustments for natural mortality in a drug-assay, the Wadley's problem of non-observable number of experimental units and the staircase method of estimation are some of the interesting instances of this nature. They are discussed by Finney [5].

The general quantal assay experiment can be formulated as follows: The purpose of such an experiment is primarily to estimate the parameters (or certain parametric functions) of the frequency distribution of x, say, $f(x, \lambda)$, of a characteristic x, λ being the parameters. Let the assay be performed over m levels. Corresponding to each level, a subset E_i of the domain of x (or of a suitable transform of x) is chosen and fixed. Considering n_i experimental units, the number of them, say r_i, having tolerance in E_i is observed. In the familiar drug-assay, E_i is $(-\infty, x_i)$ where $(-\infty, \infty)$ is taken as the domain of x, the log dose. The estimation procedure depends on the nature of the

E_i's and the form of f. Methods of analysis for this case, taking particular forms of f, such as the normal and the logistic, are discussed by Finney [5] and Berkson [1].

In quantal assay, truncated distributions are frequently come across, especially in time mortality. Meynell [6] discusses the genesis of truncated distributions of biological tolerances and gives an account of such phenomena. Though, in most cases, the points of truncation are unknown, there are instances in which they are predetermined by the method of experiment or by the selection of data. In case the points of truncation are unknown, tentative values of them are to be used in iterative procedures for determining other parameters, as shown below. Bliss and Stevens [3] have considered truncation at known points in a different context. We shall consider here truncation at known and unknown points, considering a somewhat general form of the density function—the scale and location parametric form—of which the normal, logistic and many other known laws are special cases.

A problem which is conceptually different but statistically related is that of 'censoring.' By censored observations in quantal assay, we mean that the set E_i is not $(-\infty, x_i)$ but (a_i, x_i). In truncation, the domain of the tolerance variable is truncated to a subset E (independent of i) of the domain of a variable x and the tolerance distribution is derived from the distribution of x; E_i's are of the form (k, x_i) where k is the lower point of truncation. In censoring, the tolerance variable has a domain (which may be a truncated domain or not) and the sets E_i are of the form (a_i, x_i). We shall reserve problems of censoring for a future communication.

The maximum likelihood (M.L.) and minimum chi-square (M.C.) methods of estimation of the parameters of the distribution of tolerance, truncated at known points and whose untruncated density function is of the form $\beta f(\alpha+\beta x)$, α and β being the parameters, are considered in sections 3 and 4, after explaining the notations in section 2; section 3 deals with lower truncation and section 4 deals with lower and upper truncation simultaneously. In section 5, the problem of unknown points of truncation is solved; the solution depends upon the solution to the problem of known truncation points in an iterative process. In section 6, numerical examples are worked out. Section 7 contains tables of truncated probits and of truncated logits, which are useful in following the rough graphical iterative procedure suggested in section 3.

2. Notations

Let us consider an experiment in which m sets of n_i subjects are stimulated to levels x_i, $i=1, 2, \cdots, m$, the response being of a quantal nature. Thus at stimulant level x_i, the number of subjects giving one type of response is r_i out of n_i. Let $p_i = r_i / n_i$. In general, denoting $P(E_i)$ by Φ_i, one can write the likelihood of quantal assay, in the form,

$$L = \prod_{i=1}^{n} \binom{n_i}{r_i} \Phi_i^{r_i} (1-\Phi_i)^{n_i - r_i} . \tag{2.1}$$

Let us consider the basic distribution

$$F^*(x) = \int_{-\infty}^{\alpha+\beta x} f(z) dz , \tag{2.2}$$

and denote the lower and upper points of truncation by k and k' respectively. The following is a list of notations that will be of subsequent use:

$$\begin{aligned}
&\varepsilon = \alpha + \beta k , \quad \varepsilon' = \alpha + \beta k' , \\
&N = \sum_{i=1}^{m} n_i , \\
&1/\theta = 1 - F^*(k) , \\
&1/\theta^* = F^*(k') - F^*(k) , \\
&f_i = f(\alpha + \beta x_i), \quad f_\cdot = f(\alpha + \beta k), \quad f^\cdot = f(\alpha + \beta k') \\
&F^*(k) = F_\cdot^* ; \ F^*(k') = F^{\cdot *} , \\
&F(x) = \theta[F^*(x) - F^*(k)] , \ F(x_i) = F_i , \\
&f_i^* = f_\cdot (1 - F_i) , \\
&Z_i = f_i - f_i^* , \\
&y_i = (p_i - F_i) / \theta Z_i , \\
&w_i = Z_i^2 / F_i(1 - F_i) , \\
&w_i' = f_i / Z_i , \\
&w_i'' = f_i^* / Z_i ,
\end{aligned} \tag{2.3}$$

so that $\quad w_i' - w_i'' = 1 ,$

$$u_i = w_i' x_i - w_i'' k .$$

3. Lower truncation at a known point

3.1 *Exact M.L. and M.C. estimates*

Let us first consider truncation below at k. The function Φ in (2.1) takes the form $F(x)$ as in (2.3). One obtains, by differentiating (2.3),

$$\frac{\partial F_i}{\partial \alpha}=\theta[f_i-f.(1-F_i)]\,, \tag{3.1.1}$$

$$\frac{\partial F_i}{\partial \beta}=\theta[f_i x_i-kf.(1-F_i)]\,. \tag{3.1.2}$$

The scores (S_α, S_β) and the information $(I_{\alpha\alpha}, I_{\alpha\beta}, I_{\beta\beta})$ are obtained by the usual procedure of finding the first derivatives and the expected values of the second derivatives (using $E(p_i)=F_i$). Using the notations in section 2, and dropping subscripts i for all elements inside the $\sum$ sign for convenience, one easily obtains

$$S_\alpha=\sum nwy \tag{3.1.3}$$

$$S_\beta=\sum nwyu \tag{3.1.4}$$

$$I_{\alpha\alpha}=\sum nw \tag{3.1.5}$$

$$I_{\alpha\beta}=\sum nwu \tag{3.1.6}$$

$$I_{\beta\beta}=\sum nwu^2\,. \tag{3.1.7}$$

The usual iterative solution for the maximum likelihood estimates α^*, β^* for α, β with α_0, β_0 as initial values, are obtained from

$$S_\alpha=(\alpha^*-\alpha_0)I_{\alpha\alpha}+(\beta^*-\beta_0)I_{\alpha\beta} \tag{3.1.8}$$

$$S_\beta=(\alpha^*-\alpha_0)I_{\alpha\beta}+(\beta^*-\beta_0)I_{\beta\beta}\,. \tag{3.1.9}$$

By Berkson's [2] procedure, the minimum chi-square normal equations are obtained by multiplying each term of the maximum likelihood equations, by the corresponding factor,

$$[F_i(1-p_i)+p_i(1-F_i)]\,/\,F_i(1-F_i)\,, \tag{3.1.10}$$

since they minimize the chi-square statistic

$$\sum_{i=1}^{m}[n_i(p_i-F_i)^2\,/\,F_i(1-F_i)]\,. \tag{3.1.11}$$

3.2 *Simpler approximate iterative solution*

Following Berkson [2] it is easy to show that the maximum likelihood and minimum chi-square estimates can be approached in the limit if, in the normal equations, $(p-F)$ is replaced by

$$Z[(\alpha_0+\beta_0 x)-(\alpha+\beta x)] . \tag{3.2.1}$$

To show this, let

$$Y=\alpha_0+\beta_0 x+y , \quad y^*=\alpha+\beta x .$$

Writing

$$F_0=\int_{-\infty}^{\alpha_0+\beta_0 x} f(z)dz ,$$

and noting that

$$\frac{\partial F}{\partial(\alpha+\beta x)}=\theta f , \tag{3.2.2}$$

$$\frac{\partial F}{\partial(\alpha+\beta k)}=-\theta f.(1-F) , \tag{3.2.3}$$

and considering F as a function of $(\alpha+\beta x)$ and of $(\alpha+\beta k)$ and replacing $(p-F)$ by the first term in the Taylor expansion, one obtains

$$p-F=\theta Z[Y-y^*] . \tag{3.2.4}$$

The equations to be solved are thus, for the M.L. estimates,

$$\sum nw(Y-y^*)=0 , \tag{3.2.5}$$

$$\sum nwu(Y-y^*)=0 , \tag{3.2.6}$$

and for the M.C. estimates, similar equations with each term multiplied by the corresponding factor of the form of (3.1.10).

3.3 Iterative graphical procedure

The observed proportion p, corresponding to a given x, equated to its theoretical value, gives

$$[p+(1-p)\int_{-\infty}^{\alpha+\beta k} f(z)dz]=\int_{-\infty}^{\alpha+\beta x} f(z)dz . \tag{3.3.1}$$

This can be used for an iterative process that would lead to good initial values for α and β for other methods. If tentative values for α_0 and β_0 (such as those corresponding to the untruncated distribution, that is, by taking p as the left side), are available, then from the observed p's and the known k, the left side of (3.3.1) can be calculated and by referring to the table of the distribution function of the standardized variable, values of $\alpha+\beta x$ can be obtained. By plotting $\alpha+\beta x$ against x, the values of α and β can be estimated. The value can again be improved by the same procedure. This method is illustrated in section 6.

The graphical method is justified as follows: Let p' denote the left side of (3.3.1) and p'_0 be its value at (α_0, β_0). Denoting the values of F^*, $f.$ and f at (α_0, β_0) by F^*_0, $f._0$ and f_0 respectively and noting that

$$p'-F^*=[1-F^*(k)](p-F)\,, \tag{3.3.2}$$

$$p-F=\theta(p'-F^*) \tag{3.3.3}$$

$$p'-F^*=(p'-p_0)+(p'_0-F^*_0)+(F^*_0-F^*)\,, \tag{3.3.4}$$

and replacing $(p'-p'_0)$ and $(F^*_0-F^*)$ by the first terms in their respective Taylor expansions, one obtains

$$p'-p'_0 \doteq (1-p)f._0[(\alpha+\beta k)-(\alpha_0+\beta_0 k)]\,, \tag{3.3.5}$$

$$F^*_0-F^* \doteq f_0[(\alpha_0+\beta_0 x)-(\alpha+\beta x)]\,. \tag{3.3.6}$$

Using (3.3.5) and (3.3.6) in (3.3.4) and using (3.3.3) and (3.3.4) in the scores (3.1.3) and (3.1.4), one obtains

$$(nw/Z)[f_0\overline{(p_0-F^*_0)}/f_0+(\alpha_0+\beta_0 x)-(\alpha+\beta x)] \\ +f._0[(\alpha+\beta k)-(\alpha_0+\beta_0 k)]\sum(nw/Z)(1-p)=0 \tag{3.3.7}$$

and similarly modified equations.

Noting that

$$F^*=(1/\theta)(F+F^*_0) \text{ and } 1-F^*=(1/\theta)(1-F)$$

and using the notation

$$w^*=f^2/[F^*(1-F^*)]\,,$$

the graphical procedure suggested, using the initial values, in so far as it takes care of the weights, is equivalent to the following equation

$$\sum(nw^*/f_0)[f_0(p'_0-F^*_0)f_0+(\alpha_0+\beta_0 x)-(\alpha+\beta x)]=0\,, \tag{3.3.8}$$

and a similar equation. If the point $\alpha_0+\beta_0 k$ is also looked upon as an observed point in drawing eye-fit lines, the two sets of equations are similar, and, though, the graphical procedure is not equivalent to the solution of these equations, it helps in obtaining estimates which are reasonably good and which can be profitably used as starting values for the more difficult iterative processes. While computing, it would be helpful, if the likelihood or the chi-square is computed at each stage to determine when exactly to stop the process, for otherwise, by an unsuitable eye-fit line one may get involved in a divergent graphical iterative procedure.

3.4 Tables needed for analysis

The procedures described above can be followed easily if the four quantities,

$$(1/\theta Z),\ (F/\theta Z),\ w,\ w'$$

are tabulated for different values of $(\alpha+\beta x)$ and of $(\alpha+\beta k)$. To follow the graphical procedure, tables of truncated probits and of truncated logits which give values of $(\alpha+\beta x)$, corresponding to normal and logistic tolerance distributions respectively, for given values of $(\alpha+\beta k)$ and p in the equation (3.3.1), have been prepared. Skeleton forms of these tables for

$$
\begin{aligned}
100\,p &= 01\,(01)\,99 && \text{and} \\
5+\alpha+\beta k &= 1.0\,(1.0)\,7.0 && \text{for probits} \\
&= 2.0\,(1.0)\,7.0 && \text{for logits,}
\end{aligned}
$$

are presented at the end of this paper.

Extensive tables for

$$
\begin{aligned}
100\,p &= 01\,(01)\,99 && \text{and} \\
5+\alpha+\beta k &= 0.4\,(0.1)\,7.3 && \text{for probits and} \\
&= 1.6\,(0.1)\,6.0 && \text{for logits,}
\end{aligned}
$$

are available with the author.

4. Lower and upper truncation at known points

This case requires modification only with regard to the weighting coefficients. Working on lines similar to the last section, one comes across

$$Z_i^* = f_i - f_{\cdot}(1-F_i) + f^{\cdot}(1-F_i)\,,$$

and the following weighting coefficients (with subscripts dropped)

$$
\begin{aligned}
& w^{(1)} = f/Z^*\,, \\
& w^{(2)} = -f_{\cdot}(1-F)/Z^*\,, \\
& w^{(3)} = f^{\cdot}(1-F)/Z^*\,, \\
\text{so that} \quad & w^{(1)}+w^{(2)}+w^{(3)} = 1\,, \qquad \text{and} \\
& y^* = (p-F)/\theta^* Z^*\,, \\
& u^* = w^{(1)}x + w^{(2)}k + w^{(3)}k'\,,
\end{aligned}
$$

which are to be used just as the corresponding elements of the last section.

The graphical procedure is also similar, the only change being that the left side is to be taken as

$$pF^*(k') - F^*(k)(1-p) ,$$

since on equating p to its theoretical value, one gets

$$p = [F^*(x) - F^*(k)]/[F^*(k') - F^*(k)] .$$

5. Truncation at unknown points

5.1 Estimation of truncation points when other parameters are known

Considering truncation above at k' and below at k, the equations giving estimates of k, k' are

$$\frac{\partial \log L}{\partial s} = \sum [n(p-F)/F(1-F)]\left[\frac{\partial F}{\partial s}\right] = 0 , \quad s = \varepsilon, \varepsilon' . \tag{5.1.1}$$

Using

$$\frac{\partial F}{\partial \varepsilon} = -\theta f.(1-F_i)$$

and

$$\frac{\partial F}{\partial \varepsilon'} = -\theta f^{\cdot} F_i ,$$

in (5.1.1), one obtains the scores as

$$S_{\varepsilon} = \sum [n(p-F)/F]$$

and

$$S_{\varepsilon'} = \sum n[(1-p)-(1-F)]/(1-F) ,$$

so that the equations to be solved are

$$\sum (np/F) = N ,$$

and

$$\sum [n(1-p)/(1-F)] = N .$$

Writing these explicitly in terms of the unknowns, one obtains

$$N - \prod (F_i^* - F_{\cdot}^*) - (F^{\cdot *} - F_{\cdot}^*)[\sum_{i=1}^{m} r_i \prod_{j \neq i} (F_j^* - F_{\cdot}^*)] = 0 ,$$

and an equation in which r_i is replaced by $(n_i - r_i)$ and $F^{\cdot *}$ and $F_{\cdot}^*$ are interchanged; because of this symmetry, we do not sometimes write the second equation separately. It could be seen that when α, β are known, this amounts to solving $F_{\cdot}^*$ and $F^{\cdot *}$ from these two simultaneous equations, from which k, k' can be uniquely obtained, using a table of the standardized integral.

These equations are polynomial equations of mth degree and of first degree in the two arguments $F_{\cdot}^*$ and $F^{\cdot *}$. We shall show that there is one and only one admissible set of solutions. An admissible value of k is less than x_1, the minimum level of stimulus chosen for the experiment and similarly an admissible value of k' is greater than x_m. Let us substitute in the two functions the sets of values $(0, F_1^*, \cdots, F_m^*)$ and $(F_1^*, \cdots, F_m^*, 1)$ respectively for $F_\cdot^*$ and $F^{\cdot *}$; these values are in increasing order. The signs of the polynomial in one of $F_\cdot^*$ and $F^{\cdot *}$ could be seen to be alternating between successive values, the value of the other being fixed. So there is an odd number of solutions for both k and k' lying between each (x_i, x_{i+1}) and since there are only m roots, this odd number is one. The remaining root for k is in the range $(-\infty, x_1)$ and for k' in the range (x_m, ∞).

Several methods for solving these equations are available, and they are described by Scarborough [7]. However, we suggest here one method, that of finer and finer tabulation, using simultaneous linear interpolation, for two variables, to locate the root at each stage. The equations can be written as follows:

$$g_m(F_\cdot^*)+F^{\cdot *}g_{m-1}(F_\cdot^*)=0 ,$$

$$g_m^*(F^{\cdot *})+F_\cdot^* g_{m-1}^*(F^{\cdot *})=0 ,$$

where g and g^* are polynomials of degree indicated by their suffixes. Now, each polynomial is to be evaluated at two tentative points (conveniently 0 and F_1^* for $F_\cdot^*$ and F_m^* and 1 for $F^{\cdot *}$) at which the left sides change signs; these two points are to be used to obtain linear approximations as is done in linear interpolation. Then on substitution, there are two linear equations in two unknowns $F_\cdot^*$ and $F^{\cdot *}$, which can be solved simultaneously to obtain improved values. This is to be used again with one or other of the initial values such that the left sides are of different signs and the process continued until stable values are obtained.

5.2 *Estimation of truncation points when other parameters are unknown*

The procedure to be adopted for the case of (α, β) unknown is the iterative procedure for obtaining (α, β) given tentative values k, k' as suggested in the foregoing section, and the improvement of the values (k, k') with the help of these (α, β). Let the values at the ith stage be denoted by s_i for (α, β) and F^i for $(F_\cdot^*, F^{\cdot *})$. The convergence is proved thus: Consider the likelihood function. Substitute a trial value F^0 to determine s_0 by the graphical procedure or by solving the likelihood equations. Using the s_0 to determine an improved value F^1 from the polynomials, the process is continued. Let us consider any set F^i, s_i. Since F^{i+1} is obtained from s_i by maximizing likelihood L,

$$L_{F^{i+1},\, s_i} \geq L_{F^i,\, s_i} ,$$

and since F^{i+1} is used to obtain s_{i+1} by maximizing L,

$$L_{r^{i+1}, s_{i+1}} \geq L_{r^{i+1}, s_i}.$$

Hence L at the ith stage is less than L at the $(i+1)$st stage.

6. Numerical examples

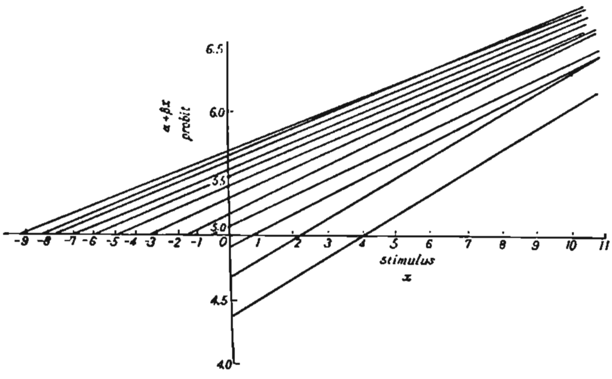
6.1 Truncation at known point

Caldecott, Frolik and Morris [4] compared the effects of X-rays and thermal neutrons on dormant seeds of barley. The stimulus in this case is a flux of 4.65×10^8 Nth/cm^2./sec. at various levels of designated time, the response being the incidence or otherwise of abnormal cells. Cells with one or more of bridges or fragments are considered abnormal. The following is an extract from their Table 1.

Table 1. The effect of thermal neutrons on frequencies of bridges and fragments in treated dormant seeds of barley.

Treatment	Cells observed	Abnormal cells (%)
3	102	46
6	88	65
9	93	81
12	96	92

The tables of truncated probits, whose summary is presented in section 7 are used. The value of k, the lower point of truncation, is taken



Graph 1. Iterative graphical procedure.

as zero. The initial values of α, β are obtained by plotting the untruncated probits against x. Graph 1 gives the eye-fit lines and Table 2 presents the computations.

Table 2. Computations for iterative graphical procedure

T.P. : Truncated probit

First stage (Initial values)

x	n	p	Probit	$\alpha+\beta x$	F	$F(1-F)$	χ^2
3	102	46	4.8996	4.88	0.2554	0.1902	22.45
6	88	65	5.3853	5.39	0.5266	0.2493	5.38
9	93	81	5.8779	5.90	0.7498	0.1876	1.80
12	96	92	6.4051	6.41	0.8922	0.0962	0.77
$\alpha=4.37$, $\beta=0.17$.							30.40

Second stage

T.P.	$\alpha+\beta x$	F	$F(1-F)$	χ^2
5.2611	5.16	0.3147	0.2157	9.98
5.6526	5.64	0.5900	0.2419	1.31
6.0803	6.12	0.7937	0.1637	0.15
6.5632	6.60	0.9139	0.0787	0.05
$\alpha=4.68$, $\beta=0.16$.				11.49

This process is continued to the twelfth stage, the chi-square values decreasing up to the eleventh and increasing at the twelfth. The values of α and β and that of chi-square are given below for these stages:

Stage	α	β	χ^2
3	4.92	0.16	6.25
4	5.09	0.15	4.96
5	5.20	0.14	4.41
6	5.33	0.13	4.07
7	5.44	0.13	3.25

Stage	α	β	χ^2
8	5.50	0.12	3.13
9	5.55	0.12	3.11
10	5.63	0.12	2.59
11	5.70	0.12	2.20
12	5.74	0.11	2.88

The eleventh stage estimates of α and β can be taken as the minimum chi-square estimates, since the fit is good or if more accuracy is required the normal equations can be solved with these values as initial values.

6.2 *Truncation at unknown point*

We shall use the same data to estimate the point of truncation as a parameter.

Table 3. Computations to estimate α, β, and k

Stage I: $\epsilon=-\infty$, $\alpha=4.37$, $\beta=0.17$, ϵ' assumed $+\infty$ throughout.

Probit	$\alpha+\beta x$	F_i	$r_i \prod_{j \neq i} F_j$	F_j-F_i
4.90	4.88	0.452	23.0300	—
5.39	5.39	0.652	19.3629	0.200
5.88	5.90	0.816	20.3550	0.364
6.41	6.41	0.925	21.1640	0.469

Stage II: $\epsilon=-2.09$, $\alpha=4.40$, $\beta=0.17$, $\log L=C-87.26$, C a constant.

T.P.	$\alpha+\beta x$	F_i	$r_i \prod_{j \neq i} F_j$	F_j-F_i
4.92	4.91	0.464	23.7508	—
5.40	5.42	0.663	20.1587	0.199
5.89	5.93	0.824	21.4807	0.360
6.42	6.44	0.925	22.4520	0.461

Stage III: $\epsilon=-0.65$, $\alpha=4.82$, $\beta=0.16$, $\log L=C-93.37$.

T.P.	$\alpha+\beta x$	F_i	$r_i \prod_{j \neq i} F_j$	F_j-F_i
5.25	5.30	0.618	31.5814	—
5.87	5.78	0.782	30.2684	0.164
6.26	6.26	0.896	34.7596	0.278
6.71	6.74	0.959	40.7846	0.341

Since the likelihood decreases at the third stage, the iterative process is stopped. The convergence of ϵ to stable values cannot be reached since the graphical method is approximate and that is the reason for the likelihood to decrease. Further, the method of estimating ϵ for given (α, β) is fairly accurate and the second stage (α, β) has led to $\epsilon=-0.65$ with $\log L=C-93.37$. Since the probit yields actual α plus 5, the final estimates are:

$$\alpha=-0.60, \quad \beta=0.17 \text{ and } k=-0.31 \text{ (from } \epsilon=-0.65).$$

The value of k, taken as zero is fairly justified by this result.

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Table 4. Truncated probits

Standardised lower point of truncation

P	1.0	2.0	3.0	4.0	5.0	6.0	7.0
.01	2.6744	2.7207	3.1546	4.0342	5.0125	6.0066	7.0047
.02	2.9465	2.9724	3.2750	4.0674	5.0250	6.0132	7.0089
.03	3.1192	3.1377	3.3745	4.0975	5.0375	6.0199	7.0132
.04	3.2493	3.2638	3.4602	4.1307	5.0500	6.0266	7.0176
.05	3.3551	3.3671	3.5358	4.1611	5.0625	6.0333	7.0220
.06	3.4452	3.4555	3.6039	4.1908	5.0751	6.0401	7.0264
.07	3.5241	3.5332	3.6661	4.2197	5.0876	6.0470	7.0308
.08	3.5949	3.6030	3.7236	4.2480	5.1002	6.0539	7.0353
.09	3.6592	3.6666	3.7771	4.2758	5.1128	6.0609	7.0399
.10	3.7184	3.7252	3.8273	4.3029	5.1254	6.0679	7.0445
.11	3.7735	3.7797	3.8747	4.3296	5.1380	6.0750	7.0491
.12	3.8250	3.8308	3.9197	4.3558	5.1507	6.0821	7.0538
.13	3.8737	3.8791	3.9627	4.3816	5.1633	6.0893	7.0585
.14	3.9198	3.9249	4.0038	4.4069	5.1760	6.0965	7.0633
.15	3.9637	3.9685	4.0433	4.4319	5.1888	6.1038	7.0681
.16	4.0057	4.0102	4.0813	4.4565	5.2015	6.1111	7.0729
.17	4.0460	4.0503	4.1181	4.4808	5.2143	6.1185	7.0778
.18	4.0848	4.0890	4.1537	4.5048	5.2271	6.1260	7.0828
.19	4.1223	4.1263	4.1883	4.5286	5.2400	6.1335	7.0878
.20	4.1586	4.1624	4.2219	4.5520	5.2529	6.1411	7.0929
.21	4.1939	4.1975	4.2547	4.5752	5.2659	6.1488	7.0980
.22	4.2281	4.2316	4.2867	4.5982	5.2789	6.1565	7.1032
.23	4.2615	4.2648	4.3180	4.6210	5.2919	6.1643	7.1084
.24	4.2941	4.2973	4.3486	4.6435	5.3050	6.1722	7.1137
.25	4.3259	4.3290	4.3786	4.6659	5.3182	6.1802	7.1190

P	1.0	2.0	3.0	4.0	5.0	6.0	7.0
.26	4.3571	4.3601	4.4081	4.6881	5.3314	6.1882	7.1244
.27	4.3676	4.3905	4.4371	4.7102	5.3447	6.1963	7.1299
.28	4.4176	4.4204	4.4656	4.7321	5.3580	6.2045	7.1355
.29	4.4471	4.4498	4.4936	4.7539	5.3714	6.2127	7.1411
.30	4.4761	4.4787	4.5213	4.7756	5.3849	6.2211	7.1467
.31	4.5046	4.5072	4.5486	4.7971	5.3984	6.2295	7.1525
.32	4.5328	4.5353	4.5756	4.8186	5.4120	6.2380	7.1583
.33	4.5606	4.5630	4.6022	4.8400	5.4257	6.2466	7.1642
.34	4.5880	4.5904	4.6286	4.8613	5.4395	6.2553	7.1701
.35	4.6152	4.6175	4.6548	4.8825	5.4533	6.2641	7.1762
.36	4.6420	4.6443	4.6806	4.9037	5.4673	6.2730	7.1823
.37	4.6686	4.6708	4.7063	4.9248	5.4813	6.2820	7.1885
.38	4.6950	4.6972	4.7318	4.9459	5.4954	6.2911	7.1948
.39	4.7212	4.7233	4.7571	4.9669	5.5097	6.3003	7.2012
.40	4.7471	4.7492	4.7822	4.9880	5.5240	6.3096	7.2076
.41	4.7729	4.7749	4.8072	5.0090	5.5384	6.3191	7.2142
.42	4.7985	4.8005	4.8321	5.0301	5.5530	6.3286	7.2209
.43	4.8240	4.8259	4.8569	5.0511	5.5677	6.3383	7.2276
.44	4.8494	4.8513	4.8815	5.0722	5.5825	6.3481	7.2345
.45	4.8747	4.8765	4.9061	5.0933	5.5974	6.3580	7.2415
.46	4.8998	4.9016	4.9306	5.1145	5.6125	6.3681	7.2485
.47	4.9250	4.9267	4.9551	5.1357	5.6277	6.3783	7.2557
.48	4.9500	4.9517	4.9796	5.1570	5.6430	6.3887	7.2630
.49	4.9750	4.9767	5.0040	5.1784	5.6585	6.3992	7.2705
.50	5.0000	5.0017	5.0284	5.1998	5.6742	6.4099	7.2780

P	1.0	2.0	3.0	4.0	5.0	6.0	7.0
.51	5.0250	5.0267	5.0529	5.2213	5.6900	6.4207	7.2857
.52	5.0501	5.0516	5.0774	5.2430	5.7060	6.4317	7.2936
.53	5.0751	5.0767	5.1019	5.2648	5.7222	6.4429	7.3016
.54	5.1002	5.1018	5.1266	5.2867	5.7386	6.4542	7.3097
.55	5.1254	5.1269	5.1513	5.3087	5.7552	6.4658	7.3180
.56	5.1507	5.1522	5.1761	5.3309	5.7720	6.4775	7.3264
.57	5.1761	5.1775	5.2010	5.3533	5.7890	6.4895	7.3350
.58	5.2016	5.2030	5.2260	5.3758	5.8062	6.5016	7.3438
.59	5.2272	5.2286	5.2512	5.3985	5.8237	6.5140	7.3528
.60	5.2530	5.2543	5.2765	5.4215	5.8415	6.5267	7.3620
.61	5.2789	5.2803	5.3021	5.4447	5.8595	6.5395	7.3713
.62	5.3051	5.3064	5.3278	5.4681	5.8778	6.5527	7.3809
.63	5.3314	5.3327	5.3538	5.4918	5.8964	6.5661	7.3907
.64	5.3580	5.3593	5.3800	5.5157	5.9153	6.5798	7.4008
.65	5.3849	5.3862	5.4065	5.5400	5.9345	6.5938	7.4111
.66	5.4121	5.4133	5.4332	5.5646	5.9541	6.6081	7.4216
.67	5.4395	5.4407	5.4603	5.5895	5.9741	6.6228	7.4324
.68	5.4673	5.4685	5.4877	5.6148	5.9944	6.6378	7.4436
.69	5.4955	5.4966	5.5155	5.6405	6.0152	6.6532	7.4550
.70	5.5240	5.5252	5.5437	5.6666	6.0364	6.6690	7.4668
.71	5.5530	5.5541	5.5724	5.6932	6.0581	6.6852	7.4789
.72	5.5825	5.5836	5.6015	5.7203	6.0804	6.7019	7.4914
.73	5.6125	5.6136	5.6312	5.7480	6.1031	6.7191	7.5043
.74	5.6430	5.6441	5.6614	5.7762	6.1265	6.7367	7.5176
.75	5.6742	5.6753	5.6922	5.8051	6.1504	6.7550	7.5314

P	1.0	2.0	3.0	4.0	5.0	6.0	7.0
.76	5.7061	5.7071	5.7237	5.8346	6.1751	6.7738	7.5457
.77	5.7386	5.7396	5.7560	5.8649	6.2005	6.7933	7.5605
.78	5.7720	5.7730	5.7890	5.8960	6.2267	6.8136	7.5759
.79	5.8063	5.8072	5.8229	5.9280	6.2537	6.8345	7.5919
.80	5.8415	5.8424	5.8578	5.9610	6.2817	6.8564	7.6087
.81	5.8778	5.8787	5.8938	5.9950	6.3108	6.8791	7.6262
.82	5.9153	5.9162	5.9310	6.0303	6.3410	6.9028	7.6445
.83	5.9541	5.9550	5.9695	6.0668	6.3724	6.9277	7.6638
.84	5.9944	5.9953	6.0095	6.1049	6.4053	6.9539	7.6841
.85	6.0365	6.0373	6.0512	6.1446	6.4398	6.9814	7.7056
.86	6.0804	6.0812	6.0948	6.1862	6.4761	7.0105	7.7285
.87	6.1265	6.1273	6.1406	6.2300	6.5144	7.0414	7.7528
.88	6.1751	6.1759	6.1889	6.2763	6.5551	7.0745	7.7789
.89	6.2267	6.2275	6.2401	6.3254	6.5985	7.1099	7.8071
.90	6.2817	6.2825	6.2948	6.3780	6.6452	7.1482	7.8376
.91	6.3410	6.3417	6.3537	6.4347	6.6958	7.1900	7.8711
.92	6.4053	6.4060	6.4177	6.4964	6.7511	7.2359	7.9081
.93	6.4761	6.4768	6.4881	6.5645	6.8123	7.2872	7.9496
.94	6.5551	6.5558	6.5667	6.6406	6.8812	7.3452	7.9969
.95	6.6452	6.6459	6.6563	6.7276	6.9604	7.4124	8.0521
.96	6.7511	6.7517	6.7617	6.8300	7.0542	7.4927	8.1184
.97	6.8812	6.8818	6.8913	6.9563	7.1705	7.5932	8.2022
.98	7.0542	7.0547	7.0637	7.1246	7.3268	7.7297	8.3172
.99	7.3268	7.3273	7.3354	7.3909	7.5762	7.9508	8.5061

Table 5. Truncated logits

Standardised lower point of truncation

P	2.0	3.0	4.0	5.0	6.0	7.0
.01	2.1931	3.0813	4.0369	5.0200	6.0137	7.0114
.02	2.3579	3.1580	4.0731	5.0400	6.0275	7.0229
.03	2.5021	3.2307	4.1089	5.0600	6.0414	7.0345
.04	2.6305	3.2998	4.1440	5.0800	6.0554	7.0462
.05	2.7466	3.3657	4.1787	5.1001	6.0695	7.0580
.06	2.8527	3.4288	4.2130	5.1201	6.0837	7.0700
.07	2.9505	3.4895	4.2468	5.1402	6.0980	7.0820
.08	3.0415	3.5478	4.2802	5.1603	6.1124	7.0942
.09	3.1267	3.6041	4.3132	5.1805	6.1269	7.1064
.10	3.2068	3.6586	4.3458	5.2007	6.1415	7.1188
.11	3.2826	3.7114	4.3781	5.2209	6.1562	7.1313
.12	3.3546	3.7627	4.4101	5.2412	6.1710	7.1439
.13	3.4233	3.8125	4.4419	5.2615	6.1860	7.1567
.14	3.4890	3.8611	4.4733	5.2819	6.2010	7.1696
.15	3.5520	3.9084	4.5045	5.3023	6.2162	7.1826
.16	3.6127	3.9547	4.5355	5.3228	6.2315	7.1958
.17	3.6712	4.0000	4.5662	5.3433	6.2470	7.2091
.18	3.7278	4.0443	4.5968	5.3640	6.2626	7.2225
.19	3.7827	4.0878	4.6271	5.3847	6.2783	7.2361
.20	3.8360	4.1305	4.6573	5.4055	6.2941	7.2499
.21	3.8878	4.1725	4.6873	5.4263	6.3101	7.2637
.22	3.9383	4.2138	4.7172	5.4473	6.3263	7.2778
.23	3.9876	4.2544	4.7470	5.4684	6.3426	7.2920
.24	4.0358	4.2945	4.7767	5.4895	6.3590	7.3064
.25	4.0830	4.3340	4.8062	5.5108	6.3757	7.3210

P	2.0	3.0	4.0	5.0	6.0	7.0
.26	4.1292	4.3731	4.8357	5.5322	6.3925	7.3357
.27	4.1746	4.4117	4.8651	5.5537	6.4094	7.3506
.28	4.2192	4.4498	4.8945	5.5754	6.4265	7.3657
.29	4.2631	4.4876	4.9238	5.5971	6.4439	7.3810
.30	4.3062	4.5250	4.9530	5.6190	6.4614	7.3965
.31	4.3488	4.5621	4.9823	5.6411	6.4791	7.4122
.32	4.3908	4.5989	5.0115	5.6633	6.4970	7.4281
.33	4.4323	4.6355	5.0408	5.6857	6.5151	7.4442
.34	4.4734	4.6718	5.0700	5.7082	6.5334	7.4605
.35	4.5140	4.7079	5.0993	5.7309	6.5519	7.4771
.36	4.5542	4.7438	5.1287	5.7538	6.5707	7.4939
.37	4.5940	4.7795	5.1581	5.7768	6.5897	7.5109
.38	4.6336	4.8151	5.1875	5.8001	6.6089	7.5282
.39	4.6728	4.8506	5.2171	5.8236	6.6284	7.5457
.40	4.7118	4.8860	5.2467	5.8473	6.6481	7.5635
.41	4.7506	4.9213	5.2764	5.8712	6.6681	7.5816
.42	4.7893	4.9565	5.3063	5.8954	6.6884	7.6000
.43	4.8277	4.9918	5.3363	5.9198	6.7090	7.6187
.44	4.8660	5.0270	5.3665	5.9445	6.7298	7.6377
.45	4.9043	5.0623	5.3968	5.9694	6.7510	7.6570
.46	4.9424	5.0976	5.4273	5.9946	6.7725	7.6766
.47	4.9805	5.1329	5.4580	6.0201	6.7944	7.6965
.48	5.0186	5.1683	5.4889	6.0460	6.8165	7.7169
.49	5.0568	5.2039	5.5201	6.0721	6.8391	7.7376
.50	5.0949	5.2395	5.5514	6.0986	6.8620	7.7586

P	2.0	3.0	4.0	5.0	6.0	7.0
.51	5.1332	5.2754	5.5831	6.1255	6.8853	7.7801
.52	5.1715	5.3114	5.6150	6.1527	6.9090	7.8020
.53	5.2099	5.3476	5.6473	6.1803	6.9331	7.8243
.54	5.2485	5.3840	5.6799	6.2083	6.9577	7.8471
.55	5.2873	5.4207	5.7128	6.2368	6.9828	7.8703
.56	5.3263	5.4576	5.7461	6.2657	7.0083	7.8940
.57	5.3656	5.4949	5.7798	6.2950	7.0343	7.9183
.58	5.4051	5.5325	5.8140	6.3249	7.0609	7.9431
.59	5.4450	5.5705	5.8486	6.3553	7.0880	7.9684
.60	5.4852	5.6089	5.8836	6.3863	7.1157	7.9944
.61	5.5258	5.6477	5.9192	6.4178	7.1441	8.0209
.62	5.5668	5.6870	5.9554	6.4500	7.1730	8.0482
.63	5.6083	5.7268	5.9921	6.4828	7.2027	8.0761
.64	5.6503	5.7672	6.0295	6.5163	7.2331	8.1047
.65	5.6928	5.8082	6.0675	6.5506	7.2642	8.1341
.66	5.7360	5.8498	6.1063	6.5856	7.2962	8.1644
.67	5.7799	5.8922	6.1458	6.6215	7.3290	8.1955
.68	5.8244	5.9353	6.1862	6.6582	7.3627	8.2275
.69	5.8698	5.9792	6.2274	6.6959	7.3974	8.2605
.70	5.9160	6.0241	6.2696	6.7346	7.4331	8.2945
.71	5.9632	6.0699	6.3129	6.7744	7.4699	8.3296
.72	6.0113	6.1167	6.3572	6.8153	7.5079	8.3659
.73	6.0606	6.1647	6.4027	6.8575	7.5472	8.4035
.74	6.1111	6.2139	6.4495	6.9010	7.5878	8.4425
.75	6.1629	6.2645	6.4977	6.9459	7.6300	8.4830

P	2.0	3.0	4.0	5.0	6.0	7.0
.76	6.2161	6.3166	6.5475	6.9924	7.6737	8.5250
.77	6.2710	6.3702	6.5988	7.0407	7.7191	8.5688
.78	6.3275	6.4257	6.6520	7.0907	7.7664	8.6145
.79	6.3860	6.4830	6.7072	7.1429	7.8158	8.6622
.80	6.4467	6.5426	6.7646	7.1972	7.8674	8.7122
.81	6.5097	6.6045	6.8244	7.2541	7.9215	8.7648
.82	6.5753	6.6691	6.8870	7.3136	7.9784	8.8200
.83	6.6439	6.7367	6.9525	7.3763	8.0384	8.8784
.84	6.7158	6.8076	7.0214	7.4423	8.1019	8.9403
.85	6.7915	6.8823	7.0942	7.5123	8.1692	9.0060
.86	6.8716	6.9614	7.1714	7.5867	8.2410	9.0762
.87	6.9566	7.0455	7.2536	7.6662	8.3179	9.1515
.88	7.0475	7.1355	7.3417	7.7515	8.4007	9.2328
.89	7.1452	7.2323	7.4367	7.8439	8.4905	9.3210
.90	7.2511	7.3373	7.5399	7.9444	8.5886	9.4175
.91	7.3669	7.4523	7.6531	8.0550	8.6967	9.5241
.92	7.4950	7.5796	7.7787	8.1781	8.8172	9.6431
.93	7.6388	7.7225	7.9200	8.3168	8.9535	9.7778
.94	7.8031	7.8860	8.0818	8.4761	9.1104	9.9332
.95	7.9955	8.0776	8.2718	8.6636	9.2955	10.1167
.96	8.2286	8.3099	8.5025	8.8918	9.5213	10.3410
.97	8.5262	8.6067	8.7976	9.1846	9.8117	10.6299
.98	8.9414	9.0212	9.2106	9.5951	10.2199	11.0366
.99	9.6442	9.7233	9.9111	10.2933	10.9157	11.7309