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A LAW OF ITERATED LOGARITHM FOR ONE-SAMPLE RANK ORDER STATISTICS AND AN APPLICATION[1]

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For one sample rank order statistics, a law of iterated logarithm and almost sure convergence to Wiener processes are established here. For the one-sample location problem, a sequential test procedure based on rank order statistics is proposed, and with the aid of the earlier results, it is shown that this has power one and arbitrarily small type I error.

1. Summary and introduction. Motivated by a martingale property of one-sample rank order statistics and the recent works of Strassen (1967) and Stout (1970) on martingales, we consider here the following problems. First, under minimal assumptions on the score function, a law of iterated logarithm for one-sample rank order statistics is derived (see Theorem 2.1). Certain bounds and asymptotic expressions for probabilities of moderate deviations for these statistics are also derived (see Theorems 2.2 and 2.3). Second, under slightly more stringent regularity conditions, almost sure (a.s.) convergence of one-sample rank order statistics to appropriate Wiener processes is studied (see Theorem 2.4). The proofs of these results are outlined in Section 3. In the last section, for the classical one-sample location problem, a sequential test procedure based on one-sample rank order statistics is proposed along the lines of Darling and Robbins (1968). Such tests have zero type II error and arbitrarily small type I error. Results of Section 2 along with those in Sen (1970) on the strong convergence of rank order statistics are utilized in the study of the properties of the proposed test procedure.

2. Statement of the main theorems. Let $\{X_i, i \geqq 1\}$ be a sequence of independent and identically distributed random variables (i.i.d. rv) defined on a measure space $(\Omega, \mathscr{A}, P)$, having a distribution function (df) $F_\theta(x) = F(x-\theta)$, where θ is a location parameter (unknown) and $F \in \mathscr{F}_0$, the class of all df's continuous with respect to Lebesgue measure and symmetric about 0, i.e., for every $F \in \mathscr{F}_0$, $F(x) + F(-x) = 1$ for all real x. For each positive integer n, let

$$(2.1)\qquad \mathbf{X}_n = (X_1, \cdots, X_n), \qquad R_{ni} = \tfrac{1}{2} + \textstyle\sum_{j=1}^{n} c(|X_i| - |X_j|), \quad i = 1, \cdots, n,$$

where $c(u) = 1, \frac{1}{2}$ or 0 according as u is $>, =$ or < 0. Also, let

$$(2.2)\qquad T_n = T(\mathbf{X}_n) = n^{-1} \textstyle\sum_{i=1}^{n} \operatorname{sgn} X_i J_n(R_{ni}/(n+1)), \qquad \operatorname{sgn} u = 2c(u) - 1,$$

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where $J_n(i/(n+1)) = EJ(U_{ni})$, $i = 1, \cdots, n$; $U_{n1} < \cdots < U_{nn}$ are n ordered random variables (rv) from a rectangular (0, 1) df, and $J(u)$, $0 < u < 1$, is a non-decreasing score function satisfying the condition that

$$(2.3) \qquad 0 < A^2 = \int_0^1 J^2(u)\, du < \infty .$$

Note that, by definition, $J_n(i/(n+1)) = i\binom{n}{i} \int_0^1 J(u) u^{i-1}(1-u)^{n-i}\, du$, $i = 1, \cdots, n$, so that by some routine steps

$$(2.4) \qquad J_n(i/(n+1)) = [i/(n+1)]J_{n+1}((i+1)/(n+2)) + [(n-i+1)/(n+1)]J_{n+1}(i/(n+2)),$$

for $i = 1, \cdots, n$. In passing, we may remark that for $J(u) = u$ and $\Phi^{-1}((1+u)/2)$, $\Phi(x)$ being the standard normal df, the corresponding T_n are the *Wilcoxon signed rank* and the *normal scores statistics.*

THEOREM 2.1. *If* $F \in \mathscr{F}_0$, $\theta = 0$ *and* (2.3) *holds, then*

$$(2.5) \qquad \limsup_{n\to\infty} n^{\frac12} T_n [A^2(2 \log\log n)]^{-\frac12} = 1 \quad \text{a.s.},$$

$$(2.6) \qquad \liminf_{n\to\infty} n^{\frac12} T_n [A^2(2 \log\log n)]^{-\frac12} = -1 \quad \text{a.s.}$$

The proof of the theorem is based on a martingale property of $\{nT_n, n \geqq 1\}$ and a second theorem which we state below. Let $\psi(t)$ be a non-decreasing positive function on $[0, \infty)$ such that there exists a positive t_0 $(< \infty)$ for which

$$(2.7) \qquad \gamma(t) = \psi^2(t) - \log\log t \ \text{ is nonnegative and non-decreasing in } \ t\ (\geqq t_0).$$

THEOREM 2.2. *If* $F \in \mathscr{F}_0$, $\theta = 0$, *and* (2.3) *and* (2.7) *hold, then for every* $\varepsilon > 0$, *there exist positive numbers* K $(< \infty)$ *and* η, *such that*

$$(2.8) \qquad P\{m^{\frac12} T_m \geqq A[2(1+\varepsilon)]^{\frac12}\psi(m), \ \textit{for some} \ \ m \geqq n\} \leqq K\{\exp[-\gamma(n) - \eta\psi^2(n)]\}, \quad \textit{for every} \ \ n \geqq t_0,$$

$$(2.9) \qquad P\{m^{\frac12} T_m \leqq -A[2(1+\varepsilon)]^{\frac12}\psi(m), \ \textit{for some} \ \ m \geqq n\} \leqq K\{\exp[-\gamma(n) - \eta\psi^2(n)]\}, \quad \textit{for every} \ \ n \geqq t_0 .$$

In fact, in the same fashion as in Cramér (1938), we may provide an asymptotic expression for the left-hand side of (2.8) and (2.9) under slightly more stringent regularity conditions.

THEOREM 2.3. *If* $J \in L_r$ *for some* $r > 2$, $F \in \mathscr{F}_0$, $\theta = 0$, *and in addition to* (2.7),

$$(2.10) \qquad \lim_{t\to\infty} (\log\log t)/\psi^2(t) = 0 \qquad \textit{and} \qquad t^{-\frac12}\psi(t) = o(t^{-\frac14}),$$

then as $n \to \infty$,

$$(2.11) \qquad [\psi(n)]^{-2} \log P\{m^{\frac12} T_m > A\psi(m), \ \textit{for some} \ \ m \geqq n\} \to -\tfrac12 ,$$

$$(2.12) \qquad [\psi(n)]^{-2} \log P\{m^{\frac12} T_m < -A\psi(m), \ \textit{for some} \ \ m \geqq n\} \to -\tfrac12 .$$

Finally, we note that if we assume that $J \in L_r$ for some $r > 2$, then, as in Theorem 4.4 of Strassen (1967), we may strengthen the results of Theorem 2.1 as follows.

THEOREM 2.4. *If $J \in L_r$ for some $r > 2$, $F \in \mathscr{F}_0$ and $\theta = 0$, then on defining $T_0 = 0$ and*

$$\tilde{T}_t = [t]T_{[t]} + (t - [t])(T_{[t+1]} - T_{[t]}), \qquad t > 0, \tag{2.13}$$

($[s]$ being the integer part of $s\ (\geq 0)$), there exists a standard Brownian motion $\xi = \{\xi(t) : 0 \leq t < \infty\}$, such that

$$\tilde{T}_t = A\xi(t) + o[(t^s(\log t)^{1+\eta}] \quad \text{a.s.}, \qquad \text{as } t \to \infty, \eta > 0, \tag{2.14}$$

where $s = (r + 2)/4r\ (< \frac{1}{2})$. In fact, (2.14) implies (2.5) and (2.6).

3. Proofs of the theorems. For convenience, we start with the proof of Theorem 2.2. Let $\mathscr{B}_n$ be the σ-field generated by $(\mathbf{S}_n, \mathbf{R}_n)$, where $\mathbf{S}_n = (\text{sgn } X_1, \cdots, \text{sgn } X_n)$ and $\mathbf{R}_n = (R_{n1}, \cdots, R_{nn})$, $n \geq 1$; clearly, $\mathscr{B}_n$ is $\uparrow$ in n. Note that for $F \in \mathscr{F}_0$, $\theta = 0$, $\mathbf{S}_n$ and $\mathbf{R}_n$ are stochastically independent [viz., Hájek and Šidák (1967, page 40)]. Write $\tilde{T}_n = nT_n$ for $n \geq 0$ and E_0 for $E_{\mathscr{F}_0}$. Then, by (2.2),

$$E_0(\tilde{T}_n) = 0,$$

$$E_0(\tilde{T}_n^2) = nA_n^2; \qquad A_n^2 = n^{-1} \sum_{i=1}^n J_n^2(i/(n+1)) \leq A^2 < \infty, \quad \text{for all } n \geq 1. \tag{3.1}$$

Also, conditional on $\mathscr{B}_n$, $R_{n+1\,n+1}$ can assume all the values $1, \cdots, n+1$ with the common probability $1/(n+1)$, and sgn X_{n+1} can assume the values $+1$ and -1 with the same probability $\frac{1}{2}$, independently of $R_{n+1\,n+1}$. Finally, conditional on $\mathscr{B}_n$, $R_{n+1\,i}$ $(1 \leq i \leq n)$ can assume the two values R_{ni} and $R_{ni} + 1$ with respective probabilities $(n + 1 - R_{ni})/(n+1)$ and $R_{ni}/(n+1)$. Hence, for every $n \geq 1$,

$$\begin{aligned} E_0(\tilde{T}_{n+1} \mid \mathscr{B}_n) &= \sum_{i=1}^n \text{sgn } X_i E_0[J_{n+1}(R_{n+1\,i}/(n+2)) \mid \mathscr{B}_n] \\ &\quad + E_0[\text{sgn } X_{n+1} J_{n+1}(R_{n+1\,n+1}/(n+2)) \mid \mathscr{B}_n] \\ &= \sum_{i=1}^n \text{sgn } X_i E_0[J_{n+1}(R_{n+1\,i}/(n+2)) \mid \mathscr{B}_n] \\ &= \sum_{i=1}^n (\text{sgn } X_i)\{[1 - (n+1)^{-1}R_{ni}]J_{n+1}(R_{ni}/(n+2)) \\ &\quad + (n+1)^{-1}R_{ni}J_{n+1}((R_{ni}+1)/(n+2))\} \\ &= \sum_{i=1}^n \text{sgn } X_i J_n(R_{ni}/(n+1)) = \tilde{T}_n, \qquad \text{by (2.4)}. \end{aligned} \tag{3.2}$$

Thus, $\{\tilde{T}_n, \mathscr{B}_n, n \geq 1\}$ forms a martingale sequence when $F \in \mathscr{F}_0$ and $\theta = 0$.

Note that, by definition, $\psi^*(n) = n^{\frac{1}{2}}\psi(n)$ is increasing in n, and by (3.2), for every real t, $\{\exp(t\tilde{T}_n), \mathscr{B}_n, n \geq 1\}$ is a nonnegative submartingale on which the classical Kolmogorov inequality yields the following:

$$\begin{aligned} &P_0\{\tilde{T}_m \geq (2A^2(1+\varepsilon))^{\frac{1}{2}}\psi^*(m), \text{ for some } m \geq n\} \\ &\quad \leq \sum_{k=0}^\infty P_0\{\max_{n_k \leq m < n_{k+1}} \exp(t_k \tilde{T}_m) \geq \exp[t_k\{2A^2(1+\varepsilon)\}^{\frac{1}{2}}\psi^*(n_k)]\} \\ &\quad \leq \sum_{k=0}^\infty \exp\{-t_k(2A^2(1+\varepsilon))^{\frac{1}{2}}\psi^*(n_k)\}E_0\{\exp(t_k\tilde{T}_{n_{k+1}})\}, \end{aligned} \tag{3.3}$$

where we let $n_k = [n(1+\varepsilon/2)^k]$ and $t_k = (2(1+\varepsilon))^{\frac{1}{2}}\psi^*(n_k)/An_k$, for $k = 0, 1, \cdots$, and P_0 denotes the probability computed under the hypothesis that $F \in \mathscr{F}_0$ and $\theta = 0$. Also, note that for $F \in \mathscr{F}_0$, $\theta = 0$, $\mathbf{S}_n$ and $\mathbf{R}_n$ are stochastically independent and that for every real x, $(e^x + e^{-x})/2 \leq \exp(x^2/2)$. Hence, on using

(3.1), we obtain that for every $t \geqq 0$,

$$\begin{aligned} E_0[\exp(t\hat{T}_n)] &= E_0[\exp\{t \textstyle\sum_{i=1}^n \operatorname{sgn} X_i J_n(R_{ni}/(n+1))\}] \\ &= E_0[E(\textstyle\prod_{i=1}^n \exp[t \operatorname{sgn} X_i J_n(R_{ni}/(n+1))] \mid \mathbf{R}_n)] \\ &= E_0[\textstyle\prod_{i=1}^n \{\frac{1}{2}[\exp\{tJ_n(R_{ni}/(n+1))\} \\ &\qquad + \exp\{-tJ_n(R_{ni}/(n+1))\}]\}] \\ &\leqq E_0[\textstyle\prod_{i=1}^n \exp\{\frac{1}{2}t^2J_n^2(R_{ni}/(n+1))\}] \\ &= E_0[\textstyle\prod_{i=1}^n \exp\{\frac{1}{2}t^2J_n^2(i/(n+1))\}] \\ &= \exp[(t^2/2)\textstyle\sum_{i=1}^n J_n^2(i/(n+1))] \\ &= \exp(nt^2A_n^2/2) \leqq \exp(nt^2A^2/2)\,. \end{aligned} \tag{3.4}$$

Therefore, by (3.3) and (3.4),

$$\begin{aligned} &P_0\{\hat{T}_m \geqq [2A^2(1+\varepsilon)]^{\frac{1}{2}}\psi^*(m), \text{ for some } m \geqq n\} \\ &\quad\leqq \textstyle\sum_{k=0}^\infty \exp\{-2(1+\varepsilon)\psi^2(n_k) + (n_{k+1}/n_k)(1+\varepsilon)\psi^2(n_k)\} \\ &\quad\doteqdot \textstyle\sum_{k=0}^\infty \exp\{-(1+\eta)\psi^2(n_k)\}\,, \quad \eta = \varepsilon(1-\varepsilon)/2\,. \end{aligned} \tag{3.5}$$

(Note that for $0 < \varepsilon < 1$, $0 < \eta < \varepsilon/2$.) Then, by (2.7), the right-hand side of (3.5) can be expressed as

$$\begin{aligned} &\textstyle\sum_{k=0}^\infty \exp\{-(1+\eta)\gamma(n_k) - (1+\eta)\log\log n_k\} \\ &\quad\leqq \exp[-(1+\eta)\gamma(n)]\textstyle\sum_{k=0}^\infty (\log n_k)^{-(1+\eta)} \qquad (\text{for } n \geqq t_0) \\ &\quad\doteqdot \exp[-(1+\eta)\gamma(n)]\textstyle\sum_{k=0}^\infty [\log n + k\log(1+\varepsilon/2)]^{-(1+\eta)} \\ &\quad= \exp[-(1+\eta)\gamma(n)][O((\log n)^{-\eta})] \\ &\quad= \{\exp[-\eta\psi^2(n) - \gamma(n) + \eta\log\log n]\}\{O((\log n)^{-\eta})\} \\ &\quad= \{\exp[-\gamma(n) - \eta\psi^2(n)]\}[O(1)]\,, \end{aligned} \tag{3.6}$$

which completes the proof of (2.8). The proof of (2.9) follows on the same line by working with $-T_n$ instead of T_n. Hence, the proof of Theorem 2.2 is complete.

Returning now to the proof of Theorem 2.1, we only prove (2.5), as (2.6) follows similarly. By letting $\psi^2(n) = \log\log n$, we have from (2.8) that for every $\varepsilon > 0$ and $n \geqq t_0$,

$$\begin{aligned} &P_0\{m^{\frac{1}{2}}T_m \geqq A(2(1+\varepsilon)\log\log m)^{\frac{1}{2}} \text{ for some } m \geqq n\} \\ &\quad\leqq K(\log n)^{-\eta}\,, \qquad (\to 0 \text{ as } n\to\infty)\,; \quad \eta = \varepsilon(1-\varepsilon)/2\,, \end{aligned} \tag{3.7}$$

and hence,

$$\limsup_{n\to\infty} \{n^{\frac{1}{2}}T_n(2A^2\log\log n)^{-\frac{1}{2}}\} \leqq 1 \quad \text{a.s.} \tag{3.8}$$

So, to complete the proof of (2.5), we only need to show that

$$\limsup_{n\to\infty} [n^{\frac{1}{2}}T_n(2A^2\log\log n)^{-\frac{1}{2}}] \geqq 1 \quad \text{a.s.} \tag{3.9}$$

For this, define $Z_1 = T_1$ and $Z_k = \hat{T}_k - \hat{T}_{k-1}$, $k \geqq 2$, and let

$$V_n = E_0(Z_1^2) + \textstyle\sum_{k=2}^n E_0(Z_k^2 \mid \mathscr{B}_{k-1})\,. \tag{3.10}$$

Then, (i) $E_0(Z_1) = 0$, $E_0(Z_k \mid \mathscr{B}_{k-1}) = 0$ [by (3.2)], (ii) $0 \leqq E_0(Z_1^2) = A_1^2 \leqq A^2 < \infty$ [by (3.1)], and (iii) $E_0(Z_k^2 \mid \mathscr{B}_{k-1}) = E_0([\sum_{i=1}^{k-1} \operatorname{sgn} X_i\{J_k(R_{ki}/(k+1)) - J_{k-1}(R_{k-1i}/k)\} + \operatorname{sgn} X_k J_k(R_{kk}/(k+1))]^2 \mid \mathscr{B}_{k-1}) = E_0([\sum_{i=1}^{k-1} \operatorname{sgn} X_i\{J_k(R_{ki}/(k+1)) - J_{k-1}(R_{k-1i}/k)\}]^2 \mid \mathscr{B}_{k-1}) + E_0\{J_k^2(R_{kk}/(k+1)) \mid \mathscr{B}_{k-1}\} + 0 \geqq E_0\{J_k^2(R_{kk}/(k+1)) \mid \mathscr{B}_{k-1}\} = A_k^2$, $k \geqq 2$, as $\mathbf{S}_k$ and $\mathbf{R}_k$ are stochastically independent, $E_0(\operatorname{sgn} X_k \mid \mathscr{B}_{k-1}) = 0$ and $E_0\{J_k^2(R_{kk}/(k+1)) \mid \mathscr{B}_{k-1}\} = k^{-1} \sum_{i=1}^{k} J_k^2(i/(k+1)) = A_k^2$, $k \geqq 2$. Thus, on noting that

$$d_n^2 = [\textstyle\sum_{i=1}^{n} A_i^2]/(nA^2) \to 1 \qquad \text{as } n \to \infty, \tag{3.11}$$

we have from (3.10) through (3.11) that

$$V_n \geqq \textstyle\sum_{i=1}^{n} A_i^2 = (nA^2)d_n^2 \to \infty \qquad \text{as } n \to \infty. \tag{3.12}$$

Let us now define $u_n^2 = \max[1, 2 \log\log V_n]$ and $K_n = 4d_n^{-1}u_n^{-\frac{1}{2}}$, so that both u_n and K_n are $\mathscr{B}_{n-1}$ measurable with

$$K_n \to 0 \quad \text{as} \quad n \to \infty, \qquad \text{by (3.12).} \tag{3.13}$$

Further, by assumption, $J(u)$ is non-decreasing, so that by (2.4),

$$\begin{aligned} J_n(i/(n+1)) &\leqq J_{n-1}(i/n) \\ &\leqq J_n((i+1)/(n+1)), \qquad \text{for all } i = 1, \cdots, n-1. \end{aligned} \tag{3.14}$$

Then, $|Z_1| = |J_1(\frac{1}{2})| = A_1 < A$, and for $k \geqq 2$,

$$\begin{aligned} |Z_k| &= |\hat{T}_k - \hat{T}_{k-1}| \\ &\leqq \textstyle\sum_{i=1}^{k-1} |J_k(R_{ki}/(k+1)) - J_{k-1}(R_{k-1i}/k)| + |J_k(R_{kk}/(k+1))| \\ &\leqq \textstyle\sum_{i=1}^{k-1} \max\{|J_k(i/(k+1)) - J_{k-1}(i/k)|, |J_k((i+1)/(k+1)) \\ &\qquad - J_{k-1}(i/(k+1))|\} + \max\{|J_k(1/(k+1))|, |J_k(k/(k+1))|\} \\ &\leqq \textstyle\sum_{i=1}^{k-1} [J_k(i+1)/(k+1)) - J_k(1/(k+1))] \\ &\qquad + \max\{|J_k(1/(k+1))|, |J_k(k/(k+1))|\} \\ &\leqq 2\{|J_k(k/(k+1))| + |J_k(1/(k+1))|\}. \\ &= 2k|\textstyle\int_0^1 J(u)u^{k-1}\,du| + 2k|\int_0^1 J(u)(1-u)^{k-1}\,du| \\ &\leqq 4kA/(2k-1)^{\frac{1}{2}}, \qquad \text{by the Schwarz inequality.} \end{aligned} \tag{3.15}$$

Thus, by (3.11), (3.12), (3.13) and (3.15), we obtain that

$$|Z_n| \leqq K_n V_n^{\frac{1}{2}} u_n \qquad \text{for all } n \geqq 1, \tag{3.16}$$

where K_n is $\mathscr{B}_{n-1}$ measurable and converges to 0 as $n \to \infty$, and $V_n \to \infty$ as $n \to \infty$. Hence, by Theorem 2 of Stout (1970), we obtain that

$$\limsup_{n\to\infty} \{\hat{T}_n[2V_n \log\log V_n]^{-\frac{1}{2}}\} \geqq 1. \tag{3.17}$$

On the other hand,

$$\begin{aligned} &\hat{T}_n[2nA^2 \log\log n]^{-\frac{1}{2}} \\ &\quad = \{\hat{T}_n[2V_n \log\log V_n]^{-\frac{1}{2}}\}\{[V_n \log\log V_n]/[nA^2 \log\log n]\}^{\frac{1}{2}}, \end{aligned} \tag{3.18}$$

where by (3.11) and (3.12), for every $\eta > 0$, there exists a positive integer $n_0(\eta)$, such that for $n \geqq n_0(\eta)$, the second factor on the right-hand side of (3.18) can be made $\geqq 1 - \eta$. Consequently, (3.9) follows from (3.17) and (3.18), and the proof of Theorem 2.1 is complete.

Since, under (2.10), $\gamma(n)/\psi^2(n) \to 1$ as $n \to \infty$, by choosing ε (> 0) sufficiently small and noting that for $0 < \varepsilon < 1$, $\eta = \varepsilon(1-\varepsilon)/2$, we obtain from (2.8) that

$$\lim_{n\to\infty} ([\psi(n)]^{-2} \log P_0\{m^{\frac{1}{2}}T_m > A\psi(m) \text{ for some } m \geqq n\}) \leqq -\tfrac{1}{2}. \tag{3.19}$$

So, to prove (2.11), it suffices to show that

$$\lim_{n\to\infty} ([\psi(n)]^{-2} \log P_0\{m^{\frac{1}{2}}T_m > A\psi(m) \text{ for some } m \geqq n\}) \geqq -\tfrac{1}{2}. \tag{3.20}$$

For this, we note that

$$\begin{aligned} &P_0\{m^{\frac{1}{2}}T_m > A\psi(m) \text{ for some } m \geqq n\} \\ &\qquad \geqq P_0\{n^{\frac{1}{2}}T_n > A\psi(n)\} = P_0\{\hat{T}_n > A\psi^*(n)\} \\ &\qquad = E[P_0\{\textstyle\sum_{i=1}^n \operatorname{sgn} X_i J_n(R_{ni}/(n+1)) > A\psi^*(n) \mid \mathbf{R}_n\}] \\ &\qquad = P_0[\textstyle\sum_{i=1}^n \operatorname{sgn} X_i J_n(i/(n+1)) > A\psi^*(n)], \end{aligned} \tag{3.21}$$

as conditional on $\mathbf{R}_n$, the distribution of $\hat{T}_n$ is generated by the 2^n equally likely sign inversions of $\mathbf{S}_n$, and hence, agrees with the distribution of $T_n^* = \sum_{i=1}^n \operatorname{sgn} X_i J_n(i/(n+1))$ ($= \sum_{i=1}^n U_{ni}$, say), whose distribution does not depend on $\mathbf{R}_n$, and for which

$$P\{U_{ni} = \pm J_n(i/(n+1))\} = \tfrac{1}{2} \qquad \text{for } i = 1, \cdots, n. \tag{3.22}$$

Thus, $EU_{ni} = 0$, $i = 1, \cdots, n$, and

$$n^{-1} \textstyle\sum_{i=1}^n EU_{ni}^2 = A_n^2 \quad (\leqq A^2); \qquad \lim_{n\to\infty} A_n^2 = A^2, \quad 0 < A^2 < \infty. \tag{3.23}$$

Let us write $\nu_n^2 = nA_n^2(\log nA_n^2)^{-3}$, so that

$$\nu_n = O(n^{\frac{1}{2}}(\log n)^{-\frac{3}{2}}). \tag{3.24}$$

On the other hand, by the assumption that $J \in L_r$, $r > 2$, and the Hölder inequality,

$$\max_{1\leqq i\leqq n}\left|J_n\left(\frac{i}{n+1}\right)\right| \leqq n[\textstyle\int_0^1 |J(u)|^r\, du]^{r^{-1}}[\int_0^1 u^{s(n-1)}\, du]^{s^{-1}} \leqq Kn^{1/r}, \tag{3.25}$$

where $1/r + 1/s = 1$ and $K < \infty$. Consequently, by (3.24) and (3.25), for every $\varepsilon > 0$, there exists a positive integer $n_0(\varepsilon)$, such that for $n \geqq n_0(\varepsilon)$, $\max_{1\leqq i\leqq n} |J_n(i/(n+1))| \leqq \varepsilon\nu_n$, so that by (3.22),

$$P\{|U_{ni}| > \varepsilon\nu_n\} = 0 \qquad \text{for all } i = 1, \cdots, n \quad (\geqq n_0(\varepsilon)). \tag{3.26}$$

Finally, as in (3.4), for every real t,

$$E_0[\exp(t \textstyle\sum_{i=1}^n U_{ni})] \leqq \exp(nt^2A^2/2). \tag{3.27}$$

Hence, by the same technique as in Rubin and Sethuraman (1965), one can extend a classical result of Cramer (1938) to a double sequence of random variables, and obtains that under (2.10),

$$[P_0\{T_n^* > A\psi^*(n)\}]/[1 - \Phi(\psi(n))] \to 1 \qquad \text{as } n \to \infty, \tag{3.28}$$

where $\Phi(x)$ is the standard normal df, so that for large x,

$$1 - \Phi(x) = (2\pi)^{-\frac{1}{2}}[\exp(-x^2/2)]x^{-1}\{1 + O(x^{-2})\}\,. \tag{3.29}$$

Thus, (3.20) follows directly from (3.21), (3.28) and (3.29), and this completes the proof of (2.11). (2.12) follows on similar lines. Hence, the proof of Theorem 2.5 is complete.

Now, proceeding as in (3.15) but using (3.25), it follows that for $J \in L_r$, for some $r > 2$,

$$|Z_n| < KAn^{1/r}\,, \qquad \text{for every } n \geqq 1, K < \varepsilon\,, \tag{3.30}$$

and hence, if we define a function $g(t)$ on $[0, \infty)$ by

$$g(t) = t^{2/r}(\log t)^{\eta}\,, \qquad \eta > 0\,, \tag{3.31}$$

it follows from (3.11), (3.12), (3.30) and (3.31) that

$$P\{Z_n^2 > \{g(V_n)\} \mid \mathscr{B}_{n-1}\} = 0 \quad \text{a.s.}\,, \qquad \text{as } n \to \infty\,. \tag{3.32}$$

Consequently, Theorem 2.4 can be proved directly from Theorem 4.4 of Strassen (1967). For brevity, the details are therefore omitted.

Theorem 2.4 is of interest in providing asymptotic expressions for the OC and ASN of sequential tests for $H_0: \theta = 0$ (vs. $H: \theta > 0$, say) based on $\{\hat{T}_n\}$ where the Wiener process approximation simplifies the expressions considerably.

4. Sequential rank order tests for location with power one. We start with the set up as in Section 2, and assume that $F \in \mathscr{F}_0$ and $J(u)$ is continuous and strictly increasing. Consider the null hypothesis

$$H_0: \theta = 0 \quad \text{vs. either} \quad H_1: \theta > 0 \quad \text{or} \quad H_2: \theta \neq 0\,. \tag{4.1}$$

It follows from Sen (1970) that if $J \in L_1$, then $\lim_{n\to\infty} T_n = \eta(\theta)$ a.s. (P_θ), where in our notations,

$$\eta(\theta) = 2\int_0^\infty J[F(x-\theta) - F(-x-\theta)]\,dF(x-\theta) - \int_0^1 J(u)\,du\,, \tag{4.2}$$

and the strict monotonicity of $J(u)$ implies that $\eta(\theta)$ is $>$, $=$ or $<$ 0 according as θ is $>$, $=$ or $<$ 0. Now, to test H_0 vs. H_1, define

(4.3) $N =$ first integer $n \geqq n_0$ such that $T_n \geqq n^{-1}c_n$; ∞ if no such n occurs,

where $\{c_n\}$ is some sequence of positive constants such that $n^{-1}c_n \to 0$ as $n \to \infty$. If H_0 is false, $T_n \to \eta(\theta)$ (> 0) a.s., as $n \to \infty$, and hence,

$$\begin{aligned} P_\theta(N = \infty) &= \lim_{n\to\infty} P_\theta(N > n) \\ &\leqq \lim_{n\to\infty} P_\theta(T_n < n^{-1}c_n) = 0\,. \end{aligned} \tag{4.4}$$

Hence, if we agree to reject H_0 as soon as we observe that $N < \infty$, while if $N = \infty$, we do not reject H_0; then since $P_\theta(N < \infty) = 1$ for every $\theta > 0$, the test has power 1. Again, when H_0 is true, by the law of iterated logarithm in

Section 2, if $\limsup_{n\to\infty} [(2A^2 n \log\log n)/c_n^2] \leq 1$, then,

$$(4.5)\qquad \begin{aligned} P_0(N < \infty) &= P_0(T_n \geq n^{-1}c_n \text{ for some } n \geq n_0) \\ &= P_0(n^{\frac12}T_n/(A(2n \log\log n)^{\frac12} \geq c_n/(2\log\log n)^{\frac12} \text{ for some } n \geq n_0) \\ &\leq P_0(n^{\frac12}T_n/(A(2\log\log n)^{\frac12}) \geq 1 \text{ for some } n \geq n_0) \end{aligned}$$

which can be made arbitrarily small by choosing n_0 adequately large. In fact, if we let $c_n = A(1+\varepsilon)[2n \log\log n]^{\frac12}$ where $\varepsilon > 0$, we obtain from Theorem 2.2 that the type I error can be bounded by $K(\log n_0)^{-\eta}$, $\eta = \varepsilon(1-\varepsilon)/2$, and this can be made smaller than any preassigned α, $0 < \alpha < 1$, by proper choice of n_0.

Now, as $J(u)$ is assumed to be continuous inside $(0, 1)$, we may as in Theorems 2 and 3 of Sen (1970), prove the following result:

If $J \in L_r$ for some $r > 2$, then for every (fixed) $c > 0$, there exist positive constants C $(< \infty)$ and $n_0(c)$, such that for $n \geq n_0(c)$,

$$(4.6)\qquad P_\theta\{|T_n - \eta(\theta)| > c\} \leq Cn^{-s},$$

where $s = \min(r-1, k)$ if $2(k-1) < r \leq 2k$, $k \geq 1$. If further, $E(\exp tJ(u)) < \infty$ for some $t > 0$, then, for every $\varepsilon > 0$, there exist positive constants C and $\rho(c)$, $0 < \rho(c) < 1$ and an n_0, such that for $n \geq n_0$,

$$(4.7)\qquad P_\theta\{|T_n - \eta(\theta)| > c\} \leq C[\rho(c)]^n .$$

Consequently, for $J \in L_r$, $r > 2$, denoting by

$$(4.8)\qquad m_0(\theta) = \{\min n : n^{-1}c_n - \eta(\theta) < -\eta(\theta)/2 = -c(\theta)\},$$

we have from (4.3) and (4.6), that for every (fixed) $\theta > 0$,

$$(4.9)\qquad \begin{aligned} P_\theta\{N > n\} &< P_\theta\{T_n < n^{-1}c_n\} \\ &\leq P_\theta\{|T_n - \eta(\theta)| > c(\theta)\} \\ &\leq Cn^{-s}, \qquad s > 1, \quad \text{for all } \quad n \geq n^* = \max\{m_0(\theta), n_0(c(\theta))\}, \end{aligned}$$

where n^* depends on θ and F, but is finite for every finite θ. Therefore,

$$(4.10)\qquad \begin{aligned} E_\theta(N) &= \textstyle\sum_{n=0}^\infty nP_\theta(N = n) = \sum_{n=0}^\infty P_\theta(N > n) \\ &\leq n^* + O(\textstyle\sum_{n=n^*+1}^\infty n^{-s}) = n^* + O((n^*)^{-s+1}) < \infty, \end{aligned}$$

for every (fixed) $\theta > 0$. Thus, for $F \in \mathscr{F}_0$, $\theta > 0$, and $J \in L_r$, $r > 2$, $E_\theta(N) < \infty$. Similarly, if $E(\exp\{tJ(u)\}) < \infty$ for some $t > 0$, we have by (4.7) and (4.10) that $E[\exp\{uN\}] < \infty$ for some $u > 0$.

We may remark that Darling and Robbins (1968) while dealing with the same testing problem involving the Kolmogorov-Smirnov statistics confined themselves to c_n behaving as $(1+\varepsilon)[n \log n]^{\frac12}$ instead of $(1+\varepsilon)[n \log\log n]^{\frac12}$. This difficulty can be avoided by means of some recent results of Brillinger (1969) on a.s. behavior of empirical processes. We may also remark that unlike the case of sample mean, we do not need the assumption of a finite moment generating function.

Finally, consider the testing problem H_0 vs. H_1. Define

(4.11) $N =$ first integer $n \geqq n_0$ such that $|T_n| \geqq n^{-1}c_n$; ∞ if no such n occurs,

where c_n is defined in the same manner as before. If H_1 holds, $T_n \to \eta(\theta)$ a.s. as $n \to \infty$, where $|\eta(\theta)| > 0$ for $\theta \neq 0$ [by (4.2)]. Hence, $|T_n| \to |\eta(\theta)| > 0$ a.s., as $n \to \infty$. Then, for every $\theta \neq 0$,

$$P_\theta(N = \infty) = \lim_{n\to\infty} P_\theta(N > n) \leqq \lim_{n\to\infty} P_\theta(|T_n| < n^{-1}c_n) = 0 . \tag{4.12}$$

Results analogous to (4.5) and (4.10) can be obtained in a similar manner.

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