Two Inequalities for the Perron Root

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ABSTRACT

If A, B are irreducible, nonnegative $n \times n$ matrices with a common right eigenvector and a common left eigenvector corresponding to their respective spectral radii $r(A)$, $r(B)$, then it is shown that for any $t \in [0,1]$, $r(tA+(1-t)B') \geqslant tr(A)+(1-t)r(B)$, where B' is the transpose of B. Another inequality is proved that involves $r(A)$ and $r(\Sigma_l D^l A E^l)$, where A is a nonnegative, irreducible matrix and D^l, E^l are positive definite diagonal matrices. These inequalities generalize previous results due to Levinger and due to Friedland and Karlin.

1. INTRODUCTION

The purpose of this paper is to prove two inequalities for the spectral radius of nonnegative matrices. These inequalities generalize previous results due to Levinger [3] and Friedland and Karlin [2].

Before giving a description of our results, let us recall the main aspects of the well-known Perron-Frobenius theory that will be used in the sequel. If A is a nonnegative $n \times n$ matrix, we will denote by $r(A)$ the spectral radius of A. If A is nonnegative, then $r(A)$ is an eigenvalue of A and we refer to $r(A)$ as the Perron root of A. Furthermore, A has nonnegative right and left eigenvectors corresponding to $r(A)$. If A is a nonnegative, irreducible matrix, then $r(A)>0$ and A has positive right and left eigenvectors corresponding to $r(A)$, which are unique up to a scalar multiple.

If A is a nonnegative, irreducible $n \times n$ matrix, then according to a result announced by Levinger [3], the function $\phi(t)=r(tA+(1-t)A')$ is either constant in $[0,1]$, or increasing in $(0,\frac{1}{2})$ and decreasing in $(\frac{1}{2},1)$. (Here A'

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denotes the transpose of A.) Furthermore, it is constant in $[0,1]$ if and only if A and A' have a common right eigenvector corresponding to $r(A)$. As noted in [4], there is no elementary proof of Levinger's result available in the literature.

LEMMA 1. *If A is a nonnegative $n\times n$ matrix, then for any t, $0\leqslant t\leqslant 1$, one has $r(tA+(1-t)A')\geqslant r(A)$. Furthermore, if A is irreducible and if $0<t<1$, then equality holds in the above inequality if and only if any right eigenvector of A corresponding to $r(A)$ is also a left eigenvector of A.*

It can be seen that Levinger's result mentioned above can be deduced from Lemma 1. For, if A is a nonnegative, irreducible $n\times n$ matrix and if $0<t_1<t_2<\frac{1}{2}$, then

$$\begin{aligned}\phi(t_2)&=r\big(t_2A+(1-t_2)A'\big)\\&=r\big(\alpha\big[t_1A+(1-t_1)A'\big]+(1-\alpha)\big[t_1A'+(1-t_1)A\big]\big),\end{aligned}$$

where $\alpha=(t_2+t_1-1)/(2t_1-1)$. Since $t_1A+(1-t_1)A'$ is irreducible, it follows by Lemma 1 that $\phi(t_2)>\phi(t_1)$. Similarly, it may be shown that ϕ is decreasing in $(\frac{1}{2},1)$. The result of Lemma 1 is generalized in our Theorem 3.

It has been shown by Friedland and Karlin [2] that if A is a nonnegative, irreducible $n\times n$ matrix with v and u as its right and left eigenvectors corresponding to $r(A)$ and if $D=\operatorname{diag}(d_1,\dots,d_n)$ is a positive definite diagonal matrix, then

$$r(DA)\geqslant r(A)\prod_i d_i^{u_iv_i}.$$

This result is considerably strengthened in Theorem 4.

The proofs of Theorems 3 and 4 are elementary and are based on a well-known inequality that has applications in information theory (see, for example, [5, p. 58]). For several other applications of the same inequality to nonnegative matrices, see [1].

2. RESULTS

The next inequality is known, but we include a short proof for the sake of completeness.

LEMMA 2. *If $x=(x_1,\dots,x_n)'$ and $y=(y_1,\dots,y_n)'$ are nonnegative, nonzero vectors, then*

$$\prod_i x_i^{x_i} \geqslant \left(\frac{\sum_i x_i}{\sum_i y_i}\right)^{\sum_i x_i} \prod_i y_i^{x_i}. \tag{1}$$

Equality holds in (1) *if and only if $x=\alpha y$ for some $\alpha>0$.*

Proof. If $x_i=0$ for some i, then $x_i^{x_i}=y_i^{x_i}=1$. If $y_i=0$ and $x_i>0$ for some i, then (1) clearly holds with a strict inequality. So we may assume that x and y are positive vectors. Let $\sum_i x_i=p$, $\sum_i y_i=q$. By the generalized arithmetic-mean–geometric-mean inequality,

$$\prod_i\left(\frac{y_i}{x_i}\right)^{x_i/p} \leqslant \sum_i \frac{x_i}{p}\cdot\frac{y_i}{x_i}=\frac{q}{p}.$$

The assertion about equality is also clear. ∎

Now we state our first main result.

THEOREM 3. *Let A,B be irreducible, nonnegative $n\times n$ matrices that have a common right eigenvector v and a common left eigenvector u corresponding to their spectral radii. Then, for any t, $0\leqslant t\leqslant 1$,*

$$r\big(tA+(1-t)B^t\big)\geqslant tr(A)+(1-t)r(B). \tag{2}$$

Furthermore, if $0<t<1$, then equality holds in (2) *if and only if v and u are linearly dependent.*

Proof. We assume, after normalizing if necessary, that $\sum u_i v_i=1$. For any positive vectors λ,μ, an application of Lemma 2 to the numbers $a_{ij}\lambda_i\mu_j$ and $a_{ij}u_iv_j$, $i,j=1,2,\dots,n$, give, after some simplication, the following:

$$\left(\sum_i\sum_j a_{ij}\lambda_i\mu_j\right)\prod_i(u_iv_i)^{u_iv_i}\geqslant r(A)\prod_i(\lambda_i\mu_i)^{u_iv_i}. \tag{3}$$

Similarly, applying Lemma 2 to the numbers $b_{ij}\mu_i\lambda_j$ and $b_{ij}u_iv_j$, $i,j=$

$1,2,\dots,r$ gives

$$\left(\sum_i \sum_j b_{ij}\mu_i\lambda_j\right)\prod_i (u_i v_i)^{u_i v_i} \geq r(B)\prod_i (\lambda_i \mu_i)^{u_i v_i} \tag{4}$$

Hence, for $0 \leq t \leq 1$,

$$\left\{\sum_i \sum_j \left[ta_{ij} + (1-t)b_{ij}\right]\lambda_i\mu_j\right\}\prod_i (u_i v_i)^{u_i v_i}$$

$$\geq \left[tr(A) + (1-t)r(B)\right]\prod_i (\lambda_i \mu_i)^{u_i v_i} \tag{5}$$

Now set μ equal to a right eigenvector of $tA + (1-t)B'$ corresponding to its spectral radius. Since $tA + (1-t)B'$ is irreducible, $\mu > 0$. Now, let $\lambda_i = u_i v_i / \mu_i$, $i = 1,2,\dots,n$. Then from (5) we get the inequality (2).

Now suppose $0 < t < 1$. Equality occurs in (2) if and only if it occurs in both (3) and (4), and that, according to Lemma 2, happens if and only if for some positive α, β,

$$a_{ij}\lambda_i\mu_j = \alpha a_{ij}u_i v_j, \quad b_{ij}\mu_i\lambda_j = \beta b_{ij}u_i v_j, \qquad i,j = 1,2,\dots,n.$$

Since $\lambda_i = u_i v_i / \mu_i$, $i = 1,2,\dots,n$, we have

$$a_{ij}\frac{v_i}{v_j} = \alpha a_{ij}\frac{\mu_i}{\mu_j}, \quad b_{ij}\frac{u_j}{u_i} = \beta b_{ij}\frac{\mu_j}{\mu_i}, \qquad i,j = 1,2,\dots,n. \tag{6}$$

Since A is irreducible, for any i, j there exist $i = i_1, i_2, \dots, i_k = j$ such that $a_{i_1 i_2} > 0$, $a_{i_2 i_3} > 0, \dots,$ $a_{i_{k-1} i_k} > 0$. From (6) we have

$$a_{i_1 i_2}a_{i_2 i_3}\cdots a_{i_{k-1} i_k}\frac{v_i}{v_j} = \alpha^{k-1}a_{i_1 i_2}a_{i_2 i_3}\cdots a_{i_{k-1} i_k}\frac{\mu_i}{\mu_j}$$

and hence

$$\frac{v_i}{v_j} = \alpha^{k-1}\frac{\mu_i}{\mu_j}.$$

Using this fact for any $j = i$, we get $\alpha = 1$, and then it follows that v and μ

are linearly dependent. Similarly, u and μ are linearly dependent, and hence so are v and u.

Conversely, if v and u are linearly dependent, equality clearly holds in (2) and the proof is complete. ■

An examination of the proof of Theorem 3 will reveal that the inequality (2) can be proved if A, B are nonnegative $n \times n$ matrices satisfying the following weaker conditions:

(i) A, B have a common right eigenvector v and a common left eigenvector u corresponding to their spectral radii and $\Sigma u_i v_i > 0$;

(ii) for $t \in (0,1)$, $tA + (1-t)B'$ has a positive (right or left) eigenvector corresponding to $r(tA + (1-t)B')$.

If condition (i) fails, then the inequality may not hold, as the following example shows. I do not have a similar example to show that condition (ii) is also necessary.

Let

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & \frac{1}{3} \end{bmatrix}, \qquad B = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \frac{1}{3} \end{bmatrix}.$$

Then $v = (1,1,0)'$ and $u = (0,0,3)'$ are right and left eigenvectors, respectively, of both A and B, but $\Sigma u_i v_i = 0$. Also, $r\left(\frac{1}{2}(A + B')\right) = \frac{1}{2}$, whereas $r(A) = r(B) = 1$, so that (2) fails.

THEOREM 4. *Let A be a nonnegative, irreducible $n \times n$ matrix with v and u as its right and left eigenvectors corresponding to $r(A)$, respectively, and suppose $\Sigma_i u_i v_i = 1$. Let*

$$D^l = \operatorname{diag}\left(\xi_1^{(l)}, \dots, \xi_n^{(l)}\right), \qquad E^l = \operatorname{diag}\left(\eta_1^{(l)}, \dots, \eta_n^{(l)}\right),$$

$l = 1, 2, \dots, k$, be positive definite diagonal matrices. Then

$$r\left(\sum_{l=1}^{k} D^l A E^l\right) \geqslant r(A) \sum_{l=1}^{k} \prod_{i=1}^{n} \left(\xi_i^{(l)} \eta_i^{(l)}\right)^{u_i v_i}. \tag{7}$$

Furthermore, if $A'A$ is irreducible, then equality holds in (7) if and only if there exist constants p_l, q_l, $l = 1, 2, \dots, k$, and α_i, $i = 1, 2, \dots, n$, such that $\eta_i^{(l)} = p_l \alpha_i$ and $\xi_i^{(l)} = q_l / \alpha_i$, $l = 1, 2, \dots, k$, $i = 1, 2, \dots, n$.

Proof. For any positive vectors λ, μ and for any l, $l=1,2,\dots,k$, an application of Lemma 2 to the numbers $a_{ij}\xi_i^{(l)}\eta_j^{(l)}\lambda_i\mu_j$ and $a_{ij}u_iv_i$, $i,j=1,2,\dots,n$, gives, after some simplification,

$$\left(\sum_i\sum_j a_{ij}\xi_i^{(l)}\eta_j^{(l)}\lambda_i\mu_j\right)\prod_i (u_iv_i)^{u_iv_i} \geqslant r(A)\prod_i \left(\xi_i^{(l)}\eta_i^{(l)}\lambda_i\mu_i\right)^{u_iv_i}. \quad (8)$$

Sum the inequalities (8) with respect to l, $l=1,2,\dots,k$. Then set μ equal to a right eigenvector of $\sum_l D^lAE^l$ with respect to its spectral radius, and set $\lambda_i = u_iv_i/\mu_i$, $i=1,2,\dots,n$. That results in the desired inequality (7).

If equality holds in (7), it must hold in (8) for each l, and that implies, by Lemma 2,

$$a_{ij}\xi_i^{(l)}\eta_j^{(l)}\lambda_i\mu_j = \theta_l a_{ij}u_iv_j, \qquad i,j=1,2,\dots,n, \quad l=1,2,\dots,k,$$

where θ_l are positive constants. Using $\lambda_i = u_iv_i/\mu_i$, we have

$$a_{ij}\xi_i^{(l)}\eta_j^{(l)} = \theta_l a_{ij}\frac{\mu_iv_j}{\mu_jv_i}, \qquad i,j=1,2,\dots,n, \quad l=1,2,\dots,k.$$

Fix l, $1\leqslant l\leqslant k$, and let

$$x_i = \frac{\xi_i^{(l)}v_i}{\mu_i\sqrt{\theta_l}}, \quad y_i = \frac{\eta_i^{(l)}\mu_i}{v_i\sqrt{\theta_l}}, \qquad i=1,2,\dots,n.$$

Then

$$a_{ij}x_iy_j = a_{ij}, \qquad i,j=1,2,\dots,n. \quad (9)$$

Now suppose $A'A$ is irreducible, and fix $p,q\in\{1,2,\dots,n\}$, $p\neq q$. If there exists a row, say the ith row, such that $a_{ip}, a_{iq}>0$, then using (9) we conclude that $y_p = y_q$. Otherwise, since $A'A$ is irreducible, there exist $p = j_1, j_2,\dots,j_s = q$ and $i_1,\dots,i_{s-1}$ such that the entries of A in positions (i_1,j_1), (i_1,j_2), (i_2,j_2), $(i_2,j_3),\dots,(i_{s-1},j_{s-1}),(i_{s-1},j_s)$ are positive. Now using (9) repeatedly, we conclude that $y_p=y_q$. Thus $y_1=\cdots=y_n$, and similarly it may be shown that $x_1=\cdots=x_n$. Now set $\alpha_i = v_i/\mu_i$, $i=1,2,\dots,n$ and observe that $\eta_i^{(l)}/\alpha_i$ depends only on l, so set it equal to p_l, while $\xi_i^{(l)}\alpha_i$ also depends only on l, so set it equal to q_l, and the proof of the "only if" part in the assertion about equality is complete. The "if" part is easily verified. ∎

The next result, due to Friedland and Karlin [2], is a simple corollary of Theorem 4.

COROLLARY 5. *Let A be an $n \times n$ irreducible, nonnegative matrix with v and u as its right and left eigenvectors corresponding to $r(A)$, respectively, and suppose $\sum u_i v_i = 1$. Then for any positive definite diagonal matrix $D = \operatorname{diag}(d_1, \ldots, d_n)$,*

$$r(DA) \geqslant \left(\prod_i d_i^{u_i v_i}\right) r(A).$$

We conclude by giving a short proof of an inequality that has been proved in [2] and that is used there to deduce the result of Corollary 5. The proof of this inequality given in [2, Section 3], although interesting, is quite involved.

LEMMA 7. *Let A be a nonnegative, irreducible $n \times n$ matrix with $r(A) = 1$, and let v and u be the right and left eigenvectors of A corresponding to $r(A)$. Suppose $\sum_i u_i v_i = 1$. Then for any $x > 0$,*

$$\sum_i u_i v_i \log\left(\frac{\sum_j a_{ij} x_j}{x_i}\right) \geqslant 0.$$

Proof. Since A is irreducible, u and v are positive. Set $y_i = x_i / u_i$, $i = 1, 2, \ldots, n$. Using the concavity of the log function, we get

$$\begin{aligned}
\sum_i u_i v_i \log\left(\frac{\sum_j a_{ij} u_j y_j}{u_i}\right) &\geqslant \sum_i u_i v_i \sum_j \frac{a_{ij} u_j \log y_j}{u_i} \\
&= \sum_i \sum_j a_{ij} v_i u_j \log y_j \\
&= \sum_j (\log y_j) u_j \sum_i a_{ij} v_i \\
&= \sum_j u_j v_j \log y_j .
\end{aligned}$$

Now the result follows. ∎

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