ON PARTIALLY BALANCED LINKED BLOCK DESIGNS

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1. Summary. The computations in the analysis of any equireplicate design can be carried out very easily if the number of treatments common to any two blocks is constant. A design with this property is called a Linked Block (LB) design and was introduced by Youden [9]. It is well known that for a Balanced Incomplete Block (BIB) design to have a constant number of treatments in common between any two blocks, it is necessary and sufficient that it is symmetric, that is, the number of blocks is equal to the number of treatments. In this paper, necessary and sufficient conditions are derived for any design with a given treatment-structure matrix to be of the LB type and the results applied to Partially Balanced Incomplete Block (PBIB) designs. Finally a list is prepared of all LB designs in the class of two-associate PBIB designs enumerated by Bose, Shrikhande and Clatworthy [2].

2. Introduction. An arrangement of v treatments in b blocks, each of k plots, $k < v$, such that each treatment occurs at most once in any block and altogether in r blocks is called an incomplete block design and denoted by $D(v, b, k, r)$. Obviously $vr = bk$. A design with $b = v$ is called symmetric. A $D(v, b, k, r)$ is completely characterised by its 'incidence matrix' $N = ((n_{ij}))$ where $n_{ij} = 1$ if the ith treatment occurs in the jth block and $n_{ij} = 0$ otherwise $i = 1, 2, \cdots, v, j = 1, 2, \cdots, b$. The matrix $\Lambda = NN'$ where $\lambda_{ii} = r$ and $\lambda_{ij} =$ the number of blocks in which the treatments i and j occur together $i \neq j = 1, 2, \cdots, v$ is called the 'treatment-structure matrix' of the design. A design is balanced if $\lambda_{ij} = \lambda$ for all $i \neq j$. The design obtained from $D(v, b, k, r)$ by considering its blocks as treatments and treatments as blocks is called its dual. The number of treatments common to the ith and the jth blocks will be denoted by μ_{ij}, $i, j = 1, 2, \cdots, b$. The matrix $M = N'N$ has been called the 'structural matrix' by Connor [5] and Connor and Hall [7]. We shall however call M the 'block-structure matrix'. A design is of the LB type if $\mu_{ij} = \mu$ for all $i \neq j$. Obviously a design is of the LB type if and only if its dual is balanced. For a definition of PBIB design the reader is referred to Bose and Shimamoto [4].

3. Conditions for a design with given treatment-structure matrix to be of the LB type. We first give two lemmas which are useful in deriving necessary and sufficient requirements on the treatment-structure matrix.

LEMMA 3.1. *If A and B are two matrices of the form $m \times n$ and $n \times m$ respectively, the non-zero latent roots of AB are identical with those of BA and if corresponding to a latent root θ, ξ is a latent vector of AB, $\eta = \xi A$ will be a latent vector*

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of BA corresponding to the same root θ. If θ is a non-zero repeated latent root of AB of multiplicity r, it is so for BA also.

PROOF. If θ is a non-zero latent root of AB of multiplicity r, we can always find r linearly independent vectors ξ_i satisfying $\xi_i AB = \theta \xi_i$ $(i = 1, 2, \cdots, r)$ Post-multiplication by A gives $\eta_i BA = \theta\eta_i$ where $\eta_i = \xi_i A$. That η_i's are linearly independent follows from the linear independence of ξ_i's because for any set of constants c_i's, $\sum c_i\eta_i B = \theta \sum c_i \xi_i$. This also shows that θ is also a latent root of BA of multiplicity r and η_i's are a corresponding set of linearly independent latent vectors.

LEMMA 3.2. *The necessary and sufficient condition for a symmetric matrix A of order n to have all its diagonal elements equal and all its off-diagonal elements equal is that it has only two latent roots, one of multiplicity* $(n - 1)$ *and the vector* $(1, 1, \cdots, 1)$ *is a latent vector corresponding to the other latent root.*

PROOF. Necessity is obvious. To prove that the conditions are sufficient, let us write

$$\alpha = \left(\frac{1}{\sqrt{n}}, \frac{1}{\sqrt{n}}, \cdots, \frac{1}{\sqrt{n}}\right).$$

Since A is symmetric there exists an orthogonal matrix

$$C = \begin{bmatrix} \alpha \\ P \end{bmatrix}$$

such that

$$CAC' = \begin{bmatrix} \theta_1 & 0 \\ 0 & \theta_2 I_{n-1} \end{bmatrix},$$

where θ_1 and θ_2 are the latent roots, θ_2 with multiplicity $(n - 1)$ and I_n is the identity matrix of order n. Premultiplying by C' and post-multiplying by C we get

$$A = \theta_1\alpha'\alpha + \theta_2 P'P = (\theta_1 - \theta_2)\alpha'\alpha + \theta_2 I_n,$$

which has diagonal elements equal to $\{\theta_1 + (n - 1)\theta_2\}/n$ and off-diagonal elements equal to $(\theta_1 - \theta_2)/n$.

We are now in a position to prove

THEOREM 3.1. *The necessary and sufficient condition for a design* $D(v, b, k, r)$ *to be of the* LB *type is that* $k - \mu$ *is a latent root of the treatment-structure matrix* Λ, *of multiplicity* $(b - 1)$ *where* $\mu = k(r - 1)/(b - 1)$.

PROOF. The necessity is obvious. To prove the sufficiency of the conditions, let us write N for the incidence matrix of the given design. Then we have to show that the block-structure matrix $M = N'N$ has all off-diagonal elements equal. Since it is given that $k - \mu$ is a latent root of multiplicity $(b - 1)$ of the treatment-structure matrix $\Lambda = NN'$ by Lemma 3.1 it will be so for $M = N'N$ also. Again since the total of each column of Λ is rk, $\epsilon = (1, 1, \cdots, 1)$ is a latent vector of Λ corresponding to the latent root rk. Again by Lemma 3.1, $\epsilon N = (k, k,$

$\cdots, k)$ is a latent vector of M corresponding to the latent root rk, therefore so is also the vector $(1, 1, \cdots, 1)$. Thus M satisfies all the conditions of Lemma 3.2. Hence it has all diagonal elements equal to k and all off-diagonal elements equal to μ. But since $\mu = \sum_{t=1}^{v} n_{ti}n_{tj}$ for all $i \neq j$ and $n_{ti} = 1$ or 0, μ must be integral. The number of treatments common to any two blocks is thus μ.

COROLLARY 3.1. *If the treatment-structure matrix of any design $D(v, b, k, r)$ has only two non zero latent roots, rk and $k(b-r)/(b-1)$ and rk is not a repeated root, then the design must be of the* LB *type.*

PROOF. If t is the multiplicity of the root $k(b-r)/(b-1)$ equating the sum of the diagonal elements of the treatment-structure matrix to the sum of the latent roots, we get $t = (b-1)$. Hence the result.

COROLLARY 3.2. *If $D(v, b, k, r)$ is balanced, the necessary and sufficient condition that it is of the* LB *type is that $v = b$.*

PROOF: Since the design is balanced, its treatment-structure matrix has a latent root of multiplicity $v - 1$, while if it is of the LB type the multiplicity must be $b - 1$. Hence $b = v$.

4. Partially balanced linked block designs. We now apply the results of Section 3 to the special case of PBIB design $D(v, b, k, r)$ having m associate classes with parameters $n_i, \lambda_i, p_{js}^i (i, j, s = 1, 2, \cdots, m)$ as defined in Bose and Shimamoto [4]. It follows from the results of Connor and Clatworthy [6] that latent roots other than rk of the treatment-structure matrix for such a design are, except for repetitions, the same as the latent roots of the reduced matrix $A = ((a_{ij}))$ of order m where

$$(4.1)\quad \begin{cases} a_{ii} = r + \lambda_1 p_{i1}^i + \lambda_2 p_{i2}^i + \cdots + \lambda_m p_{im}^i - \lambda_i n_i, \\ a_{ij} = \lambda_1 p_{i1}^j + \lambda_2 p_{i2}^j + \cdots + \lambda_m p_{im}^j - \lambda_i n_i, \\ \qquad\qquad i \neq j = 1, 2, \cdots m. \end{cases}$$

Hence we have the

THEOREM 4.1. *The necessary and sufficient condition for a* PBIB *design with m associate classes to be of the* LB *type is that the matrix A defined in* (4.1) *has only one non zero latent root $k(b-r)/(b-1)$.*

COROLLARY 4.1. *The necessary and sufficient condition for a* PBIB *design with two associate classes to be of the* LB *type is that*

$$(4.2)\quad a_{11}a_{22} - a_{21}a_{12} = 0,$$

$$(4.3)\quad a_{11} + a_{22} = k(b-r)/(b-1),$$

where

$$(4.4)\quad \begin{cases} a_{11} = r + \lambda_1 p_{11}^1 + \lambda_2 p_{12}^1 - \lambda_1 n_1, \\ a_{12} = \lambda_1 p_{11}^2 + \lambda_2 p_{12}^2 - \lambda_1 n_1, \\ a_{21} = \lambda_1 p_{21}^1 + \lambda_2 p_{22}^1 - \lambda_2 n_2, \\ a_{22} = r + \lambda_1 p_{21}^2 + \lambda_2 p_{22}^2 - \lambda_2 n_2. \end{cases}$$

PROOF. The matrix $\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$ cannot have two equal latent roots.

We shall now apply this result to some special types of PBIB designs with two associate classes.

4.1 *Group Divisible* (GD) *Designs*. A GD design as defined by Bose and Connor [1] is specified by the parameters

$$v = mn, \qquad n_1 = (n - 1), \qquad n_2 = m(n - 1),$$

$$((p_{js}^1)) = \begin{bmatrix} n - 2 & 0 \\ 0 & n(m - 1) \end{bmatrix}, \qquad ((p_{js}^2)) = \begin{bmatrix} 0 & n - 1 \\ n - 1 & n(m - 2) \end{bmatrix}$$

They have classified the GD designs as (i) Singular when $r = \lambda_1$ (ii) Semi-Regular when $r > \lambda_1$ and $rk = v\lambda_2$ and (iii) Regular when $r > \lambda_1$ and $rk > v\lambda_2$. They have also shown that for a Singular GD design, $b \geqq m$ and for a Semi-Regular GD design $b \geqq v - m + 1$. In order that a GD design may be of the LB type the condition (4.2) gives

$$(r - \lambda_1)\,(rk - v\lambda_2) = 0,$$

so that a Regular GD design is never of the LB type. The condition (4.3) gives for the case $r = \lambda$, $b = m$ and for the case $rk \neq v\lambda_2$, $b = v - m + 1$. We now summarise these results in the form of the

THEOREM 4.2. *A Regular* GD *design cannot be of the* LB *type. The necessary and sufficient condition for a Singular* GD *to be of the* LB *type is that* $b = m$ *and that for a Semi-Regular* GD *is that* $b = v - m + 1$.

Bose and Connor [1] have shown that a Singular GD design can always be derived from a BIB design with m treatments by replacing each treatment by a group of n treatments. Hence the condition for a Singular GD design to be of the LB type is that the BIB design from which it is generated should be symmetric.

4.2 *Triangular Designs.* In a Triangular design (Bose and Shimamoto [4])

$$v = \tfrac{1}{2}n(n - 1), \qquad n_1 = 2(n - 2), \qquad n_2 = \tfrac{1}{2}(n - 2)\,(n - 3),$$

$$((p_{js}^1)) = \begin{bmatrix} n - 2 & n - 3 \\ n - 3 & \tfrac{1}{2}(n - 3)(n - 4) \end{bmatrix}, \; ((p_{js}^2)) = \begin{bmatrix} 4 & 2(n - 4) \\ 2(n - 4) & \tfrac{1}{2}(n - 4)(n - 5) \end{bmatrix}$$

In order that a Triangular design may be of the LB type the condition (4.2) gives

$$(r - 2\lambda_1 + \lambda_2)\{r + (n - 4)\lambda_1 - (n - 3)\lambda_2\} = 0.$$

From the other condition (4.3) we get $b = n$ if $r - 2\lambda_1 + \lambda_2 = 0$ and $b = \tfrac{1}{2}(n - 1)\,(n - 2)$ if $r + (n - 4)\lambda_1 - (n - 3)\lambda_2 = 0$. We thus get the

THEOREM 4.3. *The necessary and sufficient condition for a Triangular design to be of the* LB *type is that either* (i) $r = 2\lambda_1 - \lambda_2$ *and* $b = n$ *or* (ii) $r = (n - 3)\lambda_2 - (n - 4)\lambda_1$ *and* $b = \tfrac{1}{2}(n - 1)(n - 2)$.

It is interesting to note that in case (i) if $r = 2$ we get the Triangular Singly Linked Block (TSLB) designs and in case (ii) if $r = n - 2$ the Triangular Doubly Linked Block (TDLB) designs as defined by Bose and Shimamoto

[4]. The results of Theorem 4.3 may thus be considered as generating the class of Triangular Multiply Linked Block designs.

4.3 *Two associate* PBIB *designs with* $k > r \geqq 2$ *and* $\lambda_1 = 1, \lambda_2 = 0$ (*Simple* PBIB). Bose and Clatworthy [3] have shown that all designs of this class are characterized by the parameters

$$v = k[(r-1)(k-1)+t]/t, \qquad b = r[(r-1)(k-1)+t]/t,$$

$$n_1 = r(k-1), \qquad n_2 = (r-1)(k-1)(k-t)/t$$

$$((p^1_{jk})) = \begin{bmatrix} (t-1)(k-1)+k-2 & (r-1)(k-t) \\ (r-1)(k-t) & (r-1)(k-t)(k-t-1)/t \end{bmatrix}$$

$$((p^2_{jk})) = \begin{bmatrix} rt & r(k-t-1) \\ r(k-t-1) & [(r-1)(k-1)(k-2t)+t(rt-k)]/t \end{bmatrix}$$

where $1 \leqq t \leqq r$. It is interesting to note that in this case the condition for an LB design is $t = r$ and then $\mu = 1$. Hence we have the

THEOREM 4.4. *The only* LB *designs in the class of two-associate* PBIB *designs with* $k > r \geqq 2$ *and* $\lambda_1 = 1, \lambda_2 = 0$ *are those which are duals of* BIB *designs in which any two treatments occur together in just one block.*

Shrikhande [8] showed that the dual of any BIB design with $\lambda = 1$ is a two associated PBIB design with $\lambda_1 = 1$, and $\lambda_2 = 0$. Our result shows that no two-associate PBIB design with $\lambda_1 = 1$ and $\lambda_2 = 0$ and $k > r \geqq 2$ can be obtained by dualising BIB designs other than those with $\lambda = 1$.

4.4 *List of two-associate* PBIB *designs of the* LB *type.* We give below a list of LB designs in the class of two associate PBIB designs enumerated by Bose, Clatworthy and Shrikhande [2]. The reference number for a design is the one used by the above authors and μ denotes other number of treatments common to any two blocks.

TABLE 4.1

List of two-associate PBIB *designs of the* LB *type.* (S = *Singular* GD. SR = *Semi-Regular* GD, Sl = *Simple*, T = *Triangular*.)

Reference No.	μ	Reference No.	μ	Reference No.	μ
S 1	2	SR 1	1	Sl 25	1
S 7	4	SR 20	1	T 1	1
S 12	3	SR 26	5	T 9	2
S 18	6	SR 32	3	T 15	3
S 22	4	SR 51	1	T 20	1
S 24	6	SR 70	1	T 22	2
S 28	8	SR 85	1	T 25	6
S 40	2	SR 89	1	T 27	5
S 41	4	Sl 4	1	T 31	1
S 46	5	Sl 9	1	T 32	1
S 77	3	Sl 17	1	T 33	1
S 81	4	Sl 18	1	T 34	2
S 89	2	Sl 21	1	T 35	1
S 111	2	Sl 22	1	T 36	1

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