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RUN ORDERS OF TREND RESISTANT 2-LEVEL FACTORIAL DESIGNS*

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SUMMARY. In this paper we consider the problem of constructing run orders of factorial designs having all factors at two levels that are robust against smooth polynomial spatial or time trends. Bounds are obained on the number of constrasts in a complete 2^n design that can be orthogonal to a k-degree polynomial trend for $k = 1, \ldots, n-1$ and it is shown that the highest degree polynomial trend that a contrast in a complete 2^n design can be orthogonal to is $n-1$. Also, some suggestions are made as to how to construct main effects only fractional factorial designs when the number of available experimental units is a multiple of 4.

1. Introduction

In this paper we consider experimental situations in which a factorial design with all factors at two levels is to have its treatments applied to experimental units in some sequence over space or time. In such a setting, the results obtained may be affected by the particular time or space order in which treatments are applied and this should be taken into consideration both when the experiment is planned and when the results are analyzed. While the time or space order of treatment applications may not itself be an important variable, it may serve as a good approximation for other important variables that are highly correlated with it. For example, a manufacturer may be interested in determining the best chemical mixture to be used in a new product. However, if the chemicals involved are mixed over time using the same equipment, it may be that time could serve as an approximating variable for the effects of mixing equipment wear-out. Or if the experimental material to which treatments are to be applied is put together at the beginning of the

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experiment and treatments are administered over time, then time might serve as an approximating variable for the effects of aging on the experimental material.

Cox (1951) began the study of systematic designs for the efficient estimation of treatment effects in the presence of a smooth polynomial trend in the context of variety trials. Other papers on the sujbect include Daniel and Wilcoxon (1966), Hill (1960), Draper and Stoneman (1968), Dickinson (1974), Joiner and Campbell (1976) Cheng (1985), Coster and Cheng (1986), and Cheng and Jacroux (1988). In particular, Cheng and Jacroux (1988) showed that in the standard order of a complete 2^n design, any k-factor interaction contrast is orthogonal to any $(k-1)$-degree polynomial trend. Thus, by designating some high-order interaction terms as main effects, one can derive from the standard order a run order of a complete 2^n design in which the main effects are orthogonal to high degree polynomial trends.

In this paper we further consider determination of run orders of factorial designs having factors at two levels. In section 2 we give a summary of the notation and terminology that is used throughout the paper. In section 3, some extensions of a result given in Cheng and Jacroux (1988) are given. More specifically, bounds are given for the number of contrasts in a complete 2^n design that can be orthogonal to a p-degree polynomial trend and a bound is also given for the highest degree of polynomial that a contrast in a complete 2^n design can be orthogonal to. Finally, in section 4, we discuss the problem of constructing main effects only fractions of 2^n designs where the number of observations to be obtained in some positive multiple of 4.

2. Preliminary notation, definitions and lemmas

We shall refer to the two levels of each factor as high level and low level. With each factor i we associate the letter $\boldsymbol{A}_i$ and call $\boldsymbol{A}_i$ the main effect of factor i. The product $\boldsymbol{A}_{i_1} \ldots \boldsymbol{A}_{i_t}$ of main effect letters shall be called the t-factor interaction term for factors $i_1, i_2, \ldots, i_t$. The set of 2^n products $\boldsymbol{A}_1^{x_1} \ldots \boldsymbol{A}_n^{x_n}$ where $x_i \in \{0, 1\}$ form an abelian multiplicative group which we denote by $\boldsymbol{A}^n$.

In a complete 2^n design a particular treatment combination is represented as the product of some subset of letters out of $a_1, \ldots, a_n$ with the presence of a_i in the product indicating that factor i occurs at its high level and the absence of a_i from the product indicating that factor i occurs at its low level. We use (1) to denote that treatment having all factors occur at their low levels.

We shall also consider *some* fractions of a complete 2^n design denoted by $d(n, m)$ where $n =$ the number of factors and $m =$ the number of different combinations out of the 2^n available that are in $d(n, m)$. Any sequential applications of the treatments in $d(n, m)$ to experimental units over space or time is called a run order of $d(n, m)$. In the present context m is not necessarily 2^{n-p} as discussed in Cheng and Jacroux (1988).

We now refer to Cheng and Jacroux (1988) replacing N and $\boldsymbol{T}_p^n$ of that paper by m and $d(n, m)$ respectively in the present context for some definitions that are central to our discussion e.g. componentwise multiplication operator 'o' (Definition 2.1), main effect contrast u_i associated with $\boldsymbol{A}_i$ and the contrast $\boldsymbol{u}_{i_1}\, o\, \boldsymbol{u}_{i_2}\, o \ldots o\, \boldsymbol{u}_{i_t}$ associated with the t-factor interaction $\boldsymbol{A}_{i_1} \boldsymbol{A}_{i_2} \ldots \boldsymbol{A}_{i_t}$ (Definition 2.2).

In a complete 2^n design, we note that the set of all 2^n products that can be formed from $u_1, \ldots, u_n$ under the opertion of '0' forms a group that we denote by $\boldsymbol{U}^n$. Clearly $\boldsymbol{U}^n$ and $\boldsymbol{A}^n$ are isomorphic to the additive abelian group of n-dimensional vectors $\boldsymbol{V}^n$ over the field $Z_n = \{0, 1\}$ where vector addition is carried out modulo 2, From this isomorphism it follows that since V^n is an n-dimensional vector space, every basis consisting of a minimal number of generator elements for $\boldsymbol{A}^n(\boldsymbol{U}^n)$ contains exactly n elements.

Suppose we let $\boldsymbol{y} = (y_1, \ldots, y_m)'$ denote the vector of ordered observations obtained after applying the treatments in $d(n, m)$ to experimental units. The model assumed here for analyzing the data obtained in $d(n, m)$ is the usual model that can be written as

$$\boldsymbol{y} = \boldsymbol{X}\boldsymbol{\beta}+\boldsymbol{\epsilon} = \boldsymbol{X}_1\boldsymbol{\beta}_1+\boldsymbol{X}_2\boldsymbol{\beta}_2+\boldsymbol{\epsilon} \qquad \ldots \quad (0.1)$$

where $\boldsymbol{\epsilon}$ is an $m\times 1$ vector of independent error terms having expectation zero and constant variance σ^2. The parameters in $\boldsymbol{\beta}_1$ correspond to the usual factorial main effect terms $\boldsymbol{A}_i$ and interaction terms $\boldsymbol{A}_{i_1} \ldots \boldsymbol{A}_{i_s}$ and the parameters in $\boldsymbol{\beta}_2$ correspond to smooth polynomial trend effects. We shall assume throughout the sequel that the levels of each factor have been coded so that they correspond to $+1$ or -1 values. Thus if we let $\boldsymbol{X} = (x_1, \ldots, x_t) = (x_{ij})$, then the column of $\boldsymbol{X}_1$ corresponding to the main effect term $\boldsymbol{A}_i$ is precisely $\boldsymbol{u}_i$ for $i = 1, \ldots, n$ and the column of $\boldsymbol{X}_1$ corresponding to the interaction term $\boldsymbol{A}_{i_1} \ldots \boldsymbol{A}_{i_s}$ is $\boldsymbol{u}_{i_1}\, o \ldots o\, \boldsymbol{u}_{i_s}$. The columns of $\boldsymbol{X}_2$ correspond to some space or time trend effect. We shall assume that any trend that might effect the observations can be represented by

$$\text{trend effect} = \alpha_0+\alpha_1 z+\alpha_2 z^2+ \ldots +\alpha_t z^t \qquad \ldots \quad (2.2)$$

and the values that z assumes correspond to equally spaced positions at which observations are obtained.

In the partitioned form of the design matrix $\boldsymbol{X} = (\boldsymbol{X}_1, \boldsymbol{X}_2)$ given above, if the columns of $\boldsymbol{X}_1$ are orthogonal, then we shall say that $d(n, m)$ is an orthogonal design. The ultimate goal of this paper is to study run orders of 2-level fractional factorial designs that yield estimates for main effects that are not contaminated by unknown trend effects of the form given in (2.2). But the only way this can happen if the columns of $\boldsymbol{X}_1$ corresponding to the factorial effects of interest are orthogonal to the columns of $\boldsymbol{X}_2$ corresponding to trend effects. With this in mind, we refer to the definition of a k-trend free main effect or t-factor interaction discussed in Cheng and Jacroux (1988) e.g. definition 2.4 with an obvious replacement of N by m in the present context.

One method that sometimes proves useful for constructing trend-free contrasts involves the Kronecker product between matrices (Raghavarao, 1971). When the Kronecker product is applied to k-trend free vectors, we have the following result.

Lemma 1. (a) *If* $\boldsymbol{p}_1$ *is an* $m \times 1$ *vector of* $+1$*'s and* -1*'s such that* $\boldsymbol{p}_1' T_0 \neq 0$ *and* $\boldsymbol{p}_2$ *is an* $n \times 1$ *vector of* $+1$*'s and* -1*'s that is q-trend free, then* $\boldsymbol{p}_1 \otimes \boldsymbol{p}_2$ *is q-trend free.*

(b) *If* $\boldsymbol{p}_1$ *is a p-trend free* $m \times 1$ *vector of* $+1$*'s and* -1*'s and* $\boldsymbol{p}_2$ *is an* $n \times 1$ *vector of* $+1$*'s and* -1*'s such that* $\boldsymbol{p}_2' T_0 \neq 0$, *then* $\boldsymbol{p}_1 \otimes \boldsymbol{p}_2$ *is p-trend free.*

(c) *If* $\boldsymbol{p}_1$ *is an* $m \times 1$ *vector of* $+1$*'s and* -1*'s that is p-trend free,* $p \geqslant 0$ *and* $\boldsymbol{p}_2$ *is an* $n \times 1$ *vector of* $+1$*'s and* -1*'s that is q-trend free,* $q \geqslant 0$, *then* $\boldsymbol{p}_1 \otimes \boldsymbol{p}_2$ *is* $(p+q+1)$*-trend free.*

Proof. The proof for (a) is obvious and the proof for (b) is similar to the proof of (c). Thus we shall only prove (c). So let $\boldsymbol{p}_1 = (e_1, e_2, \ldots, e_m)'$ and $\boldsymbol{p}_2 = (d_1, d_2, \ldots, d_n)'$. Since $\boldsymbol{p}_1$ is p-trend free,

$$\sum_{i=1}^{m} i^x e_i = 0 \text{ for } x = 0, 1, \ldots, p. \qquad \ldots \quad (2.3)$$

Upon expanding $i^x = ((i-1)+1)^x$ in (2.3) binomially and proceeding sequentially for succeeding values of x, we see that

$$\sum_{i=1}^{m} (i-1)^x e_i = 0 \text{ for } x = 1, 2, \ldots, p. \qquad \ldots \quad (2.4)$$

Now consider

$$\boldsymbol{p}_1 \otimes \boldsymbol{p}_2 = (e_1 d_1, e_1 d_2, \ldots, e_1 d_n, e_2 d_1, e_2 d_2, \ldots, e_2 d_n, \ldots, e_m d_1, e_m d_2, \ldots, e_m d_n)'$$

and the vector

$$T_x = (1^x, 2\ , \ldots, n^x, (n+1)^x, (n+2)^x, \ldots, (n+n)^x, \ldots, ((m-1)n+1)^x,$$
$$((m-1)n+2)^x, \ldots, ((m-1)n+n)^x)^1.$$

Using these last two expressions and binomially expanding terms of the form $(an+b)^x$ in T_x, we see that

$$(p_1 \otimes p_2)'\, T_x = e_1 \sum_{i=1}^{n} i^x d_i + \sum_{j=2}^{m} e_j \left\{ \sum_{i=1}^{n} d_i \left(\sum_{l=0}^{x} \binom{x}{l} ((j-1)n)^{x-l}\, i^l \right) \right\} \ldots \quad (2.5)$$

For $x = 0, 1, \ldots, q$, it follows from (2.5) and the fact p_2 is q-trend free that for each fixed value of j and ι,

$$\sum_{i=1}^{n} i^x d_i = 0 \text{ and } \sum_{i=1}^{n} d_i \left(\sum_{l=0}^{x} \binom{x}{l} ((j-1)n)^{x-l}\, i^l \right) = 0.$$

Thus $(p_1 \otimes p_2)'\, T_x = 0$ for $x = 0, 1, \ldots, q$. For $x = q+1, q+2, \ldots, p+q+1$, upon collecting cosfficients in (2.5) for each i^l term separately, $i = 1, \ldots, n$, $\iota = q+1, \ldots, x$, it follows from (2.4) that (2.5) is equal to zero. Hence, $(p_1 \otimes p_2)'\, T_x = 0$ for $x = q+1, \ldots, p+q+1$, and we have the desired result.

3. Bounds for trend resistent contrasts in complete 2^n designs

In this section we extend a result given in Cheng and Jacroux (1988). In particular, Cheng and Jacroux (1988) give an upper bound of $2^n - n - 1$ as the maximal number of 1-trend free orthogonal contrasts that a complete 2^n design can have. In this section we derive an upper bound for the number of p-trend free contrasts that a complete 2^n design can have as well as a bound on the degree of trend resistance that any contrast in a complete 2^n design can have. We begin by giving a Lemma.

Lemma 2. *Let $T_1' = (1, 2, 3, \ldots, h)$ and let $\mathbf{1}_h, W_1, \ldots, W_t$ be a set of vectors where $\mathbf{1}_h$ is the $h \times 1$ vector of ones and $W_1, \ldots, W_t$ are $h \times 1$ vectors whose components are all $+$ 1's and $-$ 1's. If $T_1 = c_0 \mathbf{1}_h + c_1 W_1 + \ldots + c_t W_t$ for appropriate constants $c_0, c_1, \ldots, c_t$, then $t \geqslant n$ where $2^n \leqslant h \leqslant 2^{n+1}$.*

Proof. Assume $t \leqslant n-1$, let $W = (W_1, \ldots, W_t)$ and let $T_1 - c_0 \mathbf{1}_h = c_1 W_1 + \ldots + c_t W_t$. Since all the rows of $T_1 - c_0 \mathbf{1}_h$ are different, it follows that none of the rows in W can be identical. But since all of the entries in W are $+1$'s and -1's and $t \leqslant n-1$, it follows that there can be at most $2^t \leqslant 2^{n-1} < h$ distinct rows in W. Hence we have a contradiction and we see that $t \geqslant n$.

Theorem 1. *In a complete 2^n design $d(n, m)$ with $m = 2^n$, the maximal number of mutually orthogonal contrasts that are t-trend free is* $2^n-1-\sum_{x=1}^{t}\binom{n}{x}$.

Proof. We begin by observing that in a complete 2^n design, all of the contrasts, main effect and interaction, are mutually orthogonal and under the operation of '0', these contrasts form the group $\boldsymbol{U}^n$ defined in the previous section. Suppose that p of the contrasts in $d(n, m)$ are not orthogonal to $\boldsymbol{T}_1$ and suppose that we denote them by $\boldsymbol{W}_1, \ldots, \boldsymbol{W}_p$. It then follows that we can write $\boldsymbol{T}_1 = c_{01}1_m+c_1\boldsymbol{W}_1+ \ldots +c_p\boldsymbol{W}_p$ for appropriate nonzero scalars $c_0, c_1, \ldots, c_p$ and from Lemma 2 that $p \geqslant n$.

Claim : The set of vectors $\boldsymbol{W}_1, \ldots, \boldsymbol{W}_p$ generates the group U^n.

Suppose the claim is not true and $\boldsymbol{W}_1, \ldots, \boldsymbol{W}_p$ generate a proper subgroup of $\boldsymbol{U}^n$, say $\boldsymbol{U}_0^n$. Then $\boldsymbol{U}_0^n$ can have at most 2^{n-1} elements in it and all elements in $\boldsymbol{U}^n$ not contained in $\boldsymbol{U}_0^n$ are orthogonal to all of the elements in $\boldsymbol{U}_0^n$. Using the properties of component wise multiplication, we see that

$$\boldsymbol{T}_x = \overbrace{\boldsymbol{T}_1 o \ldots o \boldsymbol{T}_1}^{x} = (c_0 1_m+c_1\boldsymbol{W}_1+ \ldots +c_p\boldsymbol{W}_p)o \ldots o(c_0 1_m+c_1\boldsymbol{W}_1+ \ldots +c_p\boldsymbol{W}_p)$$

$$= a_0 1_m+ \sum_{i_1=1}^{p} \ldots \sum_{i_p=1}^{p} a_{i_1, \ldots, i_p}(\boldsymbol{W}_1 o\boldsymbol{W}_2 o \ldots o\boldsymbol{W}_p) \text{ for } x = 1, 2, \ldots, m,$$

and for appropriate scalars $a_{i_1, i_2, \ldots, i_p}$. Hence $\boldsymbol{T}_x$ can be expressed as a linear combination of contrasts in the subgroup $\boldsymbol{U}_0^n$ generated by $\boldsymbol{W}_1, \ldots, \boldsymbol{W}_p$ for $x = 1, \ldots, m$ and any vector in $\boldsymbol{U}^n$ not contained in U_0^n is orthogonal to $\boldsymbol{T}_x$ for $x = 1, \ldots, m$, But this is impossible since $\boldsymbol{T}_1, \boldsymbol{T}_2, \ldots, \boldsymbol{T}_m$ form a linearly independet sent of vectors and we have a contradiction.

Since the claim is true, it follows that $\boldsymbol{W}_1, \ldots, \boldsymbol{W}_p$ must contain a basis for $\boldsymbol{U}^n$. Thus if we create the set of $\binom{p}{x}$ vectors $\boldsymbol{W}_{i_1} o \ldots o \boldsymbol{W}_{i_x}$ where $i_1, \ldots, i_x$ correspond to distinct subscripts out of $1, \ldots, p$, then this set of vectors must contain at least $\binom{n}{x}$ distinct contrasts out of $\boldsymbol{U}^n$ for $x = 1, \ldots, n$. Now, sisce

$$\boldsymbol{T}_2 = \boldsymbol{T}_1 o\boldsymbol{T}_1 = (c_0 1_m+c_1\boldsymbol{W}_1+ \ldots +c_p\boldsymbol{W}_p)o(c_0 1_m+c_1\boldsymbol{W}_1+ \ldots +c_p\boldsymbol{W}_p)$$

$$= b_0 1_m+b_1\boldsymbol{W}_1+ \ldots +b_2\boldsymbol{W}_p+ \sum_{i=1}^{p} \sum_{\substack{j=1 \\ j\neq i}}^{p} b_{ij}(\boldsymbol{W}_i o\boldsymbol{W}_j)$$

for appropriate scalars $b_i, i = 0, 1, \ldots, p$ and $b_{ij}\, i, j = 1, \ldots, p, i \neq j$, we see that $\boldsymbol{T}_2$ is a linear combination of at least $1+\binom{n}{1}+\binom{n}{2}$ distinct contrasts in U^n, hence at most $2^n-1-\binom{n}{1}-\binom{n}{2}$ vectors in U^n can be orthogonal to T_2. Similarly,

$$T_3 = T_1 o T_1 o T_1 = (c_0 1_m + c_1 W_1 + \dots + c_p W_p) o (c_0 1_m + c_1 W_1 + \dots + c_p W_p)$$
$$o(c_0 1_m + c_1 W_1 + \dots + c_p W_p)$$
$$= d_0 1_m + \sum_{i=1}^{p} d_i W_i + \sum_{i=1}^{p} \sum_{\substack{j=1\\ j\neq i}}^{p} d_{ij}(W_i \, o W_j) + \sum_{i=1}^{p} \sum_{\substack{j=1\\ j\neq i}}^{p} \sum_{\substack{k=1\\ k\neq i\\ k\neq j}}^{p} d_{ijk}(W_i \, o W_j \, o W_k)$$

for appropriate scalars d_i, d_{ij} and d_{ijk} and we see that T_3 can be expressed as a linear combination of at least $1+\binom{n}{1}+\binom{n}{2}+\binom{n}{3}$ distinct contrasts in U^n. Hence there are at most $2^n-1-\binom{n}{1}-\binom{n}{2}-\binom{n}{3}$ vectors in U^n orthogonal to T_3. Continuing in this manner, we obtain the desired formula.

Corollary 1. *In a complete* 2^n *design the degree of resistance of any contrast cannot exceed* $n-1$.

Proof. In the proof of Theorem 1, we saw that the set of contrasts W_1, ..., W_p that are not orthogonal to T_1 must contain a set of n generator vectors for U^n and that T_x is a linear combination of at least $1+\binom{n}{1}+\dots+\binom{n}{x}$ contrasts in U^n. We also saw that the only contrasts in U^n that are orthogonal to T_x are those that are not part of the linear combination of vectors in U^n that give T_x. Therefore, the largest value of x such that T_x can have contrasts in U^n orthogonal to it is the largest value of x such that $1+\binom{n}{1}+\binom{n}{2}+\dots+\binom{n}{x} < 2^n$, i.e., $x = n-1$.

With Theorem 1 and Corollary 1 in mind, we now describe a method for constructing trend free complete 2^n designs that is given in Cheng and Jacroux (1988). To begin with, write down the treatment combinations in a complete 2^n design in standard order, i.e., in the order (1), a_1, a_2, a_1a_2, a_3, a_1a_3, a_2a_3, $a_1a_2a_3$, ..., a_n, a_1a_n, ..., $a_1a_2 \dots a_n$. Let $s_1, s_2, \dots, s_n$ denote the main effect, contrasts $u_1, u_2, \dots, u_n$ derived from this standard ordering, i.e.,

$$s_1 = (-1, 1, -1, 1, \dots, -1, 1)', s_2 = (-1, -1, 1, 1, -1, -1, 1, 1, \dots, -1,$$
$$-1\ 1, 1)', \dots s_n = (-1, -1, -1, \dots, -1, 1, 1, 1, \dots, 1)'.$$

Cheng and Jacroux (1988) call $s_1, \dots, s_n$ the standard maine ffect contrasts and prove that $s_{i_1} o \dots o\, s_{i_t}$ is $(t-1)$-trend free for any subscripts $i_1, \dots, i_t$.

From this latter fact, we see that from the standard ordering of treatment combinations in a complete 2^n design we can derive $\binom{n}{x}$ contrasts that are $(x-1)$-trend free for $x = 1, \ldots, n$. Thus the standard ordering of treatment combinations in a complete 2^n design yields contrasts that attain the bounds given in Theorem 1 and Corollary 1. Now, to obtain a run order of a complete 2^n design that has main effect contrasts that are at least p_1-trend free one can use the following algorithm ;

(1) Consider the group U^n that is generated by the standard main effect contrasts $\mathbf{s}_1, \ldots, \mathbf{s}_n$. Find a set of n generators for U^n which involve products containing at least p_1+1 elements out of $s_1, \ldots, s_n$

(2) With each generator obtained in (1), identify exactly one of the main effects $\boldsymbol{A}_1, \ldots, \boldsymbol{A}_n$.

(3) Write down the run order of the 2^n complete design indicated by the assignment of main effects to generator contrasts given in (2), i.e., factor j occurs at its high or low level in run t depending upon whether the t-th entry of the generator contrast identified with A_j is a 1 or -1.

Example 1. To produce a 2^3 design where the main effects are at least 1-trend free, consider the set of three generators for $\boldsymbol{S}^n$ consisting of $\mathbf{s}_2 o \mathbf{s}_3 = (1, -1, -1, 1, 1, -1, -1, 1)'$, $\mathbf{s}_1 \, o \, \mathbf{s}_3 = (1, -1, 1, -1, -1, 1, -1, 1)'$, and $\mathbf{s}_1 \, o \, \mathbf{s}_2 \, o \, \mathbf{s}_3 = (-1, 1, 1, -1, 1, -1, -1, 1)^1$, If we assign $\boldsymbol{A}_1$, $\boldsymbol{A}_2$ and $\boldsymbol{A}_3$ to $\mathbf{s}_2 o \mathbf{s}_3$, $\mathbf{s}_1 o \mathbf{s}_3$ and $\mathbf{s}_1 o \mathbf{s}_2 o \mathbf{s}_3$, respectively, we obtain the run order $(a_1a_2,\ a_1\, a_3,\ a_2a_3,\ 1,\ a_3,\ 1,\ a_1,\ a_2\ a_1a_2a_3)$. This run order yields estimates for $\boldsymbol{A}_1$, $\boldsymbol{A}_2$ and $\boldsymbol{A}_3$ that are 1,1 and 2-trend free, respectively. To obtain the interaction contrasts, we take the appropriate products between the contrasts assigned to $\boldsymbol{A}_1$, $\boldsymbol{A}_2$ and $\boldsymbol{A}_3$. For example, the estimate for the $\boldsymbol{A}_1\boldsymbol{A}_2$ interaction term is 1-trend free since its corresponding contrasts is $(\mathbf{s}_2 o \mathbf{s}_3)\circ(\mathbf{s}_1 o \mathbf{s}_3) = \mathbf{s}_1 o \mathbf{s}_2$. However, the estimates for the interaction terms $\boldsymbol{A}_1\boldsymbol{A}_3$, $\boldsymbol{A}_2\boldsymbol{A}_3$ and $\boldsymbol{A}_1\boldsymbol{A}_2\boldsymbol{A}_3$ are not even 1-trend free since their corresponding contrasts are $\mathbf{s}_1$, $\mathbf{s}_2$ and $\mathbf{s}_3$ respectively.

4. Fractional factorials

In this section we consider the problem of constructing fractional factorial designs $d(n, m)$ that have orthogonal main effect contrasts that are at least 1-trend free. If $\boldsymbol{\mu}_t$ is an $m \times 1$ contrast that is at least 1-trend free, then $\boldsymbol{\mu}_t \boldsymbol{T}_0 = 0$. From this fact if follows that $\boldsymbol{\mu}_t$ must have an equal number of $+1$'s and -1's in it and that $(\boldsymbol{\mu}_t + \boldsymbol{T}_0)'\boldsymbol{T}_1 = -(\boldsymbol{\mu}_t - \boldsymbol{T}_0)'\boldsymbol{T}_1$. Using this latter expression and the fact that $\boldsymbol{T}_0'\boldsymbol{T}_1 = m(m+1)/2$, we see that m must be a multiple of 4. Thus we shall consider the construction of fractional factorial design

$d(n, m)$ where $m = 4s$ for some value of $s \geqslant 2$. Furthermore, we shall restrict our attention to main effects only designs. Thus the problem becomes one of constructing a set of orthogonal $m \times 1$ contrasts where $m = 4s$ for some $s \geqslant 2$ that can serve as estimates for some specified set of main effects. With this in mind, we consider two cases.

Case 1. $m = 4s$ where $s = 2p$ for some $p \geqslant 1$.

In this case, we consider even multiples of 4. One easy method of constructing a set of orthogonai contrasts for this case is via the use of Hadamard matrices. Detailed discussion on Hadamard matrices is available in Raghavarao (1971).

Without loss of generality, we can assume that the first row and column of a Hadamard matrix $\boldsymbol{H}_q$ consists of all $+1$'s. We shall make this assumption for the remainder of this section. Of course, given a Hadamard matrix $\boldsymbol{H}_q$, the last $q-1$ columns of $\boldsymbol{H}_q$ or some subset of these columns could serve as a set of mutually orthogonal contrasts for some specified set of main effects. However, there is no guarantee that these contrasts are at least 1-trend free. With this in mind, we give the following Theorem.

Theorem 2. *Let $\boldsymbol{H}_p$ and $\boldsymbol{H}_q$ be Hadamard matrices of orders p and q, and let $\bar{\boldsymbol{H}}_p$ and $\bar{\boldsymbol{H}}_q$ denote those matirces that are obtained by dropping the first column of all $+1$'s from $\boldsymbol{H}_p$ and $\boldsymbol{H}_q$, respectively. Then $\bar{\boldsymbol{H}}_p \otimes \bar{\boldsymbol{H}}_q$ is a $pq \times (p-1)(q-1)$ matrix whose columns are contrasts that are at least 1-trend free.*

Proof. This follows directly from the properties of Hadamard matrices and Lemma 1.

Corollary 2. *Suppose $m = 4s$ where $s = 2p$. Then there exists an orthogonal main effects only design $d(n, m)$ that has main effect contrasts that are at least 1-trend free for $1 \leqslant n \leqslant 4p-1$ provided there exists a Hadamard matrix $\boldsymbol{H}_q$ where $q = 4p$.*

Proof. Suppose $\boldsymbol{H}_q$ exists where $q = 4p$. Now consider the matrix $\bar{\boldsymbol{H}}_q$ defined in Theorem 1 and $(1, -1)' \otimes \bar{\boldsymbol{H}}_q$. The result now follows from Theorem 1.

Case 2. $m = 4s$ where $s = 2p+1$ for some $p \geqslant 1$

The construction of orthogonal main effects only designs for this case appears to be a much more difficult problem. In fact, the authors could find no systematic method for constructing trend free sets of multually orthogonal contrasts for this case. One logical approach to this problem is to start out with a Hadamard matrix $\boldsymbol{H}_m$ of order m, then by rearranging rows within $\boldsymbol{H}_m$, develop a set of columns that are at least 1-trend free. In proceeding along these lines, the authors have had some limited success. But there seems to be no way of determining how many trend free contrasts can be obtained using this technique.

Example 2. Consider the case where $m = 12$. For this number of observations, the authors could find at most 3 orthogonal contrasts that are at least 1-trend free. One set of such contrasts is given by

$$(1, 1, 1, -1, -1, -1, -1, -1, -1, 1, 1, 1)',$$

$$(1, -1, -1, 1, 1, -1, -1, 1, -1, 1, 1, -1)',$$

$$(1, -1, 1, -1, -1, -1, 1, 1, 1, -1, 1, -1)',$$

We note that the last contrast given above is also 2-trend free and is the only 2-trend free contrast that the authors could find for $m = 12$.

Comment : If one relaxes the condition of mutual orthogonality in the problem of constructing trend free contrasts, then a good deal more can be done with respect to constructing 1-trend free main effects only designs. For example, let $\boldsymbol{P}$ be an $a \times b$ matrix of rank b and let $\boldsymbol{Q}$ be a $c \times d$ matrix of rank d such that all the columns of $\boldsymbol{P}$ and $\boldsymbol{Q}$ each contain an equal number of $+1$'s and -1's. Then by Lemma 1 the $ac \times bd$ matrix $\boldsymbol{P} \otimes \boldsymbol{Q}$ has columns that can serve as a set of linearly independent contrasts that are at least 1-trend free. A design $d(n, m)$ that is constructed based on contrasts such as those given bv the columns of $\boldsymbol{P} \otimes \boldsymbol{Q}$ will not in general have orthogonal estimates for main effects and hence will not be optimal. However, these more general designs constructed in this manner will yield esimates for main effects that are not influenced by at least unknown linear trends.

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