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EDUCATION AND PSYCHOLOGY

Chairman: PROFESSOR HAROLD HOTELLING

STATISTICAL ANALYSIS OF EXPERIMENTS ON DIFFERENTIAL LIMEN VALUES FOR LIFTED WEIGHTS.

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Mr. Gopeswar Pal of the Calcutta University sent us for a statistical study the material collected by him in the course of certain experiments on the Differential Limen for Lifted Weights and also the following description of the experiments :—

"Since the time of Weber, experiments on lifted weights have been conducted in almost all the Psychological laboratories of the world. Various factors which influence the judgment of the observer have been sought to be analysed and measured.

In all such experiments a given standard weight is lifted by the observer and immediately after this one of the several companion weights is lifted and judgment regarding heaviness of the latter as compared with that of the standard is given in terms of "equal", "less" or "heavier". These comparison weights generally differ among themselves by a finite value and are once for all fixed before the experiment begins.

In the present investigation an apparatus was specially devised with the help of which it was possible to increase *continuously* the standard weights at a given rate, while it was being held by the observer, till the increment reached a definite value.

The same standard weight was increased continuously to different amounts in separate trials. At the end of each trial the observer's judgments regarding increase in terms of "equal" or "heavier" were recorded. For example, the standard weight of 100 gms. was increased to 105, 110, 115, 120, 125, 130 or 135 gms. in separate experiments in a random order, and this sequence of seven experiments was repeated one hundred times.

The amount of increase necessary for just noticeable heavierness is regarded as Differential Limen (or D.L.) and the just noticeable heavierness is defined as the heavierness which is as often appreciated as not. Altogether 100 x 7 experiments were performed to determine one D.L. value for a given standard weight with a given rate of increase.

Four different rates of increase were used, namely, 50, 100, 200 and 400 grams per 30 seconds. In order to study the relation between D.L. values and standard weights, seven different standard weights namely, 100, 150, 200, 250, 300, 350 and 400 grms. were used. D. L. values for each of these standard weights at each of the above four rates of increase were obtained.

The experiment was independently done by two observers. After an interval of six months the whole set of experiments was repeated with the same two observers.

In this way 112 D.L. values were obtained each value being determined by a process involving 700 repetitions of the same operation.

It was found by Weber and others who worked with discontinuous increments of the comparison weights that the D.L. value was directly proportional to the value of standard (within certain limits in the median region of the values of the standard).

The present extensive series of investigation however has not confirmed the proportional relation, so long held to be applicable in all cases; this will be clear from a statistical study of the material."

THE STATISTICAL DESIGN OF THE EXPERIMENT

The data will thus be seen to conform to a $4 \times 7 \times 2 \times 2$ factorial design without replication (since dates have been taken as a factor). Fisher's technique of analysis of variance may therefore be conveniently applied.

TOTAL ANALYSIS OF VARIANCE

Table 1 presents the analysis of variance partitioning the total 111 degrees of freedom into those for four main effects, six two-factor interactions and the 'residual'

TABLE 1. ANALYSIS OF VARIANCE OF "D.L." VALUES

Due to:	D. F.	Sum of Squares	Variance	'Error' Variance	Ratio of Variances		
					Observed	Expected	
						5%	1%
(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)
Rates (R.)	3	2891·29	963·76	2·90 (57)	332·33	2·8	4·2
Weights (W.)	6	3862·58	643·76	2·64 (60)	243·85	2·3	3·1
Observers (Ob.)	1	437·66	437·66	2·20 (45)	198·94	4·1	7·3
Dates (D.)	1	9·26	9·26	3·17 (45)	2·92	4·1	7·3
R. × W.	18	269·87	14·99	2·88 (54)	5·20	1·9	2·4
R. × Ob.	3	47·32	15·77	2·46 (39)	6·41	2·9	4·4
R. × D.	3	5·34	1·78	3·59 (39)	0·50	2·9	4·4
W. × Ob.	6	29·31	4.88	2·12 (42)	2·30	2·4	3·3
W. × D.	6	5·84	0·97	3·17 (42)	0·31	2·4	3·3
Ob. × D.	1	3·09	3·09	2·72 (27)	1·14	4·2	7·7
Residual	63	168·07	2·67				
Total	111	7729·63					

The 'residual' consists of four three-factor interactions and one four-factor interaction. The sum of squares corresponding to each of the five components of 'residual' are given in Table 2.

TABLE 2. COMPONENTS OF THE RESIDUAL SUM OF SQUARES

Due to:	D. F.	Sum of Squares	Variance
(1)	(2)	(3)	(4)
R × W × Ob.	18	25·26	1·40
R × W × D	18	69·27	3·85
R × Ob. × D	3	9·82	3·27
W × Ob. × D	6	2·78	0·46
R×W×Ob.×D	18	60·94	3·39
Residual	63	168·07	2·67

Usually interactions involving three or more factors are found to be non-existent so that a variance estimated from them can be taken to be only of the order of error. As in any case such an estimate is more likely than not to be greater than the true error variance, our tests of significance will be all the more stringent.

Specific Estimates of Error Variance

By pooling together all the three-factor and four-factor interactions the error variance works out to be 2·67 with 63 degrees of freedom. But as is evident from column (4) of Table 2 there is a certain amount of heterogeneity in variances due to each of the five components of this error. It will logically be a better procedure to get an 'error' variance separately for each of the main effects and two-factor interactions.

Thus the 'error' variance corresponding to main effect of rates will include only the interactions in which "Rates" are involved, namely R × W × Ob., R × W × D, R × Ob. × D and R × W × Ob. × D. The 'error' variance corresponding to the interaction: Rates × Weights, will include the interactions: R × W × Ob., R × W × D and R × W × Ob. × D.

Similar calculations have been done for each of the other main effects and two-factor interactions and are given in column (5) of Table 1. The number within brackets against each 'error' variance shows the degrees of freedom.

Tests of Significance

The tests of significance made in columns (6), (7) and (8) of Table 1 lead to the following conclusions:

Significant main effects:	Rates, Weights, Observers.
Significant two-factor interactions:	Rates × Weights, Rates × Observers.

We may consider these results in greater detail.

Main Effects of Rates

The main effect of Rates tells us whether the mean D.L. values for each of the four rates (the mean being taken over all the other factors) differ among themselves or are stationary. Since this main effect has been found to be highly significant, we conclude that the mean D.L. value varies significantly with the "Rate".

We may now investigate whether it is possible to describe the connexion between D.L. values and Rates by a simple mathematical expression. As we want to use the analysis of variance for testing the significance of the fit, it is convenient to use orthogonal polynomials. Fortunately in this case we can convert the independent variate (namely "Rates") into an arithmetic progression by using the logarithmic scale. That is, we use $x=1$ for Rate 50, $x=2$ for Rate 100; $x=3$ for Rate 200 and $x=4$ for Rate 400.

Using a second degree parabola, we then get the following graduation, Y being the D.L. value:—

$$Y = A_r + B_r x + C_r (x^2 - 1{\cdot}25) = 28{\cdot}693 - 2{\cdot}116\ x - 4{\cdot}490\ x^2$$

$$\text{where, } x = \log \frac{R}{100\sqrt{2}} \div \log 2$$

A_r is nothing but the mean value of Y (or D.L.). B_r and C_r are respectively the linear and quadratic coefficients, and on their magnitudes depends the appropriateness of including the linear and quadratic terms. The significance of the linear and quadratic terms are tested in Table 3 and they are both found to be highly significant.

TABLE 3. COMPONENTS OF VARIATION DUE TO RATES

Due to:	D. F.	Sum of Squares	Variance	Ratio of Variances: Observed [(4)÷2·90]	Expected 5%	Expected 1%
(1)	(2)	(3)	(4)	(5)	(6)	(7)
Linear (log. Rate)	1	626·25	626·25	215·95	4·0	7·1
Quadratic (log Rate)	1	2255·42	2255·42	777·73	4·0	7·1
Deviation from second degree parabola	1	9·62	9·62	3·32	4·0	7·1
Rates	3	2891·29				

MAIN EFFECTS OF WEIGHTS

The main effect of "Weights" has also been found to be highly significant, which shows that the mean D.L. value varied considerably from weight to weight. As the weights are already in arithmetic progression, a second degree parabola was fitted with D.L. values as the dependent variable and weight as the independent variable, and the following curve was obtained:—

$$Y = A_w + B_w\ x + C_w\ (x^2 - 4) = 23{\cdot}38 + 2{\cdot}932x - 0{\cdot}075x^2$$

$$\text{where, } x = \frac{W - 250}{50}$$

TABLE 4. COMPONENTS OF VARIATION DUE TO WEIGHTS

Due to:	D. F.	Sum of Squares	Variance	Ratio of Variances: Observed [(4)÷2·64]	Expected Ratio 5%	Expected Ratio 1%
(1)	(2)	(3)	(4)	(5)	(6)	(7)
Linear	1	3852·83	3852·83	1459·41	4·0	7·1
Quadratic	1	7·50	7·50	2·84	4·0	7·1
Deviation from second degree parabola	4	2·27	0·57	0·22	2·5	3·6
Weights	6	3862·60				

The quadratic component is insignificant; so that the relation between D. L. and weight can be adequately represented by a linear formula.—

$$Y = A_w + B_w\ x = 23{\cdot}08 + 2{\cdot}932x,$$

where, $x = (W - 250)/50$

The tests of significance of the linear and quadratic terms are shown in Table 4.

INTERACTION BETWEEN RATES AND WEIGHTS

The interaction between Rates and Weights indicates whether the pattern of variation in D.L. values with Rates remains the same or not from Weight to Weight or conversely whether the pattern of variation in D.L. values with weights remains the same or not from Rate to Rate.

We have seen from the main effect of Rates, that the pattern of variation of D.L. values with Rate, when all weights are combined, is given by a second degree parabola, when log. (Rate) is taken as the independent variable. We can fit similar curves between D.L. value and log (Rate) separately for each of the seven Weights. These equations will then represent the pattern of variation between D.L. and log (Rate) for each of the Weights. If these patterns do not differ among themselves throughout the whole range of Weights, then the 7 linear coefficients and the 7 quadratic coefficients should remain constant within the limits of errors of sampling. The seven relevant equations are given below :—

Weight	Equations connecting 'D. L.' Value and Rate y='D.L.' Value, x=log. (Rate÷100√2)/log. 2
100	$y = 16{\cdot}519 - 1{\cdot}010x - 2{\cdot}175x^2$
150	$y = 21{\cdot}413 - 1{\cdot}789x - 3{\cdot}242x^2$
200	$y = 25{\cdot}231 - 2{\cdot}225x - 3{\cdot}737x^2$
250	$y = 28{\cdot}876 - 2{\cdot}414x - 4{\cdot}605x^2$
300	$y = 32{\cdot}793 - 1{\cdot}980x - 5{\cdot}250x^2$
350	$y = 36{\cdot}681 - 2{\cdot}895x - 6{\cdot}137x^2$
400	$y = 39{\cdot}338 - 2{\cdot}495x - 6{\cdot}262x^2$

These equations are shown graphically in Chart 1.

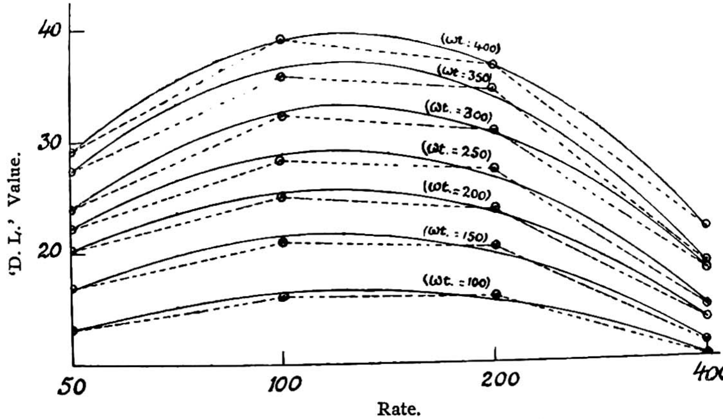

Chart I. Relation between 'D. L.' Value and Rate

We find that the B_r values and C_r values do not appear to remain the same in the equations. In fact if we plot the 7 values of B_r against the 7 weights, they are seen to lie practically on a straight line. Similarly if we plot the 7 values of C_r against the 7 weights, a linear variation is apparent. This feature of the coefficients B_r and C_r is effectively brought out in the analysis presented in Table 5.

TABLE 5. COMPONENTS OF VARIATIONS DUE TO RATES & WEIGHTS

Due to:	D. F.	Sum of Squares	Variance	Ratio of Variances		
				Observed [(4)÷2·88]	Expected 5%	Expected 1%
(1)	(2)	(3)	(4)	(5)	(6)	(7)
Rate linear × Weight linear	1	29·46	29·46	10·23	4·0	7·2
Rate linear × Weight quadratic	1	5·36	5·36	1·86	4·0	7·2
Rate quadratic × Weight linear	1	218·74	218·74	75·95	4·0	7·2
Rate quadratic × Weight quadratic	1	1·94	1·94	0·67	4·0	7·2
Residual (Rate × weight)	14	14·37	1·03	0·35	1·9	2·5
Rates × Weights	18	269·87				

We find from this table that both the linear as well as the quadratic variation of D.L. value with log (Rate) change in a linear way with weights.

These interesting properties of the B and C coefficients enable us to extend the results of the present experiment to cover other cases and to predict the D.L. values for any combination of Rate and Weight which lie within the range of observation of the present experiment, that is, for any Rate between 50 and 400 gms. per 30 seconds and any Weights between 100 and 400 gms.

Rate	Equations connecting 'D.L.' Value and Weight y='D.L.' Value, $x = \frac{\text{Weight (in gms.)} - 250}{50}$
50	$y = 22{\cdot}424 + 2{\cdot}568\,x - 0{\cdot}131x^2$
100	$y = 28{\cdot}696 + 3{\cdot}777\,x - 0{\cdot}114x^2$
200	$y = 27{\cdot}358 + 3{\cdot}486\,x - 0{\cdot}112x^2$
400	$y = 15{\cdot}062 + 1{\cdot}900\,x + 0{\cdot}057x^2$

We may now study the same Table 5 from the point of view of the change in pattern of variation in D.L. values with Weights for a given Rate, when the Rate is changed. We have seen that when averaged over all the Rates, the mean D.L. value changes with weight according to a linear formula. The four equations for the four individual Rates are given above, and represented graphically in Chart 2.

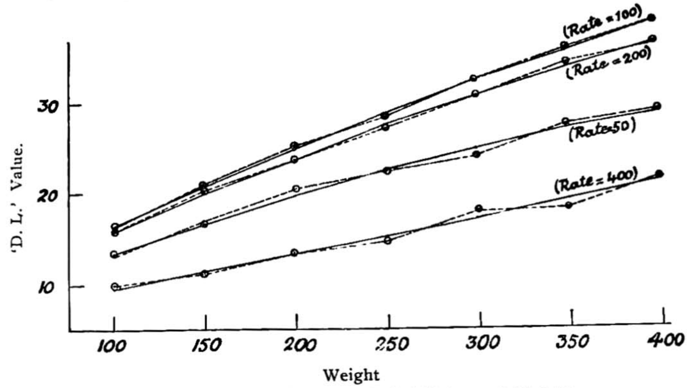

Chart 2. Relation between 'D. L.' Value and Weight

It will be seen that the four B_w coefficients can be graduated with a second degree parabola on log (Rate). We find then that there are both linear and quadratic components on log (Rate) in the variation of the linear component of the variation of D.L. value with Weight.

Table 5.1 summarises the results set forth in this section. Besides the observed mean D.L. values for each of the 28 Rate-Weight combinations, the main effects of Rates and

TABLE 5.1. SUMMARY OF RESULTS FOR RATE × WEIGHT COMBINATIONS

Rate:	50	100	200	400	Mean	ξ_1 (w)	ξ_2 (w)	A_r	B_r	C_r
Weight: 100	13·25	16·15	15·80	10·00	13·80	− 3	+ 5	13·80	−1·010	−2·175
150	16·98	20·95	20·25	11·25	17·36	− 2	0	17·36	−1·789	3·242
200	20·30	25·00	23·60	13·35	20·56	− 1	− 3	20·56	−2·225	−3·737
250	22·33	28·35	27·10	14·70	23·12	0	− 4	23·12	−2·414	−4·605
300	24·00	32·30	30·65	17·95	26·23	+ 1	− 3	26·23	1·980	5·250
350	27·45	35·90	34·40	18·30	29·01	+ 2	0	29·01	−2·895	−6·137
400	29·00	39·00	36·55	21·50	31·51	+ 3	+ 5	31·51	−2·495	−6·262
Mean	21·90	28·24	26·91	15·29	23·08			23·08	−2·116	−4·490
ξ_1 (r)	−3/2	−½	+½	+3/2						
ξ_2 (r)	+ 1	− 1	− 1	+ 1						
A_w	21·90	28·24	26·91	15·29	23·08					
B_w	+ 2·568	+ 3·777	+ 3·486	+ 1·900	+ 2·932					
C_w	− 0·131	− 0·114	− 0·112	+ 0·057	− 0·075					

Weights are shown in row 9 and column 6 respectively ; columns 9, 10 and 11 give the constants of the curve fitted to D.L. value and log (Rate) for each of the 7 Weights and mean over all Weights ; and rows 12, 13 and 14 give the constants of the curve fitted to D.L. value and Weight for each of the 4 Rates and for mean over all rates. The numbers written in rows 10 and 11 and in columns 7 and 8 are necessary in getting B_r, C_r, B_w and C_w.

It may be noted that the sums of squares of various components in Tables 3, 4 and 5 can be calculated from the figures given in the present Table. Thus to get the sums of squares due to the component: Rate-linear × Weight-quadratic, we have (*i*) to multiply each of the 28 mean D.L. values of the Rate-Weight combinations by two numbers namely values of the $\xi_1(r)$ and $\xi_2(w)$ which occur in the same column and in the same row respectively as itself: (*ii*) add up the 28 products, (*iii*) square the sum and (*iv*) divide by the product of the sum of squares of $\xi_1(r)$, namely, 5 and of the sum of squares of $\xi_2(w)$, namely, 84.

INTERACTION BETWEEN RATES AND OBSERVERS

The interaction between Rates and Observers, which was found significant in Table 1, indicates that the pattern of variations of D.L. value with log (Rate) for one observer is different from the pattern for the other observer. Relevant equations are given below for the two observers :—

Observer	Equations connecting 'D. L.' Value and Rate y='D. L.' Value, x=Log (Rate÷100√2) / Log.2
1	$y = 25{\cdot}919 - 2{\cdot}060\,x - 3{\cdot}850\,x^2$
2	$y = 31{\cdot}467 - 2{\cdot}170\,x - 5{\cdot}125\,x^2$

These equations are shown graphically in Chart 3.

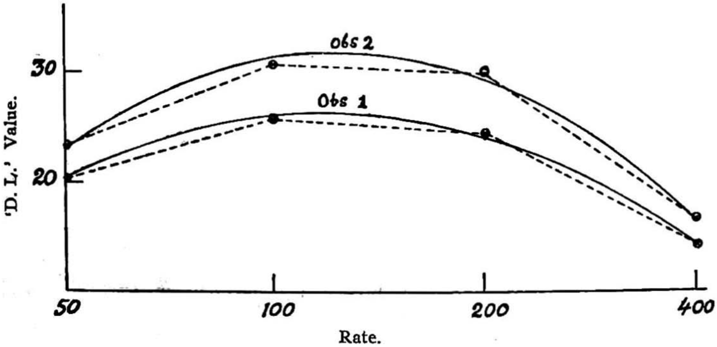

Chart 3. Relation between 'D. L.' Value and Rate for each observer

We may next test whether the two linear coefficients and the two quadratic coefficients are significantly different or not. This is done in Table 6, by breaking up the 3 degrees of freedom due to Rate × Observers.

We find that the quadratic coefficients are significantly different, but not the two linear coefficients.

TABLE 6. COMPONENTS OF VARIATION DUE TO RATES × OBSERVERS

Due to :	D. F.	Sum of Squares	Variance	Ratio of Variances		
				Observed [(4)÷2·46]	Expected	
					5%	1%
(1)	(2)	(3)	(4)	(5)	(6)	(7)
Rate linear × Observers	1	0·42	0·42	0·17	4·1	7·4
Rate quadratic × Observers	1	45·52	45·52	18·50	4·1	7·4
Rate deviation × Observers	1	1·38	1·38	0·56	4·1	7·4
Rates × Observers	3	47·32				

Table 6.1 gives the observed mean D.L. values for each of the 8 Rate × Observer combinations.

TABLE 6.1. SUMMARY OF RESULTS FOR RATE × OBSERVER COMBINATIONS

	Rate	50	100	200	400	A_r	B_r	C_r
Observer	1	20·43	25·74	24·17	14·09	21·107	−2·060	−3·850
	2	23·37	30·73	29·64	16·50	25·061	−2·170	−5·125

INTERACTION BETWEEN WEIGHTS AND OBSERVERS

The interaction between Weights and Observers was not significant in Table 1. But, since the sum of squares corresponding to this interaction is the biggest among non-significant components of variation of Table 1, the corresponding 6 degrees of freedom have been broken up by fitting separate curves for each observer, connecting D.L. value and Weight.

TABLE 7. COMPONENTS OF VARIATION DUE TO WEIGHTS × OBSERVERS

Due to :	D. F.	Sum of Squares	Variance	Ratio of Variances		
				Observed [(4)÷2·12]	Expected	
					5%	1%
(1)	(2)	(3)	(4)	(5)	(6)	(7)
Weight linear × Observers	1	1·93	1·93	0·91	4·1	7·4
Weight quadratic × Observers	1	8·87	8·87	4·18	4·1	7·4
Weight deviation × Observers	4	18·51	4·63	2·18	2·6	3·9
Weights × Observers	6	29·31				

The results are given in Table 7, and we find that the two equations are not significantly different.

Table 7.1 gives the observed mean D.L. values for each of the 14 Weights × Observer combinations.

TABLE 7.1. SUMMARY OF RESULTS FOR WEIGHT × OBSERVER COMBINATIONS

Weight		100	150	200	250	300	350	400	A_w	B_w	C_w
Observer	1	12·20	15·56	18·83	21·21	23·13	26·78	30·05	21·107	2·867	+ 0·007
	2	15·40	19·15	22·30	25·03	29·33	31·25	32·98	25·061	2·998	− 0·156

SUMMARY OF CONCLUSIONS

In this paper the method of analysis of variance and non-linear regressions have been used for studying extensive experimental material on the Differential Limen for Lifted Weights. Four different rates of increase (50, 100, 200 and 400 grams) per 30 seconds, and seven different standard weights (100, 150, 200, 250, 300, 350 and 400 grams) were used. The experiment was done independently by two observers, and was repeated by the same two observers after an interval of six months. The material thus conformed to a factorial design of the type :—4 Rates × 7 Weights × 2 Observers × 2 Dates. A summary of the results is given below :

(1) There was no appreciable difference due to Dates showing that the observations were consistent among themselves.

(2) The mean value of the Differential Limen (D.L.) varied systematically with changes in Rates, Weights, and Observers.

(3) Two-factor interactions between Rates and Weights, and between Rates and Observers were significant. The interaction between Weights and Observers was not significant.

(4) For any given Weight, the connexion between D.L. value and Rate was non-linear, and a simple parabolic curve gave satisfactory graduation. The parameters of the parabolic curve of the regression of D.L. on Rates themselves varied systematically with the Weights.

(5) The regression of D.L. values on Weights was linear in character within errors of sampling. The actual value of the linear coefficient varied systematically with the Rates.

(6) The systematic variations of the parameters of the parabolic regression on Rate with changes in Weight, and of the parameter of the linear regression on Weight with changes in Rate enable a two-way table of combinations of Rates and Weights being constructed from which the D.L. value for any particular combination within the ranges of values used in this experiment can be easily calculated with the help of suitable graduating equations.

(7) For any given Observer, the regression of D.L. values on Rates can be graduated by a simple parabolic curve and the regression on Weights by a straight line. The parameters of the parabolic curve on Rates were significantly different for the two Observers. This shows that changes in the D.L. values with Rate depended on the particular Observer concerned. The parameter of the linear regression on Weights did not differ significantly for the two Observers. For a given Observer it is possible to calculate the D.L. values corresponding to different Rates with the help of parabolic graduating curves the parameters of which can be obtained from other graduating curves depending on the Weight.

Apart from the psychological results, this paper brings out the advantages and possibilities of using suitable statistical tools in designing and analysing observations involving a large number of factors in experimental psychology.