isid/ms/2007/02
March 5, 2007
http://www.isid.ac.in/~statmath/eprints

# Torus equivariant spectral triples for odd dimensional quantum spheres coming from $C^*$-extensions

Partha Sarathi Chakraborty
Arupkumar Pal

Indian Statistical Institute, Delhi Centre
7, SJSS Marg, New Delhi–110 016, India

# Torus equivariant spectral triples for odd dimensional quantum spheres coming from $C^*$-extensions

PARTHA SARATHI CHAKRABORTY and ARUPKUMAR PAL

March 16, 2007

**Abstract**

The torus group $(S^1)^{\ell+1}$ has a canonical action on the odd dimensional sphere $S_q^{2\ell+1}$. We take the natural Hilbert space representation where this action is implemented and characterize all odd spectral triples acting on that space and equivariant with respect to that action. This characterization gives a construction of an optimum family of equivariant spectral triples having nontrivial $K$-homology class thus generalizing our earlier results for $SU_q(2)$. We also relate the triple we construct with the $C^*$-extension

$$0 \longrightarrow \mathcal{K} \otimes C(S^1) \longrightarrow C(S_q^{2\ell+3}) \longrightarrow C(S_q^{2\ell+1}) \longrightarrow 0.$$

**AMS Subject Classification No.:** 58B34, 46L87, 19K33
**Keywords.** Spectral triples, noncommutative geometry, quantum group.

## 1 Introduction

In noncommutative geometry (NCG), a geometric space is described by a triple $(\mathcal{A}, \mathcal{H}, D)$, called a spectral triple, with $\mathcal{A}$ being an involutive algebra represented as bounded operators on a Hilbert space $\mathcal{H}$, and $D$ being an unbounded selfadjoint operator with compact resolvent and having bounded commutators with the algebra elements. The operator $D$ should be nontrivial in the sense that the associated Kasparov module should give a nontrivial element in $K$-homology. A natural question is, are there enough spectral triples around us? The answer is both yes and no. If we do not demand any further properties then by a theorem of Baaj and Julg ([1]; [7], chapter 4, appendix A), given any countably generated subalgebra $\mathcal{A}$ of a $C^*$-algebra there exists a spectral triple $(\mathcal{A}, \mathcal{H}, D)$. But if we demand further properties like finite summability then given a dense subalgebra of a $C^*$-algebra it may not admit a finitely summable spectral triple ([6]). Therefore given a natural dense subalgebra of a $C^*$-algebra it is meaningful to ask whether it admits finitely summable nontrivial spectral triples. Also, the result of Baaj & Julg starts from a Fredholm module, so one has very little control over the Hilbert space or the representation.

In an earlier paper ([5]), the authors studied spectral triples for the odd dimensional quantum spheres taking the Hilbert space to be the $L_2$ space of the sphere and the representation to be the natural representation by left multiplication there. In the present article, we fix a different representation space dictated by the torus action on the sphere, and investigate spectral triples for that. The results here generalize those in [4].

We will use the method described in [5] and used implicitly in [3] and [4]. Observe that the self-adjoint operator $D$ in a spectral triple comes with two very crucial restrictions on it, namely, it has to have compact resolvent, and must have bounded commutators with algebra elements. Various analytic consequences of the compact resolvent condition (growth properties of the commutators of the algebra elements with the sign of $D$) have been used in the past by various authors. We will exploit it from a combinatorial point of view. The idea is very simple. Given a selfadjoint operator with compact resolvent, one can associate with it a certain graph in a natural way. This makes it possible to do a detailed combinatorial analysis of the growth restrictions (on the eigenvalues of $D$) that come from the boundedness of the commutators, and to characterize the sign of the operator $D$ completely.

We take a representation space where the canonical action of $(S^1)^{\ell+1}$ on $C(S_q^{2\ell+1})$ is implemented. If we further want our Dirac operator $D$ to be equivariant with respect to the torus action then $D$ should commute with the unitary implementing that action. Hence $D$ respects the spectral subspaces. This allows us to write down the form of the Dirac operator. Then using the boundedness of the commutators we completely characterize all equivariant Dirac operators. We also produce a nontrivial optimal equivariant Dirac.

Odd dimensional quantum spheres of successive dimension are related through a short exact sequence that says that the $(2\ell+3)$-dimensional sphere $C(S_q^{2\ell+3})$ is an extension of the $(2\ell+1)$-dimensional sphere $C(S_q^{2\ell+1})$ by $C(S^1)$. One can naturally associate a $KK_1(C(S_q^{2\ell+1}), C(S^1))$ element with such an extension (for a discussion on the relation between $C^*$-extensions and KK-elements, see chapter 8, section 17, Blackadar [2]). In the last section, we compute this KK-element and show that the generic spectral triple that we construct in section 3 comes from this KK-element.

## 2 Torus action on quantum spheres

Let $q \in (0,1)$. The $C^*$-algebra $A_\ell = C(S_q^{2\ell+1})$ of continuous functions on the quantum sphere $S_q^{2\ell+1}$ is the universal $C^*$-algebra generated by elements $z_1, z_2, \ldots, z_{\ell+1}$ satisfying the following relations (see [8], [10]):

$$\begin{aligned} z_i z_j &= q z_j z_i, && 1 \le j < i \le \ell+1, \\ z_i z_j^* &= q z_j^* z_i, && 1 \le i \ne j \le \ell+1, \\ z_i z_i^* - z_i^* z_i + (1-q^2) \sum_{k>i} z_k z_k^* &= 0, && 1 \le i \le \ell+1, \end{aligned} \tag{2.1}$$

$$\sum_{i=1}^{\ell+1} z_i z_i^* = 1.$$

Let $N$ be the number operator given by $N : e_n \mapsto n e_n$ on $L_2(\mathbb{N})$ and $S$ be the shift $S : e_n \mapsto e_{n-1}$. We will use the same symbol $S$ to denote shift on $L_2(\mathbb{N})$ as well as on $L_2(\mathbb{Z})$. In the case of $L_2(\mathbb{N})$, $S(e_0)$ is defined to be zero. Let

$$\mathcal{H}_\ell = \underbrace{L_2(\mathbb{N}) \otimes \cdots \otimes L_2(\mathbb{N})}_{\ell \text{ copies}} \otimes L_2(\mathbb{Z}).$$

Let $\pi_\ell$ be the representation of $A_\ell$ on the space $\mathcal{L}(\mathcal{H}_\ell)$ of bounded operators on $\mathcal{H}_\ell$ given on the generators by

$$\begin{aligned} z_k &\mapsto \underbrace{q^N \otimes \ldots \otimes q^N}_{k-1 \text{ copies}} \otimes \sqrt{1-q^{2N}} S^* \otimes \underbrace{I \otimes \cdots \otimes I}_{\ell+1-k \text{ copies}}, \quad 1 \le k \le \ell, \\ z_{\ell+1} &\mapsto \underbrace{q^N \otimes \cdots \otimes q^N}_{\ell \text{ copies}} \otimes S^*, \end{aligned}$$

Then $\pi_\ell$ gives a faithful representation of $A_\ell$ on $\mathcal{H}_\ell$ (see lemma 4.1 and remark 4.5, [8]). Observe that the $C^*$-algebra generated by the operator $S$ on $L_2(\mathbb{Z})$ is isomorphic to $C(S^1)$. Using this and the identification

$$\mathcal{L}(L_2(\mathbb{N}^\ell) \otimes C(S^1)) \cong \mathcal{L}(\mathcal{K}(L_2(\mathbb{N}^\ell)) \otimes C(S^1)),$$

where $\mathcal{L}(\cdot)$ denotes the space of bounded adjointable operators and $\mathcal{K}(\cdot)$ denotes the space of compact operators, one can see that for all $a \in A_\ell$, the operators $\pi_\ell(a)$ actually lift to adjointable operators on the Hilbert $C(S^1)$-module $L_2(\mathbb{N}^\ell) \otimes C(S^1)$.

The $K$-groups of these $C^*$-algebras have been computed by Vaksman & Soibelman and Hong & Szymanski:

**Proposition 2.1 ([10],[8])** *$K_0(A_\ell) = K_1(A_\ell) = \mathbb{Z}$.*

The group $(S^1)^{\ell+1}$ has an action on $C(S_q^{2\ell+1})$ given on the generating elements by

$$\tau_{\mathbf{w}}(z_i) = w_i z_i, \qquad \mathbf{w} = (w_1, w_2, \ldots, w_{\ell+1}) \in (S^1)^{\ell+1}.$$

If $U_{\mathbf{w}}$ denotes the unitary $w_1^N \otimes w_2^N \otimes \cdots \otimes w_{\ell+1}^N$ on $\mathcal{H}_\ell$, then one has $\pi_\ell(\tau_{\mathbf{w}}(a)) = U_{\mathbf{w}} \pi_\ell(a) U_{\mathbf{w}}^*$ for all $a \in C(S_q^{2\ell+1})$. Thus $(\pi_\ell, U)$ is a covariant representation of $(A_\ell, (S^1)^{\ell+1}, \tau)$ on $\mathcal{H}_\ell$. In the next section, we characterize all equivariant spectral triples for this representation and construct an optimal triple using this characterization.

## 3 Equivariant spectral triples

Let $\Gamma = \underbrace{\mathbb{N} \times \cdots \times \mathbb{N}}_{\ell \text{ copies}} \times \mathbb{Z}$, so that $L_2(\Gamma) = \mathcal{H}_\ell$. For $\gamma = (\gamma(1), \gamma(2), \cdots, \gamma(\ell+1)) \in \Gamma$, $e_\gamma$ denotes the basis element of $\mathcal{H}_\ell$ given by $e_{\gamma(1)} \otimes \cdots \otimes e_{\gamma(\ell+1)}$.

**Theorem 3.1** *Let $D$ be a self-adjoint operator with compact resolvent on $\mathcal{H}_\ell$ that commutes with the operators $U_{\mathbf{w}}$. Then $D$ must diagonalise with respect to the canonical basis, i. e. must be of the form*

$$e_\gamma \mapsto d(\gamma) e_\gamma, \tag{3.2}$$

*where $d(\gamma) \in \mathbb{R}$ for all $\gamma \in \Gamma$.*

*Moreover, such an operator $D$ will have bounded commutators with elements from the *-subalgebra of $C(S_q^{2\ell+1})$ generated by the $z_i$'s if and only if the $d(\gamma)$'s obey the following condition:*

$$|d(\gamma) - d(\gamma + \epsilon_k)| = O(q^{-\gamma(1)-\ldots-\gamma(k-1)}), \qquad 1 \le k \le \ell + 1, \tag{3.3}$$

*where $\epsilon_k$ stands for the vector whose $k^{th}$ coordinate is 1 and all other coordinates are 0.*

*Proof*: The first part is immediate. For the second part, just observe that

$$\begin{aligned} [D, \pi(z_k)] e_\gamma &= (d(\gamma + \epsilon_k) - d(\gamma)) q^{\gamma(1)+\ldots+\gamma(k-1)} \sqrt{1 - q^{2\gamma(k)+2}} e_{\gamma+\epsilon_k}, \quad 1 \le k \le \ell, \\ [D, \pi(z_{\ell+1})] e_\gamma &= (d(\gamma + \epsilon_{\ell+1}) - d(\gamma)) q^{\gamma(1)+\ldots+\gamma(\ell)} e_{\gamma+\epsilon_{\ell+1}}. \end{aligned}$$

□

By a compact perturbation, one can ensure that all the $d(\gamma)$'s are nonzero in the above theorem. We will assume from now on that $d(\gamma) \neq 0$ for all $\gamma$. Using (3.3) we get a constant $c$ such that $|d(\gamma) - d(\gamma + \epsilon_k)| q^{-\gamma(1)-\ldots-\gamma(k-1)} < c$, with $\epsilon_k$ as in the theorem. Now join two elements $\gamma$ and $\gamma'$ in $\Gamma$ by an edge if $|d(\gamma) - d(\gamma')| \le c$. Call the resulting graph $\mathcal{G}$ the growth graph for $D$.

**Lemma 3.2** *Let $k$ be an integer with $1 \le k \le \ell + 1$. Let*

$$\gamma = (0, \ldots, 0, r, i_{k+1}, \ldots, i_{\ell+1}), \quad \gamma' = (0, \ldots, 0, s, i_{k+1}, \ldots, i_{\ell+1}).$$

*Then there is a path in $\mathcal{G}$ of length $|r - s|$ joining $\gamma$ and $\gamma'$ such that all vertices on this path are of the form $(0, \ldots, 0, t, i_{k+1}, \ldots, i_{\ell+1})$.*

*Proof*: Assume without loss in generality that $\gamma(k) < \gamma'(k)$. Write $t = \gamma'(k) - \gamma(k)$. From (3.3), it is clear that if $\delta(i) = 0$ for $1 \le i \le k - 1$, then there is an edge joining $\delta$ and $\delta + \epsilon_k$. Thus $(\gamma, \gamma + \epsilon_k, \gamma + 2\epsilon_k, \ldots, \gamma + t\epsilon_k)$ will give us a required path. □

**Lemma 3.3** *Let $k$ be an integer with $1 \leq k \leq \ell + 2$. Let*

$$\gamma = (i_1, \ldots, i_{k-1}, i_k, \ldots, i_{\ell+1}), \quad \gamma' = (0, \ldots, 0, i_k, \ldots, i_{\ell+1}).$$

*Then there is a path of length $|i_1| + \ldots + |i_{k-1}|$ joining $\gamma$ and $\gamma'$ such that all vertices on this path are of the form $(j_1, \ldots, j_{k-1}, i_k, \ldots, i_{\ell+1})$, where each $j_n$ lies between 0 and $|i_n|$.*

*Proof*: For $1 \leq j \leq k$, let $\gamma_j$ denote the element of $\Gamma$ whose first $j-1$ coordinates are 0 and $j^{\text{th}}$ coordinate onwards coincide with those of $\gamma$. Thus $\gamma_1 = \gamma$ and $\gamma_k = \gamma'$. Now apply the previous proposition to get a path of length $|\gamma_j(j) - \gamma_{j+1}(j)| = \gamma(j)$ joining $\gamma_j$ and $\gamma_{j+1}$ for $1 \leq j \leq k-1$. Joining all these paths together, one gets the required path. $\square$

**Proposition 3.4** *Let $D$ be a Dirac operator that commutes with the operators $U_{\mathbf{w}}$. Then $D$ must be of the form $e_\gamma \mapsto d(\gamma) e_\gamma$ where*

$$|d(\gamma)| = O(\gamma(1) + \ldots + \gamma(\ell) + |\gamma(\ell+1)| + 1).$$

*Proof*: Note that if $\gamma$ is an arbitrary element of the growth graph $\mathcal{G}$, then by the previous lemmas $\gamma$ can be connected with 0 by a path of length $\gamma(1) + \ldots + \gamma(\ell) + |\gamma(\ell+1)|$, hence the result. $\square$

**Theorem 3.5** *Write $\Gamma^+ = \{\gamma \in \Gamma : d(\gamma) > 0\}$, and $\Gamma^- = \Gamma \backslash \Gamma^+$. There exist nonnegative integers $M_1, M_2, \ldots, M_{\ell+1}$ such that for each $k \in \{1, 2, \ldots, \ell\}$ and for each*

$$(i_{k+1}, i_{k+2}, \ldots, i_{\ell+1}) \in F_k := \prod_{r=k+1}^{\ell} \{0, 1, \ldots, M_r\} \times \{-M_{\ell+1}, -M_{\ell+1} + 1, \ldots, M_{\ell+1}\},$$

*none of the following sets intersect both $\Gamma^+$ and $\Gamma^-$:*

$$A_1 = \{\gamma \in \Gamma : \gamma(\ell+1) > M_{\ell+1}\}, \quad A_2 = \{\gamma \in \Gamma : \gamma(\ell+1) < -M_{\ell+1}\},$$

$$B_{k,(i_{k+1}, i_{k+2}, \ldots, i_{\ell+1})} = \{\gamma \in \Gamma : \gamma(k) > M_k, \gamma(r) = i_r \text{ for } k+1 \leq r \leq \ell+1\}.$$

*Proof*: We will construct these numbers $M_1, M_2, \cdots M_{\ell+1}$ inductively starting from $M_{\ell+1}$. Assume there are two sequences of elements $\gamma_k \in \Gamma^+$ and $\delta_k \in \Gamma^-$ such that

$$\gamma_0(\ell+1) < \delta_0(\ell+1) < \gamma_1(\ell+1) < \delta_1(\ell+1) < \cdots.$$

For each $k$, use lemma 3.3 to get a path $p_k$ from $\gamma_k$ to $\delta_k$ such that for any vertex on the path, the $(\ell+1)^{\text{th}}$ coordinate lies between $\gamma_k(\ell+1)$ and $\delta_k(\ell+1)$. This last condition would ensure that the paths $p_k$ are all disjoint. Since $p_k$ connects points of $\Gamma^+$ with $\Gamma^-$, there is a vertex $\mu_k$ in $p_k$ such that $d(\mu_k) \in [-c, c]$. Moreover disjointness of the $p_k$'s implies that the vertices $\mu_k$

are all distinct. Therefore counted with multiplicity, the compact interval $[-c, c]$ has infinitely many eigenvalues of $D$, a contradiction to compact resolvent condition for $D$. Therefore there exists $M'_{\ell+1}$ such that $\{\gamma \in \Gamma : \gamma(\ell+1) > M'_{\ell+1}\}$ does not intersect both $\Gamma^+$ and $\Gamma^-$. One can similarly show that if there are elements $\gamma_k \in \Gamma^+$ and $\delta_k \in \Gamma^-$ such that

$$\gamma_0(\ell+1) > \delta_0(\ell+1) > \gamma_1(\ell+1) > \delta_1(\ell+1) > \cdots,$$

then there is some big enough natural number $M''_{\ell+1}$ such that the set $\{\gamma \in \Gamma : \gamma(\ell+1) < -M''_{\ell+1}\}$ is either in $\Gamma^+$ or in $\Gamma^-$. Now taking $M_{\ell+1} = \max\{M'_{\ell+1}, M''_{\ell+1}\}$, we get that neither of $A_1, A_2$ intersect both $\Gamma^+$ and $\Gamma^-$.

Next, given $M_{k+1}, \ldots, M_{\ell+1}$ and $(i_{k+1}, i_{k+2}, \ldots, i_{\ell+1}) \in F_k$, if there are elements $\gamma_n \in \Gamma^+$ and $\delta_n \in \Gamma^-$ with

$$\gamma_n(j) = i_j = \delta_n(j), \quad k+1 \leq j \leq \ell+1,$$

$$\gamma_0(k) < \delta_0(k) < \gamma_1(k) < \delta_1(k) < \cdots,$$

then using lemma 3.3 again, one can join each pair $(\gamma_n, \delta_n)$ by disjoint paths and arguing as above arrive at a contradiction to the fact that $D$ has compact resolvent. Therefore the existence of $M_k$ follows. $\square$

**Theorem 3.6** *Let $D_{torus}$ be the operator $e_\gamma \mapsto d(\gamma)e_\gamma$ on $\mathcal{H}_\ell$ where the $d(\gamma)$'s are given by*

$$d(\gamma) = \begin{cases} \gamma(1) + \ldots + \gamma(\ell) + |\gamma(\ell+1)| & \text{if } \gamma(\ell+1) \geq 0, \\ -(\gamma(1) + \ldots + \gamma(\ell) + |\gamma(\ell+1)|) & \text{if } \gamma(\ell+1) < 0. \end{cases}$$

*Then $(C(S_q^{2\ell+1}), \mathcal{H}_\ell, D_{torus})$ is a nontrivial $(\ell+1)$-summable spectral triple.*

*The operator $D_{torus}$ is optimal, i. e. if $D$ is any Dirac operator acting on $\mathcal{H}$ that commutes with the $U_{\mathbf{w}}$'s, then there exist positive reals $a$ and $b$ such that*

$$|D| \leq a + b|D_{torus}|.$$

*Proof*: Clearly $D_{torus}$ is a selfadjoint operator with compact resolvent. That it has bounded commutators with the $\pi(z_j)$'s follow by direct verification.

From the commutation relations that the generators $z_j$ obey, it follows that $z_{\ell+1}$ is normal and the element $z^*_{\ell+1}z_{\ell+1}$ has spectrum $\{q^{2n} : n \in \mathbb{N}\} \cup \{0\}$. Let

$$u = \chi_{\{1\}}(z^*_{\ell+1}z_{\ell+1})(z_{\ell+1} - 1) + 1.$$

It is easy to see that $u$ is a unitary. We will now compute the pairing between $D_{torus}$ and $\pi(u)$. First observe that the action of $\pi(u)$ on $\mathcal{H}$ is given by

$$\pi(u)e_\gamma = \begin{cases} e_{\gamma+\epsilon_{\ell+1}} & \text{if } \gamma(i) = 0 \text{ for } 1 \leq i \leq \ell, \\ e_\gamma & \text{otherwise.} \end{cases}$$

Write $P = \frac{1}{2}(I + \operatorname{sign} D_{torus})$. Then $P$ is the projection onto the closed linear span of $\{e_\gamma : \gamma(\ell+1) \geq 0\}$. It follows that the index of $PuP$ is $-1$.

Summability follows from the observation that the number of elements in $\{(i_1, \ldots, i_{\ell+1}) \in \mathbb{N}^\ell \times \mathbb{Z} : \sum_{k=1}^{\ell} i_k + |i_{\ell+1}| \leq n\}$ is of the order $n^{\ell+1}$.

Optimality is a consequence of proposition 3.4. $\square$

**Theorem 3.7** *Let $D$ be a Dirac operator on $\mathcal{H}$ that commutes with the operators $U_{\mathbf{w}}$. Then either $D$ is trivial or has the same $K$-homology class as $D_{torus}$ or $-D_{torus}$.*

*Proof*: If $D$ is a self-adjoint operator with compact resolvent on $\mathcal{H}$ that commutes with the operators $U_{\mathbf{w}}$ and if $P = \frac{1}{2}(\operatorname{sign} D + I)$, then by theorem 3.5, $P$ is the projection onto the closed linear span of $\{e_\gamma : \gamma \in \Gamma^+\}$ where $\Gamma^+$ must be of one of the following form:

$$A_1 \cup (\cup_{x \in E} B_x), \tag{3.4}$$

$$A_2 \cup (\cup_{x \in E} B_x), \tag{3.5}$$

$$A_1 \cup A_2 \cup (\cup_{x \in E} B_x)\,, \tag{3.6}$$

$$\cup_{x \in E} B_x, \tag{3.7}$$

where $E$ is some finite subset of $\cup_{k=1}^{\ell} \{k\} \times F_k$. By direct calculations in the first two cases the index of $P\pi(u)P$ turns out to be $-1$ and $1$ respectively, whereas in the last two cases, the index is zero. Thus one always has

$$\langle [u], (C(S_q^{2\ell+1}), \mathcal{H}, D) \rangle = 0 \text{ or } \pm 1.$$

By [9], we have $K^1(C(S_q^{2\ell+1})) = \mathbb{Z}$. therefore the result follows. $\square$

# 4 Relation with $C^*$-extensions

In this section we will denote the generators for $A_\ell$ by $z_k$ and the generators for $A_{\ell+1}$ by $y_k$. $A_\ell^0$ will denote the *-subalgebra of $A_\ell$ generated by the $z_k$'s. Let $J_\ell^0$ denote the two-sided *-ideal in $A_\ell^0$ generated by $z_{\ell+1}$ and let $J_\ell$ denote the norm closure of $J_\ell^0$ in $A_\ell$. Thus $J_\ell$ is the ideal in $A_\ell$ generated by the element $z_{\ell+1}$.

For a Hilbert $C^*$-module $E$, we will denote by $\mathcal{L}(E)$ the $C^*$-algebra of bounded adjointable operators on $E$, and by $\mathcal{K}(E)$ its ideal of 'compact' operators. We denote by $\mathcal{K}$ the $C^*$-algebra $\mathcal{K}(\mathcal{H})$ for an infinite dimensional Hilbert space $\mathcal{H}$.

**Lemma 4.1** *Let $C^*(S)$ denote the $C^*$-algebra generated by the operator $S$ on $L_2(\mathbb{Z})$. Then one has $J_\ell \cong \mathcal{K}(L_2(\mathbb{N}^\ell)) \otimes C^*(S) \cong \mathcal{K} \otimes C(S^1)$.*

*Proof*: We will identify $A_\ell$ with $\pi_\ell(A_\ell)$.

For $1 \leq k \leq \ell$, denote by $X_k$ the operator

$$\underbrace{q^N \otimes \ldots \otimes q^N}_{k \text{ copies}} \otimes \underbrace{I \otimes \ldots \otimes I}_{\ell+1-k \text{ copies}}$$

on $\mathcal{H}_\ell$. Write $X_0 = I$. Then it is easy to check that one has the relations

$$z_k z_k^* = X_{k-1}^2 - X_k^2, \quad 1 \leq k \leq \ell.$$

It follows that $X_k \in A_\ell$ for all $1 \leq k \leq \ell$.

Write $p_{ij}$ for the rank one operator $|e_i\rangle\langle e_j|$ on $L_2(\mathbb{N})$. Then

$$p_{i_1 j_1} \otimes \ldots \otimes p_{i_\ell j_\ell} \otimes S^k$$

can be written in the form

$$f_1(X_1) \ldots f_\ell(X_\ell) z_{\ell+1}^{-k} g_1(X_1) \ldots g_\ell(X_\ell)$$

where $f_i$, $g_i$ are continuous functions on the spectrums of the respective $X_i$'s. Therefore $p_{i_1 j_1} \otimes \ldots \otimes p_{i_\ell j_\ell} \otimes S^k \in J_\ell$. It follows from this that $\mathcal{K}(L_2(\mathbb{N}^\ell)) \otimes C^*(S) \subseteq J_\ell$.

For the reverse inclusion, observe that any polynomial in the $z_i$'s and their adjoints is a finite sum of the form $\sum_j T_j \otimes S^{k_j}$ where $T_j \in \mathcal{L}(L_2(\mathbb{N}^\ell))$ and $k_j \in \mathbb{Z}$. Therefore $J_\ell^0$ is contained in $\mathcal{K}(L_2(\mathbb{N}^\ell)) \otimes C^*(S)$. Same is therefore true for its closure $J_\ell$. $\square$

**Proposition 4.2** *Let $\sigma_\ell : \mathcal{A}_{\ell+1} \to \mathcal{A}_\ell$ be the homomorphism given by*

$$y_i \mapsto \begin{cases} z_i & \text{if } 1 \leq i \leq \ell+1, \\ 0 & \text{if } i = \ell+2. \end{cases}$$

*Then we have the following short exact sequence*

$$0 \longrightarrow J_{\ell+1} \longrightarrow A_{\ell+1} \xrightarrow{\sigma_\ell} A_\ell \longrightarrow 0. \tag{4.8}$$

We will need the following lemma for the proof.

**Lemma 4.3** *Let $\mathcal{A}$ be the universal $C^*$-algebra in noncommuting variables $x_1, x_2, \cdots x_n$ subject to relations $R_1(x_1, x_2, \cdots, x_n), \cdots, R_j(x_1, x_2, \cdots, x_n)$. Let $J$ be the ideal of $\mathcal{A}$ generated by non-commutative polynomials $Q_1(x_1, x_2, \cdots, x_n), Q_2(x_1, x_2, \cdots, x_n), \cdots, Q_k(x_1, x_2, \cdots, x_n)$. Then $\mathcal{A}/J$ is isomorphic to the universal $C^*$-algebra $\mathcal{A}(J)$ generated by $x_1, x_2, \cdots, x_n$ subject to the relations $R_1, \cdots, R_j, Q_1, \cdots, Q_k$.*

Note that it is part of the hypothesis that the universal $C^*$-algebras $\mathcal{A}$ and $\mathcal{A}(J)$ exist.

*Proof*: Let $\xi_1, \cdots, \xi_n$ be the generating elements of $\mathcal{A}(J)$. Clearly we have a surjection $q : \mathcal{A}(J) \to \mathcal{A}/J$ mapping $\xi_i$ to $x_i$. To show that this is injective it is enough to show that given a polynomial $\alpha = f(\xi_1, \cdots, \xi_n) \in \mathcal{A}(J)$, one has $\|q(\alpha)\| = \|a\|$, where $a = f(x_1, \cdots, x_n)$. Now observe that

$$\begin{aligned}
\|a\| &= \sup\{\|\pi(a)\| : \pi \text{ is a representation of } \mathcal{A}, \pi(J) = 0\} \\
&= \sup\{\|\pi(a)\| : \pi \text{ is a representation of the algebra generated by } x_1, x_2, \cdots x_n \\
&\qquad \text{subject to } R_1, \cdots, R_j, Q_1, \ldots, Q_k\} \\
&= \|\alpha\|.
\end{aligned}$$

Thus the proof is complete. □

*Proof of proposition 4.2.* Clearly $J_{\ell+1} \subseteq \ker(\sigma_\ell)$ and lemma 4.3 gives $\mathcal{A}_{\ell+1}/J_{\ell+1} \cong \mathcal{A}_{\ell+1}(J_{\ell+1})$. Also note that in the defining relations for the generators for $\mathcal{A}_{\ell+1}$ if we put $y_{\ell+2} = 0$ we get the relations for $\mathcal{A}_\ell$, hence $\mathcal{A}_{\ell+1}(J_{\ell+1}) = \mathcal{A}_\ell$. Therefore $\ker(\sigma_\ell) = J_{\ell+1}$, hence the result. □

Proposition 4.2 gives a homomorphism $\psi_{\ell+1} : \mathcal{A}_{\ell+1} \to M(J_{\ell+1})$. Using lemma 4.1 we get $M(J_{\ell+1}) \cong \mathcal{L}(L_2(\mathbb{N}^{\ell+1}) \otimes C(S^1))$. Thus $\psi_{\ell+1}$ is given by:

$$\begin{aligned}
y_k &\mapsto \underbrace{q^N \otimes \cdots \otimes q^N}_{k-1 \text{ copies}} \otimes \sqrt{1 - q^{2N}} S^* \otimes \underbrace{I \otimes \cdots \otimes I}_{\ell+2-k \text{ copies}}, \quad 1 \leq k \leq \ell+1, \\
y_{\ell+2} &\mapsto \underbrace{q^N \otimes \cdots \otimes q^N}_{\ell+1 \text{ copies}} \otimes Z.
\end{aligned}$$

Here $Z : C(S^1) \to C(S^1)$ denotes the operator given by $(Zf)(z) = zf(z)$.

Define $\tilde{\sigma}_\ell : A_\ell \to \mathcal{L}(\mathcal{H}_\ell \otimes C(S^1))$ by

$$\begin{aligned}
z_k &\mapsto \underbrace{q^N \otimes \cdots \otimes q^N}_{k-1 \text{ copies}} \otimes \sqrt{1 - q^{2N}} S^* \otimes \underbrace{I \otimes \cdots \otimes I}_{\ell+2-k \text{ copies}}, \quad 1 \leq k \leq \ell, \\
z_{\ell+1} &\mapsto \underbrace{q^N \otimes \cdots \otimes q^N}_{\ell \text{ copies}} \otimes S^* \otimes I.
\end{aligned}$$

Let

$$E_\ell = \underbrace{\mathcal{K}(L_2(\mathbb{N})) \otimes \cdots \otimes \mathcal{K}(L_2(\mathbb{N}))}_{\ell \text{ copies}} \otimes C(S^1), \quad F_\ell = \underbrace{L_2(\mathbb{N}) \otimes \cdots \otimes L_2(\mathbb{N})}_{\ell \text{ copies}} \otimes C(S^1).$$

Let $U$ be the unitary from $L_2(\mathbb{N}) \oplus L_2(\mathbb{N})$ onto $L_2(\mathbb{Z})$ given by

$$e_n \oplus 0 \mapsto e_n, \qquad 0 \oplus e_n \mapsto e_{-n-1}, \qquad n \in \mathbb{N}.$$

Using this unitary in the $(\ell+1)^{\text{th}}$ copy, one can identify $\mathcal{H}_\ell \otimes C(S^1)$ with $F_{\ell+1} \oplus F_{\ell+1}$. Let $P \in \mathcal{L}(L_2(\mathbb{Z}))$ be the projection onto the $L_2(\mathbb{N})$ part and let $Q_\ell = \underbrace{I \otimes \cdots \otimes I}_{\ell \text{ copies}} \otimes P \otimes I$. Define $C_\ell : \mathcal{L}(\mathcal{H}_\ell \otimes C(S^1)) \to \mathcal{L}(F_{\ell+1})$ by $C_\ell(T) = Q_\ell T Q_\ell$. Now define $\hat{\sigma}_\ell : A_\ell \to \mathcal{L}(F_{\ell+1})$ by $\hat{\sigma}_\ell(a) = C_\ell \tilde{\sigma}_\ell(a)$. For convenience, we summarize various maps and the spaces between which they act in the following diagram:

$$\begin{tikzcd}
J_{\ell+1} \arrow[r] \arrow[d, leftrightarrow, "="] & A_{\ell+1} \arrow[r, "\sigma_\ell"] \arrow[d, "\psi_{\ell+1}"] & A_\ell \arrow[d, "\tilde{\sigma}_\ell"] \arrow[ddl, "\hat{\sigma}_\ell"'] \\
E_{\ell+1} \arrow[d, "\subseteq"] & \psi_{\ell+1}(A_{\ell+1}) \arrow[d, "\subseteq"] & \mathcal{L}(\mathcal{H}_\ell \otimes C(S^1)) \arrow[d, leftrightarrow, "\cong"] \\
M(E_{\ell+1}) \arrow[r, leftrightarrow, "\cong"] & \mathcal{L}(F_{\ell+1}) & \mathcal{L}(F_{\ell+1} \oplus F_{\ell+1}) \arrow[l, "C_\ell"']
\end{tikzcd}$$

**Theorem 4.4** *The element $(\mathcal{H}_\ell \otimes C(S^1), \tilde{\sigma}, 2Q-I)$ gives the $KK$-class in $KK^1(C(S_q^{2\ell+1}), C(S^1))$ corresponding to the extension (4.8).*

*Proof*: Let $r \in \mathbb{N}$ and let $p$ be a polynomial in noncommuting variables and their adjoints. Using the observation that $Q_\ell$ commutes with $\tilde{\sigma}_\ell(z_k)$ for $1 \le k \le \ell$, one gets

1. $\hat{\sigma}_\ell(z_{\ell+1}^r p(z_1, \cdots, z_\ell, z_1^*, \cdots, z_\ell^*)) = \hat{\sigma}_\ell(z_{\ell+1}^r)\hat{\sigma}_\ell(p(z_1, \cdots, z_\ell, z_1^*, \cdots, z_\ell^*))$.
2. $\hat{\sigma}_\ell((z_{\ell+1}^*)^r p(z_1, \cdots, z_\ell, z_1^*, \cdots, z_\ell^*)) = \hat{\sigma}_\ell((z_{\ell+1}^*)^r)\hat{\sigma}_\ell(p(z_1, \cdots, z_\ell, z_1^*, \cdots, z_\ell^*))$.

Using this one can now easily show that

1. $\hat{\sigma}_\ell(p(z_1, \cdots, z_\ell, z_1^*, \cdots, z_\ell^*)) = \psi_{\ell+1}(p(y_1, \cdots, y_\ell, y_1^*, \cdots, y_\ell^*))$.
2. $\hat{\sigma}_\ell(z_{\ell+1}^r) - \psi_{\ell+1}(y_{\ell+1}^r) \in \mathcal{K}(L_2(\mathbb{N}^{\ell+1})) \otimes C^*(S) = \psi_{\ell+1}(J_{\ell+1})$.
3. $\hat{\sigma}_\ell((z_{\ell+1}^*)^r) - \psi_{\ell+1}((y_{\ell+1}^*)^r) \in \mathcal{K}(L_2(\mathbb{N}^{\ell+1})) \otimes C^*(S) = \psi_{\ell+1}(J_{\ell+1})$.

It follows from these that for any polynomial $p$ we have

$$\begin{aligned} &\hat{\sigma}_\ell(p(z_1, \cdots, z_{\ell+1}, z_1^*, \cdots, z_{\ell+1}^*)) - \psi_{\ell+1}(p(y_1, \cdots, y_{\ell+1}, y_1^*, \cdots, y_{\ell+1}^*)) \\ &\qquad \in \mathcal{K}(L_2(\mathbb{N}^{\ell+1})) \otimes C(S^1) = \psi_{\ell+1}(J_{\ell+1}). \end{aligned} \tag{4.9}$$

Let $\tau : \mathcal{A}_\ell \to M(J_{\ell+1})/J_{\ell+1}$ be the Busby invariant for the extension (4.8), and let $\Phi : M(J_{\ell+1}) \to M(J_{\ell+1})/J_{\ell+1}$ be the quotient map. For a polynomial $p$ in noncommuting variables and their adjoints, we now have from (4.9),

$$\begin{aligned} \tau(p(z_1, \cdots, z_{\ell+1}, z_1^*, \cdots, z_{\ell+1}^*)) &= \Phi \circ \psi_\ell(p(y_1, \cdots, y_{\ell+1}, y_1^*, \cdots, y_{\ell+1}^*)) \\ &= \Phi \circ \hat{\sigma}_\ell(p(z_1, \cdots, z_{\ell+1}, z_1^*, \cdots, z_{\ell+1}^*)). \end{aligned}$$

Since such elements are dense in $\mathcal{A}_\ell$, we get

$$\tau(a) = \Phi \circ \hat{\sigma}_\ell(a), \qquad a \in \mathcal{A}_\ell.$$

Thus by (4.9) $\tau$ admits the completely positive lifting $\hat{\sigma}_\ell$ and the result follows. $\square$

Thus one now has the following commutative diagram:

$$\begin{tikzcd}
J_{\ell+1} \arrow[r] \arrow[d, leftrightarrow, "="] & A_{\ell+1} \arrow[r, "\sigma_\ell"] \arrow[d, leftrightarrow, "\psi_{\ell+1}"] & A_\ell \arrow[dl, "\hat{\sigma}_\ell"] \arrow[d, "\tilde{\sigma}_\ell"] \\
E_{\ell+1} \arrow[d, "\subseteq"] & \psi_{\ell+1}(A_{\ell+1}) \arrow[d, "\subseteq"] & \mathcal{L}(\mathcal{H}_\ell \otimes C(S^1)) \arrow[d, leftrightarrow, "\cong"] \\
M(E_{\ell+1}) \arrow[r, leftrightarrow, "\cong"] & \mathcal{L}(F_{\ell+1}) & \mathcal{L}(F_{\ell+1} \oplus F_{\ell+1}) \arrow[l, "C_\ell"']
\end{tikzcd}$$

Let $ev_1$ denote the following representation of $C(S^1)$ on $\mathbb{C}$:

$$ev_1(f) = f(1).$$

Now take the trivial grading on $\mathbb{C}$. Then $(\mathbb{C}, ev_1, 0)$ gives an even Fredholm module for $C(S^1)$.

**Lemma 4.5** *The Fredholm module* $(\mathbb{C}, ev_1, 0)$ *is a generator for the group* $KK^0(C(S^1), \mathbb{C})$.

*Proof*: This can be seen as follows. The identity projection gives a generating element for $KK^0(\mathbb{C}, C(S^1)) = K_0(C(S^1)) = \mathbb{Z}$. The pairing of this with $[(\mathbb{C}, ev_1, 0)]$ gives 1. One can conclude from this that $[(\mathbb{C}, ev_1, 0)]$ must be $\pm 1$. $\square$

**Proposition 4.6** $(\mathcal{H}_\ell, \pi, sign\, D_{torus})] = (\mathcal{H}_\ell \otimes C(S^1), \tilde{\sigma}_\ell, 2Q_\ell - I) \otimes_{ev_1} (\mathbb{C}, ev_1, 0)$.

*Proof*: For this, one needs to note that $(\mathcal{H}_\ell \otimes C(S^1)) \otimes \mathbb{C} \cong \mathcal{H}_\ell$ where the tensor product is the internal tensor product of Hilbert $C^*$-modules, and under this isomorphism, $(2Q_\ell - I) \otimes I$ is just the operator $\mathrm{sign}\, D_{torus}$. $\square$

Thus on multiplying the even Fredholm module $(\mathbb{C}, ev_1, 0)$ from the left by the $KK$-element we just computed, one gets the odd fredholm module corresponding to the spectral triple $(\mathcal{H}_\ell, \pi_\ell, D_{torus})$ we have constructed in the last section.

# References


[1] Baaj, S. ; Julg, P. : Théorie bivariante de Kasparov et opérateurs non bornés dans les $C^*$-modules hilbertiens. (French. English summary) [Bivariant Kasparov theory and unbounded operators on Hilbert $C^*$-modules] C. R. Acad. Sci. Paris Sér. I Math. 296 (1983), no. 21, 875–878.

[2] Blackadar, Bruce : *K-theory for operator algebras*, Second edition. MSRI Publications, 5. Cambridge University Press, Cambridge, 1998.

[3] Chakraborty, P. S. ; Pal, A. : Equivariant spectral triples on the quantum $SU(2)$ group, arXiv:math.KT/0201004, *K-Theory*, 28(2003), No. 2, 107-126.

[4] Chakraborty, P. S. ; Pal, A. : Spectral triples and associated Connes-de Rham complex for the quantum $SU(2)$ and the quantum sphere, arXiv:math.QA/0210049, *Commun. Math. Phys.*, 240(2003), No. 3, 447-456.

[5] Chakraborty, P. S. ; Pal, A. : Characterization of $SU_q(\ell+1)$-equivariant spectral triples for the odd dimensional quantum spheres, arXiv:math.QA/0701694.

[6] Connes, A. : Compact metric spaces, Fredholm modules, and hyperfiniteness. *Ergodic Theory Dynam. Systems* 9 (1989), no. 2, 207–220.

[7] Connes, A. : *Noncommutative Geometry*, Academic Press, 1994.

[8] Hong, Jeong Hee; Szymański, Wojciech : Quantum spheres and projective spaces as graph algebras. *Comm. Math. Phys.* 232 (2002), no. 1, 157–188.

[9] Rosenberg, J. & Schochet, C. : The Künneth theorem and the universal coefficient theorem for Kasparov's generalized $K$-functor, *Duke Math. J.* 55 (1987), no. 2, 431–474.

[10] Vaksman, L. L.; Soibelman, Yan S. : Algebra of functions on the quantum group SU($n$+1), and odd-dimensional quantum spheres. (Russian) *Algebra i Analiz* 2 (1990), no. 5, 101–120; translation in *Leningrad Math. J.* 2 (1991), no. 5, 1023–1042.



Partha Sarathi Chakraborty (parthac@imsc.res.in)

Institute of Mathematical Sciences, CIT Campus, Chennai–600 113, INDIA

Arupkumar Pal (arup@isid.ac.in)

Indian Statistical Institute, 7, SJSS Marg, New Delhi–110 016, INDIA