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On a method to construct magic rectangles of even order

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Abstract

Magic rectangles are well-known for its very interesting and entertaining combinatorics. In a magic rectangle, the integers 1 to mn are arranged in an array of m rows and n columns so that each row adds to the same total M and each column to the same total N . In the present paper we provide a simple and systematic method for constructing any even by even magic rectangle.

Keywords : magic rectangles; magic constants.

1. Introduction

Magic rectangles are well-known for its very interesting and entertaining combinatorics. A magic rectangle is an arrangement of the integers 1 to mn in an array of m rows and n columns so that each row adds to the same total M and each column to the same total N . The totals M and N are termed the magic constants. Since the average value of the integers is $A = (mn + 1)/2$, we must have $M = nA$ and $N = mA$. The total of all the integers in the array is $mnA = mM = nN$. If mn is even $mn+1$ is odd and so for $M = n(mn+1)/2$ and $N = m(mn + 1)/2$ to be integers n and m must both be even. On the other hand if mn is odd then m and n must both be odd, by simple arithmetic. Therefore, an odd by even magic rectangle is impossible. Also, it is easy to see that a 2×2 magic rectangle is impossible.

For an update on available literature on magic rectangles we refer to Hagedorn (1999) and Bier and Kleinschmidt (1997). Such magic rectangles have been used in designing experiments. For example, Phillips (1964, 1968a, 1968b) illustrated the use of these magic figures for the elimination of trend effects in certain classes of one-way, factorial, latin-square, and graeco-latin-square designs. As highly balanced structures, magic rectangles can be potential tools for use in situations yet unexplored.

In the present paper we provide a method for constructing any even by even magic rectangle. The construction involves some simple steps. The method has been shaped in form of an algorithm that is very convenient for writing a computer program for constructing such rectangles.

[1] Paper prepared during author's visit to The University of Akron, Akron, Ohio, USA.

In Section 2 we construct magic rectangle of sides $m = 2p$ and $n = 2q$. The proofs related to the construction is given in the appendix. In Section 3 we illustrate our construction method through some examples of magic rectangles.

2. The construction

We construct magic rectangle of sides $m = 2p$ and $n = 2q$ for given positive integers p and q. We consider separately the cases (i) at least one of p, q is even, and (ii) both p and q are odd. Throughout, for $z > 0$, $[z]$ stands for the largest integer contained in z.

Case A: At least one of p, q is even. Without loss of generality, let p be even.
Step A1. Write the mn consecutive integers from 1 to mn with first column having integers $1, 2, \ldots, m$; second column having integers $2m, 2m-1, \ldots, m+1$; third column having integers $2m+1, 2m+2, \ldots, 3m$; forth column having integers $4m, 4m-1, \ldots, 3m+1$; and so on, $(n-1)$-th column having integers $(n-2)m+1, (n-2)m+2, \ldots, (n-1)m$; n-th column having integers $nm, nm-1, \ldots, (n-1)m+1$. We would call this a serpentine format for writing the mn consecutive integers in n columns.
Step A2. For $1 \leq j \leq q$, swap the middle p elements of j-th column with the corresponding middle p elements of the $(n+1-j)$-th column.

Case B: Both p and q are odd.
Step B1. Same as Step A1.
Step B2. Consider the first $2[q/2]$ columns. For $1 \leq i \leq p$, swap the $2[q/2]$ elements in the i-th row with the corresponding $2[q/2]$ elements in the $(m+1-i)$-th row.
Step B3. For $1 \leq j \leq 2[q/2]$, swap any p elements of j-th column with the corresponding p elements of the $(n+1-j)$-th column. In particular, we may take the first p elements of the columns for swapping.
Step B4. Swap the middle $p-3$ elements of q-th column with the corresponding middle $p-3$ elements in the $(q+1)$-th column. Also, for $t = 1, 3$, swap the element in the $\{t + (p-3)/2, q\}$-th position with the element in the $\{t + (p-3)/2, q+1\}$-th position.

The proofs related to the construction are given in the appendix.

3. Some illustrative examples

In this section we provide some examples of magic rectangles of orders 10×8 and 10×14.

Magic rectangle of order 10×8 is the transpose of magic rectangle of order 8×10. So, we construct a magic rectangle of order 8×10. Here $p = 4$, $q = 5$. Therefore, Step A1 gives

$$\begin{pmatrix}
1 & 16 & 17 & 32 & 33 & 48 & 49 & 64 & 65 & 80 \\
2 & 15 & 18 & 31 & 34 & 47 & 50 & 63 & 66 & 79 \\
3 & 14 & 19 & 30 & 35 & 46 & 51 & 62 & 67 & 78 \\
4 & 13 & 20 & 29 & 36 & 45 & 52 & 61 & 68 & 77 \\
5 & 12 & 21 & 28 & 37 & 44 & 53 & 60 & 69 & 76 \\
6 & 11 & 22 & 27 & 38 & 43 & 54 & 59 & 70 & 75 \\
7 & 10 & 23 & 26 & 39 & 42 & 55 & 58 & 71 & 74 \\
8 & 9 & 24 & 25 & 40 & 41 & 56 & 57 & 72 & 73
\end{pmatrix},$$

and the desired magic rectangle, as given by Step A2, is

$$\begin{pmatrix}
1 & 16 & 17 & 32 & 33 & 48 & 49 & 64 & 65 & 80 \\
2 & 15 & 18 & 31 & 34 & 47 & 50 & 63 & 66 & 79 \\
78 & 67 & 62 & 51 & 46 & 35 & 30 & 19 & 14 & 3 \\
77 & 68 & 61 & 52 & 45 & 36 & 29 & 20 & 13 & 4 \\
76 & 69 & 60 & 53 & 44 & 37 & 28 & 21 & 12 & 5 \\
75 & 70 & 59 & 54 & 43 & 38 & 27 & 22 & 11 & 6 \\
7 & 10 & 23 & 26 & 39 & 42 & 55 & 58 & 71 & 74 \\
8 & 9 & 24 & 25 & 40 & 41 & 56 & 57 & 72 & 73
\end{pmatrix}.$$

Magic rectangle of order 10×14 has $p = 5$, $q = 7$. Therefore,

Step B1 gives

$$\begin{pmatrix}
1 & 20 & 21 & 40 & 41 & 60 & 61 & 80 & 81 & 100 & 101 & 120 & 121 & 140 \\
2 & 19 & 22 & 39 & 42 & 59 & 62 & 79 & 82 & 99 & 102 & 119 & 122 & 139 \\
3 & 18 & 23 & 38 & 43 & 58 & 63 & 78 & 83 & 98 & 103 & 118 & 123 & 138 \\
4 & 17 & 24 & 37 & 44 & 57 & 64 & 77 & 84 & 97 & 104 & 117 & 124 & 137 \\
5 & 16 & 25 & 36 & 45 & 56 & 65 & 76 & 85 & 96 & 105 & 116 & 125 & 136 \\
6 & 15 & 26 & 35 & 46 & 55 & 66 & 75 & 86 & 95 & 106 & 115 & 126 & 135 \\
7 & 14 & 27 & 34 & 47 & 54 & 67 & 74 & 87 & 94 & 107 & 114 & 127 & 134 \\
8 & 13 & 28 & 33 & 48 & 53 & 68 & 73 & 88 & 93 & 108 & 113 & 128 & 133 \\
9 & 12 & 29 & 32 & 49 & 52 & 69 & 72 & 89 & 92 & 109 & 112 & 129 & 132 \\
10 & 11 & 30 & 31 & 50 & 51 & 70 & 71 & 90 & 91 & 110 & 111 & 130 & 131
\end{pmatrix},$$

following which, Step B2 gives

$$\begin{pmatrix}
10 & 11 & 30 & 31 & 50 & 51 & 61 & 80 & 81 & 100 & 101 & 120 & 121 & 140 \\
9 & 12 & 29 & 32 & 49 & 52 & 62 & 79 & 82 & 99 & 102 & 119 & 122 & 139 \\
8 & 13 & 28 & 33 & 48 & 53 & 63 & 78 & 83 & 98 & 103 & 118 & 123 & 138 \\
7 & 14 & 27 & 34 & 47 & 54 & 64 & 77 & 84 & 97 & 104 & 117 & 124 & 137 \\
6 & 15 & 26 & 35 & 46 & 55 & 65 & 76 & 85 & 96 & 105 & 116 & 125 & 136 \\
5 & 16 & 25 & 36 & 45 & 56 & 66 & 75 & 86 & 95 & 106 & 115 & 126 & 135 \\
4 & 17 & 24 & 37 & 44 & 57 & 67 & 74 & 87 & 94 & 107 & 114 & 127 & 134 \\
3 & 18 & 23 & 38 & 43 & 58 & 68 & 73 & 88 & 93 & 108 & 113 & 128 & 133 \\
2 & 19 & 22 & 39 & 42 & 59 & 69 & 72 & 89 & 92 & 109 & 112 & 129 & 132 \\
1 & 20 & 21 & 40 & 41 & 60 & 70 & 71 & 90 & 91 & 110 & 111 & 130 & 131
\end{pmatrix},$$

and Step B3 gives

$$\begin{pmatrix}
140 & 121 & 120 & 101 & 100 & 81 & 61 & 80 & 51 & 50 & 31 & 30 & 11 & 10 \\
139 & 122 & 119 & 102 & 99 & 82 & 62 & 79 & 52 & 49 & 32 & 29 & 12 & 9 \\
138 & 123 & 118 & 103 & 98 & 83 & 63 & 78 & 53 & 48 & 33 & 28 & 13 & 8 \\
137 & 124 & 117 & 104 & 97 & 84 & 64 & 77 & 54 & 47 & 34 & 27 & 14 & 7 \\
136 & 125 & 116 & 105 & 96 & 85 & 65 & 76 & 55 & 46 & 35 & 26 & 15 & 6 \\
5 & 16 & 25 & 36 & 45 & 56 & 66 & 75 & 86 & 95 & 106 & 115 & 126 & 135 \\
4 & 17 & 24 & 37 & 44 & 57 & 67 & 74 & 87 & 94 & 107 & 114 & 127 & 134 \\
3 & 18 & 23 & 38 & 43 & 58 & 68 & 73 & 88 & 93 & 108 & 113 & 128 & 133 \\
2 & 19 & 22 & 39 & 42 & 59 & 69 & 72 & 89 & 92 & 109 & 112 & 129 & 132 \\
1 & 20 & 21 & 40 & 41 & 60 & 70 & 71 & 90 & 91 & 110 & 111 & 130 & 131
\end{pmatrix},$$

and the desired magic rectangle, as given by Step B4, is

$$\begin{pmatrix}
140 & 121 & 120 & 101 & 100 & 81 & 61 & 80 & 51 & 50 & 31 & 30 & 11 & 10 \\
139 & 122 & 119 & 102 & 99 & 82 & 79 & 62 & 52 & 49 & 32 & 29 & 12 & 9 \\
138 & 123 & 118 & 103 & 98 & 83 & 63 & 78 & 53 & 48 & 33 & 28 & 13 & 8 \\
137 & 124 & 117 & 104 & 97 & 84 & 77 & 64 & 54 & 47 & 34 & 27 & 14 & 7 \\
136 & 125 & 116 & 105 & 96 & 85 & 76 & 65 & 55 & 46 & 35 & 26 & 15 & 6 \\
5 & 16 & 25 & 36 & 45 & 56 & 75 & 66 & 86 & 95 & 106 & 115 & 126 & 135 \\
4 & 17 & 24 & 37 & 44 & 57 & 67 & 74 & 87 & 94 & 107 & 114 & 127 & 134 \\
3 & 18 & 23 & 38 & 43 & 58 & 68 & 73 & 88 & 93 & 108 & 113 & 128 & 133 \\
2 & 19 & 22 & 39 & 42 & 59 & 69 & 72 & 89 & 92 & 109 & 112 & 129 & 132 \\
1 & 20 & 21 & 40 & 41 & 60 & 70 & 71 & 90 & 91 & 110 & 111 & 130 & 131
\end{pmatrix}.$$

Appendix

Proof for the case when at least one of p, q is even.

i) Without loss of generality, let p be even (since a magic rectangle of order $n \times m$ is the transpose of magic rectangle of order $m \times n$ and visa verse). The snake format of generating the columns ensures that the row sums are constant.

ii) After Step A1, for $1 \leq j \leq n$, the j -th column sum is $m(m+1)/2 + (j-1)m^2$ and the sum of the middle p elements in the j column is $m^2 j/2 - m(m-1)/4$. Also, for $1 \leq j \leq q$, the j -th column sum is less from the magic constant $N = m(mn+1)/2$ by a quantity $(n+1-2j)m^2/2$ and for $q+1 \leq j \leq n$, the j -th column sum is more from the magic constant N by a quantity $(2j-n-1)m^2/2$. Thus, in Step A2, for $1 \leq j \leq q$, the swap of the middle p elements between j and $(n+1-j)$ -th columns increases and decreases the respective column totals by the desired amounts and thus reduces it to a magic rectangle.

Proof for the case when both p and q are odd.

i) Upto Step B3, the proof follows on lines similar to Case A.

ii) After Step B3, certain elements between the two middle columns are swapped. The q -th column sum is $m(m+1)/2 + (q-1)m^2$ which is less from the magic constant N by a quantity $m^2/2$. Similarly, the $(q+1)$ -th column sum is $m(m+1)/2 + qm^2$ which is more from the magic constant N by a quantity $m^2/2$. It is easy to see that the swap carried out in Step B4 increases and decreases the respective column totals by the desired amounts, thereby ensuring a magic rectangle.

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