Journal of Statistical Planning and Inference 138 (2008) 3749 – 3757

Contents lists available at ScienceDirect

Journal of Statistical Planning and Inference

journal homepage: www.elsevier.com/locate/jspi

On $E(s^2)$-optimal supersaturated designs

Ashish Das[a,*,1], Aloke Dey[b], Ling-Yau Chan[c], Kashinath Chatterjee[d]

[a] Department of Mathematics, Indian Institute of Technology Bombay, Mumbai 400076, India
[b] Theoretical Statistics and Mathematics Unit, Indian Statistical Institute, New Delhi 110 016, India
[c] Department of Industrial and Manufacturing Systems Engineering, The University of Hong Kong, Pokfulam Road, Hong Kong, China
[d] Department of Statistics, Visva Bharati University, Santiniketan, India

ARTICLE INFO

Article history:
Received 7 December 2006
Received in revised form
1 October 2007
Accepted 18 December 2007
Available online 28 March 2008

Keywords:
Effect sparsity
Lower bound
Screening designs

ABSTRACT

A popular measure to assess 2-level supersaturated designs is the $E(s^2)$ criterion. In this paper, improved lower bounds on $E(s^2)$ are obtained. The same improvement has recently been established by Ryan and Bulutoglu [2007. $E(s^2)$-optimal supersaturated designs with good minimax properties. J. Statist. Plann. Inference 137, 2250–2262]. However, our analysis provides more details on precisely when an improvement is possible, which is lacking in Ryan and Bulutoglu [2007. $E(s^2)$-optimal supersaturated designs with good minimax properties. J. Statist. Plann. Inference 137, 2250–2262]. The equivalence of the bounds obtained by Butler et al. [2001. A general method of constructing $E(s^2)$-optimal supersaturated designs. J. Roy. Statist. Soc. B 63, 621–632] (in the cases where their result applies) and those obtained by Bulutoglu and Cheng [2004. Construction of $E(s^2)$-optimal supersaturated designs. Ann. Statist. 32, 1662–1678] is established. We also give two simple methods of constructing $E(s^2)$-optimal designs.

1. Introduction

Supersaturated designs have received considerable attention in the recent past due to their usefulness in factor screening. In a factorial experiment involving m two-level factors and n runs, n is required to be at least $m+1$ for the estimability of all main effects. A design is called supersaturated if $n < m+1$. Under the assumption of effect sparsity that only a small number of factors are active, supersaturated designs can provide considerable cost saving in factor screening.

We represent an n-run supersaturated design for m two-level factors by an $n \times m$ matrix X of 1's and -1's where we assume that each column of X has an equal number of 1's and -1's and $n > 4$ is even. We also assume that for any two columns $\boldsymbol{u} = (u_1, \ldots, u_n)'$ and $\boldsymbol{v} = (v_1, \ldots, v_n)'$ of X, $\boldsymbol{u} \neq \pm \boldsymbol{v}$. The number of possible factors that can be accommodated is at most M, where

$$M = \frac{1}{2}\binom{n}{\frac{n}{2}} = \binom{n-1}{\frac{n}{2}-1}.$$

Thus we have $n-1 < m \leqslant M$. The choice of two-level supersaturated designs has mainly been based on the $E(s^2)$-optimality criterion proposed by Booth and Cox (1962). An $E(s^2)$-optimal supersaturated design is one that minimizes $E(s^2) = \sum_{i \neq j} s_{ij}^2 / \{m(m-1)\}$, where s_{ij} is the (i,j)-th entry of $X'X$.

* Corresponding author.
E-mail address: ashish@isid.ac.in (A. Das).
[1] On lien from Stat-Math Division, Indian Statistical Institute, New Delhi 110 016, India.

0378-3758/$ - see front matter
doi:10.1016/j.jspi.2007.12.014

Nguyen (1996) and Tang and Wu (1997) independently derived the lower bound (LB)

$$E(s^2) \geqslant \frac{(m-n+1)n^2}{(m-1)(n-1)}, \tag{1.1}$$

for any supersaturated design with m factors and n runs. When $n \equiv 0 \pmod 4$, this bound can be achieved only if m is a multiple of $n-1$; when $n \equiv 2 \pmod 4$, m needs to be an even multiple of $n-1$. Bulutoglu and Cheng (2004) and Butler et al. (2001) provided better LBs for $E(s^2)$ than (1.1).

In Section 2, we obtain further improved LBs on $E(s^2)$. After the first version of this paper was submitted, our attention was drawn to a recent work of Ryan and Bulutoglu (2007) who also derived similar improved bounds. However, our analysis towards finding improved LBs provides more details as it precisely identifies the situations when an improvement is possible. Such details are not provided in Ryan and Bulutoglu (2007); see the discussion in Section 2. In Section 3, we present in simpler terms the improved LBs. In the process, the equivalence of the bounds obtained by Butler et al. (2001) (in the cases where their result applies) and those obtained by Bulutoglu and Cheng (2004) is established. Finally in Section 4, we give two simple methods for constructing $E(s^2)$-optimal designs, one of which has also been described by Ryan and Bulutoglu (2007). The proofs of all the results are postponed to Section 5.

2. Improved LBs on $E(s^2)$

From Theorem 3.1 of Bulutoglu and Cheng (2004), it follows that for given m and n, $m > n-1$ (m not a multiple of $n-1$ when $n \equiv 0 \pmod 4$; or, m not an even multiple of $n-1$ when $n \equiv 2 \pmod 4$) there exists a unique integer q such that $(q-2)(n-1) < m < (q+2)(n-1)$ and $(m+q) \equiv 2 \pmod 4$. However, if we do not put the restriction of (i) m not being a multiple of $n-1$ when $n \equiv 0 \pmod 4$ and (ii) m not being an even multiple of $n-1$ when $n \equiv 2 \pmod 4$, then we show that there exists a unique non-negative integer q such that $(q-2)(n-1) \leqslant m < (q+2)(n-1)$ and $(m+q) \equiv 2 \pmod 4$. An explicit expression for q is also given. It can be verified that the Bulutoglu–Cheng proof goes through even for $m = n-1$ and for q such that $(q-2)(n-1) \leqslant m < (q+2)(n-1)$ and $(m+q) \equiv 2 \pmod 4$. Thus, subject to this minor modification, Theorem 3.1 of Bulutoglu and Cheng (2004) would hold for all m and n with $m \geqslant n-1$. Note that even though supersaturated designs have been defined only for $m > n-1$, the bounds are still meaningful for $m \geqslant n-1$. For given m and n, the following result gives an explicit expression for q. Throughout, for $z > 0$, $[z]$ stands for the largest integer contained in z.

Lemma 2.1. *For given $n \equiv 0 \pmod 2$, $m \equiv k \pmod 4$, $0 \leqslant k \leqslant 3$ and $m \geqslant n-1$, there is a unique non-negative integer q such that $(q-2)(n-1) \leqslant m < (q+2)(n-1)$ and $(m+q) \equiv 2 \pmod 4$. This unique q is given by $q = 4[(m+k(n-1))/4(n-1)] + 2 - k$.*

Let

$$g(q) = n((m+q)^2 - n(q^2+m)), \quad a_1 = (q-2)(n-1), \quad a_2 = (q-2)(n-1) + n/2,$$

$$a_3 = (q-1)(n-1), \quad a_4 = (q+1)(n-1), \quad a_5 = (q+2)(n-1) - n/2,$$

$$a_6 = (q+2)(n-1), \quad a_2' = (q-3)(n-1) + 3n/2, \quad a_5' = (q+3)(n-1) - 3n/2,$$

$$a_2'' = (q-1)(n-1) - n/2, \quad a_5'' = (q+1)(n-1) + n/2.$$

Also, let

$$\mathscr{R} = \{m : a_1 \leqslant m < a_6\}, \quad \mathscr{R}_{11} = \{m : a_3 \leqslant m \leqslant a_4\} = \mathscr{R}_{21} = \mathscr{R}_{31},$$

$$\mathscr{R}_{12} = \{m : a_2 \leqslant m < a_3 \text{ or } a_4 < m \leqslant a_5\}, \quad \mathscr{R}_{13} = \{m : a_1 \leqslant m < a_2 \text{ or } a_5 < m < a_6\},$$

$$\mathscr{R}_{22} = \{m : a_2' \leqslant m < a_3 \text{ or } a_4 < m \leqslant a_5'\}, \quad \mathscr{R}_{23} = \{m : a_1 \leqslant m < a_2' \text{ or } a_5' < m < a_6\},$$

$$\mathscr{R}_{32} = \{m : a_2'' \leqslant m < a_3 \text{ or } a_4 < m \leqslant a_5''\}, \quad \mathscr{R}_{33} = \{m : a_1 \leqslant m < a_2'' \text{ or } a_5'' < m < a_6\}.$$

Note that $\mathscr{R}$ covers the entire range of m and

$$\mathscr{R} = \mathscr{R}_{11} \cup \mathscr{R}_{12} \cup \mathscr{R}_{13} = \mathscr{R}_{21} \cup \mathscr{R}_{22} \cup \mathscr{R}_{23} = \mathscr{R}_{31} \cup \mathscr{R}_{32} \cup \mathscr{R}_{33}.$$

Bulutoglu and Cheng (2004) in their Theorem 3.1 had considered the set $\{m : a_3 < m < a_4\}$ instead of the sets $\mathscr{R}_{i1}, i = 1,2,3$. However, since $m+q$ is even, $m \neq (q \pm 1)(n-1)$, in the definition of $\mathscr{R}_{i1}$, $i = 1,2,3$, we kept the closed intervals for m, without affecting the results. The LBs given in Theorem 3.1 of Bulutoglu and Cheng (2004) can now be rephrased as below.

When $n \equiv 0 \pmod 4$ and $m \in \mathscr{R}_{1i}$, $i = 1,2,3$, $E(s^2) \geqslant B_{1i}/m(m-1)$, where

$$\begin{aligned} B_{11} &= g(q) + 2n(n-2), \\ B_{12} &= g(q) - 2n(n-2) + 4n|m - q(n-1)|, \\ B_{13} &= g(q) + 4n(n-1). \end{aligned} \tag{2.1}$$

When $n \equiv 2 \pmod 4$, $m \in \mathscr{R}_{2i}$, $i=1,2,3$, q is even, $E(s^2) \geqslant \max(B_{2i}/m(m-1), 4)$, where

$$\begin{aligned} B_{21} &= g(q) + 2n(n-2) + 8, \\ B_{22} &= g(q) - 2n(n-10) + 4(n-2)|m - q(n-1)| - 24, \\ B_{23} &= g(q) + 4n(n-1). \end{aligned} \tag{2.2}$$

When $n \equiv 2 \pmod 4$, $m \in \mathscr{R}_{3i}$, $i=1,2,3$, q is odd, $E(s^2) \geqslant \max(B_{3i}/m(m-1), 4)$, where

$$\begin{aligned} B_{31} &= g(q) + 2n(n-2), \\ B_{32} &= g(q) - 2n(n-2) + 4n|m - q(n-1)|, \\ B_{33} &= g(q) + 4n(n-3) + 8|m - q(n-1)| + 8. \end{aligned} \tag{2.3}$$

For obtaining the LBs for $E(s^2)$, Bulutoglu and Cheng (2004) and Butler et al. (2001) used structural properties of the matrix XX'. We use a property of the matrix $X'X$ to improve the abovementioned LBs of $E(s^2)$. We shall use $A(v)$ to denote a term which has a factor v. The following three lemmas are useful in the sequel.

Lemma 2.2. *For $n \equiv 0 \pmod 4$, each of B_{11}, B_{12} and B_{13} is a multiple of* 32.

Lemma 2.3. *For $n \equiv 2 \pmod 4$ and q even, each of $B_{21} - 4m(m-1)$, $B_{22} - 4m(m-1)$ and $B_{23} - 4m(m-1)$ is a multiple of* 64.

Lemma 2.4. *For $n \equiv 2 \pmod 4$ and q odd, $B_{32} - 4m(m-1)$ and $B_{33} - 4m(m-1)$ have a factor* 64, *whereas $B_{31} - 4m(m-1)$ has a factor* 32. *Moreover, $B_{31} - 4m(m-1)$ has a factor* 64 *unless* (i) *$m \equiv 1 \pmod 4$ and $(m+q) \equiv 6 \pmod 8$ or* (ii) *$m \equiv 3 \pmod 4$ and $(m+q) \equiv 2 \pmod 8$.*

Based on Lemmas 2.2–2.4, one can prove the following result.

Theorem 2.1. *Let $n \equiv 0 \pmod 2$, $m \equiv k \pmod 4$, $m \geqslant n-1$, $q = 4[(m + k(n-1))/4(n-1)] + 2 - k$, $g(q) = n((m+q)^2 - n(q^2+m))$, $\delta = m(m-1)$ and for $i=1,2,3$, B_{1i}, B_{2i}, B_{3i} are as in* (2.1)–(2.3), *respectively. Then,*

(1) *when $n \equiv 0 \pmod 4$ and $m \in \mathscr{R}_{1i}$, $E(s^2) \geqslant B_{1i}/\delta$, $i=1,2,3$;*
(2) *when $n \equiv 2 \pmod 4$, $m \in \mathscr{R}_{2i}$ and q is even, $E(s^2) \geqslant \max(B^*_{2i}/\delta, 4)$, $i=1,2,3$, where for $i=1,2,3$, $B^*_{2i} = B_{2i}$;*
(3) *when $n \equiv 2 \pmod 4$, $m \in \mathscr{R}_{3i}$ and q is odd, $E(s^2) \geqslant \max(B^*_{3i}/\delta, 4)$, $i=1,2,3$, where for $i=2,3$, $B^*_{3i} = B_{3i}$ and $B^*_{31} = B_{31} + x$ with*

$$x = \begin{cases} 32 & \text{if } m \equiv (1+2j) \pmod 4 \text{ and } (m+q) \equiv (6-4j) \pmod 8, \text{ for } j = 0 \text{ or } 1, \\ 0 & \text{if } m \equiv (1+2j) \pmod 4 \text{ and } (m+q) \equiv (2+4j) \pmod 8, \text{ for } j = 0 \text{ or } 1. \end{cases}$$

The LB improvements of Ryan and Bulutoglu (2007) are the same as above. This can be seen by noting the following:

(i) For $n \equiv 2 \pmod 4$, the Ryan–Bulutoglu bounds are $E(s^2) \geqslant 4 + 64\delta^{-1}\lceil \delta(h-4)/64 \rceil^+$, where h takes different values same as our respective B_{ij}/δ of (2.2) and (2.3);
(ii) $4 + 64\delta^{-1}\lceil \delta(h-4)/64 \rceil^+ = \max(4 + 64\delta^{-1}\lceil (B_{ij} - 4\delta)/64 \rceil, 4) = \max(B^*_{ij}/\delta, 4)$.

However, unlike that in Ryan and Bulutoglu (2007), our analysis is more transparent as it tells exactly when an improvement is possible and by how much. This also allows one to establish an equivalence result in the next section. Ryan and Bulutoglu (2007) found an $E(s^2)$-optimal supersaturated design for all cases with $n \leqslant 16$ except the 14 run and 16 factor case; see Ryan and Bulutoglu (2007) for details.

3. Equivalent form of the improved LBs

Bulutoglu and Cheng (2004) were the first to present a complete solution on LBs to $E(s^2)$ for any $m > n-1$. Earlier, Butler et al. (2001) had obtained LBs to $E(s^2)$ for $m = p(n-1) \pm r$, $0 \leqslant r < n/2$, where (i) p is positive and $n \equiv 0 \pmod 4$ and (ii) p is even and $n \equiv 2 \pmod 4$.

Bulutoglu and Cheng (2004) made a numerical comparison to see how their bounds compare with those of Butler et al. (2001). The numerical comparison suggested that Bulutoglu–Cheng bounds are in agreement with Butler et al. (2001) bounds in the cases where they are applicable. We now give an equivalent form of the improved LBs. This equivalent form also establishes the equivalence of the bounds obtained by Bulutoglu and Cheng (2004) and those obtained earlier by Butler et al. (2001) for all cases where their result applies. In the process, we present in simpler terms the LBs of Bulutoglu and Cheng (2004) and their improvements. We have the following result for the improved LBs covering the full scenario in a more elegant form which in particular, includes the case of p being odd, a case not covered explicitly by the earlier results.

Theorem 3.1. *For a supersaturated design with n runs and* $m = p(n-1) \pm r$ *factors* (p *positive*, $0 \leqslant r < n/2$), $E(s^2)$ *is greater than or equal to the LB, where LB is as defined below*:

(1) *Let* $n \equiv 0\,(\mathrm{mod}\,4)$. *Then*,

$$\mathrm{LB} = \frac{n^2(m-n+1)}{(n-1)(m-1)} + \frac{n}{m(m-1)}\left\{D(n,r) - \frac{r^2}{n-1}\right\},$$

where

$$D(n,r) = \begin{cases} n+2r-3 & \textit{for } r \equiv 1\,(\mathrm{mod}\,4),\\ 2n-4 & \textit{for } r \equiv 2\,(\mathrm{mod}\,4),\\ n+2r+1 & \textit{for } r \equiv 3\,(\mathrm{mod}\,4),\\ 4r & \textit{for } r \equiv 0\,(\mathrm{mod}\,4).\end{cases}$$

(2) *Let* $n \equiv 2\,(\mathrm{mod}\,4)$. *Then*,

$$\mathrm{LB} = \max\left\{\frac{n^2(m-n+1)}{(n-1)(m-1)} + \frac{n}{m(m-1)}\left\{D(n,r) - \frac{r^2}{n-1}\right\}, 4\right\},$$

where

(i) *when p is even*,

$$D(n,r) = \begin{cases} n+2r-3+x/n & \textit{for } r \equiv 1\,(\mathrm{mod}\,4),\\ 2n-4+8/n & \textit{for } r \equiv 2\,(\mathrm{mod}\,4),\\ n+2r+1 & \textit{for } r \equiv 3\,(\mathrm{mod}\,4),\\ 4r & \textit{for } r \equiv 0\,(\mathrm{mod}\,4),\end{cases}$$

(ii) *when p is odd*,

$$D(n,r) = \begin{cases} 2r-8r/n+n-16/n+9 & \textit{for } r \equiv 1\,(\mathrm{mod}\,4),\\ 4r-8r/n-8/n+8 & \textit{for } r \equiv 2\,(\mathrm{mod}\,4),\\ 2r+n+8/n-3 & \textit{for } r \equiv 3\,(\mathrm{mod}\,4),\\ 2n-4+x/n & \textit{for } r \equiv 0\,(\mathrm{mod}\,4),\end{cases}$$

and $x = 32$ *if* $\{(m-1-2i)/4 + [(m+(1+2i)(n-1))/4(n-1)]\} \equiv (1-i)\,(\mathrm{mod}\,2)$, *for* $i = 0$ *or* 1; *else* $x = 0$.

Note that LB in (1) and in (2) with p even (except $r \equiv 1\,(\mathrm{mod}\,4)$, $x \neq 0$) of Theorem 3.1 are the same as in Butler et al. (2001). The LB in (2) for p even, $r \equiv 1\,(\mathrm{mod}\,4)$, $x \neq 0$ and for p odd, $r \equiv 0\,(\mathrm{mod}\,4)$, $x \neq 0$ are an improvement over the earlier bounds of Bulutoglu and Cheng (2004) but the same as that of Ryan and Bulutoglu (2007).

4. Methods of constructing $E(s^2)$-optimal designs

In this section, we give two methods for constructing $E(s^2)$-optimal supersaturated designs. In the first method, Hadamard matrices are used to obtain $E(s^2)$-optimal designs for $m = n+1$ or $m = n$ factors each at 2 levels in n runs where $n \equiv 2\,(\mathrm{mod}\,4)$. In the second method we use complement of a supersaturated design and show that the complementary design is $E(s^2)$-optimal if the original supersaturated design is $E(s^2)$-optimal. This idea exists, for example, in Bulutoglu and Cheng (2004) and Eskridge et al. (2004). However, in their case, this property was attributed to augmentation of two balanced incomplete block designs. The result given in Theorem 4.2, which was also obtained by Ryan and Bulutoglu (2007), generalizes the idea to cover all cases. In fact, this result reduces the general problem of identifying $E(s^2)$-optimal designs to half. That is, one needs only to look for $E(s^2)$-optimal designs with $m \leqslant \frac{1}{4}\binom{n}{\frac{n}{2}} = M/2$. The two methods of construction follow.

Theorem 4.1. *For* $n \equiv 2\,(\mathrm{mod}\,4)$ *if a Hadamard matrix of order* $n+2$ *exists then an n run*, $(n+1)$ *factor*, $E(s^2)$*-optimal supersaturated design X with* $E(s^2) = 4$ *can be obtained. The design remaining after deleting any one column of X is an n run, n factor*, $E(s^2)$*-optimal supersaturated design with* $E(s^2) = 4$.

Note that since for a 2^{n+1} experiment in n runs ($n \equiv 2\,(\mathrm{mod}\,4)$), the design X has $E(s^2) = 4$, therefore any subset of the columns of X would give rise to a design having minimum $E(s^2)$. A useful connection can be drawn from the result of Theorem 4.1 to that in Cheng and Tang (2001). These authors show that $B(n,2) \leqslant n+2$, when $n \equiv 2\,(\mathrm{mod}\,4)$, where $B(n,2)$ denotes the maximum number of columns a supersaturated design can have under the constraint that $\max|s| \leqslant 2$. Theorem 4.1 shows that $B(n,2) \geqslant n+1$.

Theorem 4.2. *Let d be an* $E(s^2)$*-optimal supersaturated design for a* 2^m *experiment in n runs, where* $n-1 < m \leqslant M$. *Then the design* d' *having* $M-m$ *columns of Y which are not columns of d is an* $E(s^2)$*-optimal supersaturated design for a* 2^{M-m} *experiment in n runs*, *where Y is an $n \times M$ matrix, whose columns represent the M factors, such that for any two columns $\boldsymbol{u}=(u_1,\ldots,u_n)'$ and $\boldsymbol{v}=(v_1,\ldots,v_n)'$ of Y, $\boldsymbol{u} \neq \pm\boldsymbol{v}$.*

Let g' be a design obtained by taking all the $M-m$ columns of Y which are not columns (or -1 times the columns) of a given design g involving m factors. Also, let E_g be the value of $E(s^2)$ for the design g. Then using the structural properties of $Y'Y$, it can be shown that

$$E_{g'} = \frac{n^2(M-2m)M(n-1)^{-1} + m(n^2 + (m-1)E_g) - (M-m)n^2}{(M-m)(M-m-1)}$$

or

$$E_{g'} = \frac{n^2(M-2m)(M-n+1)}{(n-1)(M-m)(M-m-1)} + \frac{m(m-1)E_g}{(M-m)(M-m-1)}. \tag{4.1}$$

Based on the above observation, it would be natural to ask whether the relation between E_g and $E_{g'}$, namely (4.1), can be used to further improve the bounds of Theorem 3.1. In what follows, we show that the LBs of Theorem 3.1 agree (except when $n \equiv 2 \pmod 4$ with either $m \leqslant n+1$ or $m \geqslant M-n-1$) for the designs g' when one uses the LB of Theorem 3.1 for g in (4.1). In other words, the LB for g' can simply be obtained by substituting the LB for g in (4.1).

Let $t=n/2$. Then we can write M as $M=p^*(n-1)$, where $p^*=t^{-1}\binom{2(t-1)}{t-1}$ is a positive integer (being a Catalan number). Now, for the design g, let $m=p(n-1)\pm r$ (p positive, $0 \leqslant r < n/2$, $m \geqslant n+2$). Then, for g' the number of factors is $M-m=M-p(n-1)\mp r=(p^*-p)(n-1)\mp r$. Also, when t is odd (i.e., the case when $n \equiv 2 \pmod 4$), by taking $t=2w+1$ for some w, $p^*=(2w+1)^{-1}\binom{4w}{2w}$ is even since $\binom{4w}{2w}$ is even. Thus, when $n \equiv 2 \pmod 4$, for p even (odd), p^*-p is even (odd). Therefore, while obtaining LB using Theorem 3.1, the values of $\{D(n,r)-r^2/(n-1)\}=H$ (say), are same for g and g'. Let g attain the LB. Then, substituting the LB value in place of E_g in (4.1) gives (on simplification)

$$E_{g'} = \frac{n^2(M-m-n+1)}{(n-1)(M-m-1)} + \frac{nH}{(M-m)(M-m-1)}.$$

This establishes our claim.

When $n \equiv 2 \pmod 4$, LB $=4$ for $m \leqslant n+2$ and LB >4 for $m > n+2$, since

$$\frac{n^2(m-n+1)}{(n-1)(m-1)} + \frac{n}{m(m-1)}\left\{D(n,r) - \frac{r^2}{n-1}\right\} \begin{array}{lll} < & 4 & \text{when } m < n+2, \\ = & 4 & \text{when } m = n+2, \\ > & 4 & \text{when } m > n+2. \end{array}$$

This is the reason why, for $n \equiv 2 \pmod 4$ with $m \leqslant n+1$, on substituting $E_g=4$, (4.1) gives $E_{g'}$ (with $M-m$ factors) leading to a sharper LB than what is provided by Theorem 3.1.

In closing this section, we make some remarks on near optimal designs obtained by Eskridge et al. (2004) for $n \equiv 2 \pmod 4$ and $m=j(n-1)$, j odd, using the properties of regular graph designs (RGDs). For their designs obtained from generators of cyclic RGD,

$$E(s^2) = \frac{n^2(m-n+1)}{(n-1)(m-1)} + \frac{4(n-1)(n-2)}{m(m-1)}. \tag{4.2}$$

With $n \geqslant 10$ and $j>2$, Eskridge et al. (2004) obtained the LBs to the $E(s^2)$-efficiency (using Nguyen–Tang–Wu bounds) and also showed that their designs have $E(s^2)$-efficiency greater than 0.9493. Now, for $n \equiv 2 \pmod 4$, $m=j(n-1)$, j odd, based on Theorem 3.1, the LB for $E(s^2)$ is given by

$$E(s^2) \geqslant \frac{n^2(m-n+1)}{(n-1)(m-1)} + \frac{n(2n-4+x/n)}{m(m-1)}, \tag{4.3}$$

where $x=32$ if $\{(m-1-2i)/4 + [(m+(1+2i)(n-1))/4(n-1)]\} \equiv (1-i) \pmod 2$, for $i=0$ or 1; else $x=0$.

Using (4.2) and (4.3) we get a sharper LB to the $E(s^2)$-efficiency, given by

$$E(s^2)\text{-efficiency} \geqslant \frac{1+b}{1+a}, \tag{4.4}$$

where $a=4(n-2)(n-1)^2/\{mn^2(m-n+1)\}$, $b=(2n-4+x/n)(n-1)/\{mn(m-n+1)\}$ and $x=32$ if $\{(m-1-2i)/4+[\{m+(1+2i)(n-1)\}/\{4(n-1)\}]\} \equiv (1-i) \pmod 2$, for $i=0$ or 1; else $x=0$.

From (4.4), it follows that the designs based on an RGD have $E(s^2)$-efficiency greater than 0.9774. This shows that RGD based designs have efficiencies higher than what is presently known. However, in the light of results in Ryan and Bulutoglu (2007), it should be noted that for $n \equiv 2 \pmod 4$ and $m=j(n-1)$, j odd, the RGD based designs are not necessarily $E(s^2)$-optimal and there exist examples in which these are not so.

5. Proofs

Proof of Lemma 2.1. Given m, $q=4x+2-k$ for some integer x, since $m\equiv k\,(\mathrm{mod}\,4)$ and $(m+q)\equiv 2\,(\mathrm{mod}\,4)$. Now, $(q-2)(n-1)\leqslant m<(q+2)(n-1)$ implies $m/(n-1)-2<q\leqslant m/(n-1)+2$ which, on substituting for q, yields $\{m+k(n-1)\}/\{4(n-1)\}-1<x\leqslant\{m+k(n-1)\}/\{4(n-1)\}$. Thus, $x=[\{m+k(n-1)\}/\{4(n-1)\}]$ and we get the desired expression for q. By substituting the four possible values of k in the expression for q, it follows that q is non-negative. □

Proof of Lemma 2.2. For $t>0$, let $n=4t$. Now, since $(m+q)\equiv 2\,(\mathrm{mod}\,4)$, it follows that q^2+m is even. Let $q+m=4w+2$ for some positive w. Then $g(q)=n((m+q)^2-n(q^2+m))=4t(4w+2)^2-16t^2A(2)=64tw(w+1)+16t-A(32)=A(32)+16t$. Therefore, $B_{11}=g(q)+2n(n-2)=A(32)+16t+32t^2-16t=A(32)$, $B_{12}=g(q)-2n(n-2)+4n|m-q(n-1)|=A(32)+16t-32t^2+16t+16t|2(2w-2tq+1)|=A(32)$, $B_{13}=g(q)+4n(n-1)=A(32)+16t+64t^2-16t=A(32)$. □

Proof of Lemma 2.3. For $t>0$, let $n=4t+2$. Now, since $(m+q)\equiv 2\,(\mathrm{mod}\,4)$ and q is even, it follows that either (i) $m\equiv 2\,(\mathrm{mod}\,4)$ and $q\equiv 0\,(\mathrm{mod}\,4)$ or (ii) $m\equiv 0\,(\mathrm{mod}\,4)$ and $q\equiv 2\,(\mathrm{mod}\,4)$. This implies that for some non-negative integers y, s, and $i=0$ or 1, we have $m=4y+2i$ and $q=4s+2-2i$. Now, substituting for n, m and q, and using the fact $i^2=i$, we have after simplification, $g(q)-4m(m-1)=n((m+q)^2-n(q^2+m))-4m(m-1)=32(2t+1)\{y(y+1)-s(s+1)-4ts(s+1)+2ys-4ts\}-64\{t^2+y^2+yt(t+1)\}-48t-8+64i\{4st(t+1)+y+s\}+32it(t+1)=A(64)-48t-8$. Therefore, $B_{21}-4m(m-1)=g(q)-4m(m-1)+2n(n-2)+8=A(64)-48t-8+16t(2t+1)+8=A(64)+32t(t-1)=A(64)$, $B_{22}-4m(m-1)=g(q)-4m(m-1)-2n(n-10)+4(n-2)|m-q(n-1)|-24=A(64)-48t-8-32t^2+48t+8\pm 64t(y-s-4st+2t(i-1)+i)\mp 32t=A(64)-32t(t\pm 1)=A(64)$, $B_{23}-4m(m-1)=g(q)-4m(m-1)+4n(n-1)=A(64)-48t-8+8(8t^2+6t+1)=A(64)+64t^2=A(64)$. □

Proof of Lemma 2.4. For $t>0$, let $n=4t+2$. Now, since $(m+q)\equiv 2\,(\mathrm{mod}\,4)$ and q odd, it follows that either (i) $m\equiv 1\,(\mathrm{mod}\,4)$ and $q\equiv 1\,(\mathrm{mod}\,4)$ or (ii) $m\equiv 3\,(\mathrm{mod}\,4)$ and $q\equiv 3\,(\mathrm{mod}\,4)$. This implies that for some non-negative integers y, s, and $i=0$ or 1, we have $m=4y+2i+1$ and $q=4s+2i+1$. Now, substituting for n, m and q, and using the fact $i^2=i$, we have after simplification, $g(q)-4m(m-1)=n((m+q)^2-n(q^2+m))-4m(m-1)=64(2t+1)(ys+yi-ts-2ts^2-2tsi)-64t(y^2-s^2-ty-2ti)-64yi-32it(t+1)-32(y^2+s^2+t^2)-16t=A(64)-32(y^2+s^2+t^2)-16t$. Therefore, $B_{31}-4m(m-1)=g(q)-4m(m-1)+2n(n-2)=A(64)-32(y^2+s^2+t^2)-16t+32t^2+16t=A(64)-32(y^2+s^2)$, $B_{32}-4m(m-1)=g(q)-4m(m-1)-2n(n-2)+4n|m-q(n-1)|=A(64)-32(y^2+s^2+t^2)-16t-32t^2-16t+32(2t+1)|y-s-t-2t(2s+i)|=A(64)-64t(t-|y-s-t-2t(2s+i)|)-32\{y(y\mp 1)+s(s\pm 1)\pm 2t(2s+i)+t\pm t\}=A(64)$, $B_{33}-4m(m-1)=g(q)-4m(m-1)+4n(n-3)+8|m-q(n-1)|+8=A(64)-32(y^2+s^2+t^2)-16t+64t^2+16t+32|y-s-t-2t(2s+i)|=A(64)+64t^2-32\{y(y\mp 1)+s(s\pm 1)+t(t\pm 1)\pm 2t(2s+i)\}=A(64)$. Now, $B_{31}-4m(m-1)$ is an odd multiple of 32 if and only if y^2+s^2 is odd. Also, y^2+s^2 is odd if and only if $y+s$ is odd. Let $y+s=2w+1$ for some w. Then $m+q=4(y+s)+2+4i=8w+6+4i$ and either of the following holds:

(i) $i=0$, $(m+q)\equiv 6\,(\mathrm{mod}\,8)$,
(ii) $i=1$, $(m+q)\equiv 2\,(\mathrm{mod}\,8)$.

Similarly, it follows that $y+s$ is even when $m+q=8w+2+4i$, implying either (i) $i=0$, $(m+q)\equiv 2\,(\mathrm{mod}\,8)$ or (ii) $i=1$, $(m+q)\equiv 6\,(\mathrm{mod}\,8)$. □

Proof of Theorem 2.1. The result basically follows from Lemmas 2.2–2.4 and the following facts:

(i) For $n\equiv 0\,(\mathrm{mod}\,4)$, s_{ij} is an integral multiple of 4 for all $i\neq j$. Therefore, $\sum_{i\neq j}s_{ij}^2$ is a multiple of 32.
(ii) For $n\equiv 2\,(\mathrm{mod}\,4)$, we have $|s_{ij}|\equiv 2\,(\mathrm{mod}\,4)$. This means that $s_{ij}^2=A(32)+4$. Therefore, $\sum_{i\neq j}s_{ij}^2-4m(m-1)=A(64)$.

For $n\equiv 0\,(\mathrm{mod}\,4)$, we have already seen in Lemma 2.2 that each of B_{11}, B_{12} and B_{13} is a multiple of 32. Also, when $n\equiv 2\,(\mathrm{mod}\,4)$, Lemmas 2.3 and 2.4 show that each of $B_{21}-4m(m-1)$, $B_{22}-4m(m-1)$, $B_{23}-4m(m-1)$, $B_{32}-4m(m-1)$ and $B_{33}-4m(m-1)$ is a multiple of 64 while $B_{31}-4m(m-1)$ is a multiple of 32 but not necessarily a multiple of 64. This leads to the LB improvement for the bound involving B_{31} by increasing B_{31} to $B_{31}+32$ in all those cases where $B_{31}-4m(m-1)$ is not a multiple of 64. □

Proof of Theorem 3.1. For some integers i and r, let $m=(q\mp i)(n-1)\pm r$. Then,

$$q=\frac{m\mp r}{n-1}\pm i \quad\text{and}\quad m+q=(q\mp i)n\pm(i+r)\equiv 2\,(\mathrm{mod}\,4). \tag{5.1}$$

Substituting the value of q from (5.1) in $g(q)$ and after some simplification, we have

$$g(q)=n(m+q)^2-n^2(q^2+m)=m(m-1)T+n\left\{2ri-(n-1)i^2-\frac{r^2}{n-1}\right\}, \tag{5.2}$$

where $T = n^2(m-n+1)/\{(n-1)(m-1)\}$. We now consider the various ranges of m that have been used in Theorem 2.1, leading to different cases detailed below.

Case 1: $m \in \mathscr{R}_{11}$ if and only if $m = (q \mp i)(n-1) \pm r$ for some $r \in [0, n/2)$ and either $i = 0$ or 1 where $r \equiv (2-i) \pmod 4$. Then, using (5.2), the LB $B_{11}/\{m(m-1)\}$ in Theorem 2.1 is equivalent to

$$\frac{g(q)+2n(n-2)}{m(m-1)} = T + \frac{n}{m(m-1)}\left\{2n-4-(n-1)i^2+2ri-\frac{r^2}{n-1}\right\}. \tag{5.3}$$

For $n \equiv 0 \pmod 4$ and $m = p(n-1) \pm r$ for some positive p and $0 \leqslant r < n/2$, substituting $i = 0$ and 1 in (5.3) gives the respective LBs as

$$\mathrm{LB} = T + \frac{n}{m(m-1)}\left\{2n-4-\frac{r^2}{n-1}\right\}, \quad r \equiv 2 \pmod 4, \tag{5.4}$$

$$\mathrm{LB} = T + \frac{n}{m(m-1)}\left\{n+2r-3-\frac{r^2}{n-1}\right\}, \quad r \equiv 1 \pmod 4. \tag{5.5}$$

Case 2: $m \in \mathscr{R}_{12}$ if and only if $m = (q \mp i)(n-1) \pm r$ for some $r \in [0, n/2)$ and $i = -1$ where $r \equiv 3 \pmod 4$. Then, with $i = -1$, using (5.1) and (5.2), the LB $B_{12}/\{m(m-1)\}$ in Theorem 2.1 is equivalent to

$$\frac{g(q)-2n(n-2)+4n|m-q(n-1)|}{m(m-1)} = T + \frac{n}{m(m-1)}\left\{n+2r+1-\frac{r^2}{n-1}\right\}. \tag{5.6}$$

For $n \equiv 0 \pmod 4$ and $m = p(n-1) \pm r$ for some positive p $(0 \leqslant r < n/2)$, (5.6) gives the LB as

$$\mathrm{LB} = T + \frac{n}{m(m-1)}\left\{n+2r+1-\frac{r^2}{n-1}\right\}, \quad r \equiv 3 \pmod 4. \tag{5.7}$$

Case 3: $m \in \mathscr{R}_{13}$ if and only if $m = (q \mp i)(n-1) \pm r$ for some $r \in [0, n/2)$ and $i = 2$ where $r \equiv 0 \pmod 4$. Then, with $i = 2$, using (5.2), the LB $B_{13}/\{m(m-1)\}$ in Theorem 2.1 is equivalent to

$$\frac{g(q)+4n(n-1)}{m(m-1)} = T + \frac{n}{m(m-1)}\left\{4r-\frac{r^2}{n-1}\right\}. \tag{5.8}$$

For $n \equiv 0 \pmod 4$ and $m = p(n-1) \pm r$ for some positive p $(0 \leqslant r < n/2)$, (5.8) gives the LB as

$$\mathrm{LB} = T + \frac{n}{m(m-1)}\left\{4r-\frac{r^2}{n-1}\right\}, \quad r \equiv 0 \pmod 4. \tag{5.9}$$

Case 4: $m \in \mathscr{R}_{21}$ if and only if $m = (q \mp i)(n-1) \pm r$ for some $r \in [0, n/2)$ and either $i = 0$ or 1 where $r \equiv (2+i) \pmod 4$. Also, $(q \mp i) \equiv i \pmod 2$, $i = 0, 1$. Then, using (5.2), the LB $B_{21}/\{m(m-1)\}$ in Theorem 2.1 is equivalent to

$$\frac{g(q)+2n(n-2)+8}{m(m-1)} = T + \frac{n}{m(m-1)}\left\{2n-(n-1)i^2+2ri-4+\frac{8}{n}-\frac{r^2}{n-1}\right\}. \tag{5.10}$$

For $n \equiv 2 \pmod 4$ and $m = p(n-1) \pm r$ for some positive $p \equiv i \pmod 2$ $(0 \leqslant r < n/2)$, substituting $i = 0$ and 1 in (5.10) gives the respective LB as

$$\mathrm{LB} = T + \frac{n}{m(m-1)}\left\{2n-4+\frac{8}{n}-\frac{r^2}{n-1}\right\}, \quad p \text{ even}, \quad r \equiv 2 \pmod 4, \tag{5.11}$$

$$\mathrm{LB} = T + \frac{n}{m(m-1)}\left\{n+\frac{8}{n}+2r-3-\frac{r^2}{n-1}\right\}, \quad p \text{ odd}, \quad r \equiv 3 \pmod 4. \tag{5.12}$$

Case 5: $m \in \mathscr{R}_{22}$ if and only if $m = (q \mp i)(n-1) \pm r$ for some $r \in [0, n/2-1)$ and $i = -1$ where $r \equiv 1 \pmod 4$. Also, $(q \pm 1) \equiv 1 \pmod 2$. Then, with $i = -1$, using (5.1) and (5.2), the LB $B_{22}/\{m(m-1)\}$ in Theorem 2.1 is equivalent to

$$\begin{aligned}&\frac{g(q)-2n(n-10)+4(n-2)|m-q(n-1)|-24}{m(m-1)}\\ &\quad = T + \frac{n}{m(m-1)}\left\{2r-\frac{8r}{n}+n-\frac{16}{n}+9-\frac{r^2}{n-1}\right\}.\end{aligned} \tag{5.13}$$

Note that since $n \equiv 2 \pmod 4$ and $r \equiv 1 \pmod 4$, $(n/2-1) \equiv 0 \pmod 2$ and $r \neq n/2 - 1$. Thus in the above range of r, we can take $r < n/2$.

For $n \equiv 2 \pmod 4$ and $m = p(n-1) \pm r$ for some positive $p \equiv 1 \pmod 2$ $(0 \leqslant r < n/2)$, (5.13) gives the LB as

$$\mathrm{LB} = T + \frac{n}{m(m-1)}\left\{2r - \frac{8r}{n} + n - \frac{16}{n} + 9 - \frac{r^2}{n-1}\right\}, \quad p \text{ odd}, \quad r \equiv 1 \pmod 4. \tag{5.14}$$

Case 6: $m \in \mathscr{R}_{23}$ if and only if $m = (q \mp i)(n-1) \pm r$ for some $r \in [0, n/2]$ and $i = 2$ where $r \equiv 0 \pmod 4$. Also, $(q \mp 2) \equiv 0 \pmod 2$. Then, with $i = 2$, using (5.2), the LB $B_{23}/\{m(m-1)\}$ in Theorem 2.1 is equivalent to

$$\frac{g(q) + 4n(n-1)}{m(m-1)} = T + \frac{n}{m(m-1)}\left\{4r - \frac{r^2}{n-1}\right\}. \tag{5.15}$$

Note that since $n \equiv 2 \pmod 4$ and $r \equiv 0 \pmod 4$, $n/2 \equiv 1 \pmod 2$ and $r \neq n/2$. Thus in the above range of r, we can take $r < n/2$.

For $n \equiv 2 \pmod 4$ and $m = p(n-1) \pm r$ for some positive $p \equiv 0 \pmod 2$ $(0 \leqslant r < n/2)$, (5.15) gives the LB as

$$\mathrm{LB} = T + \frac{n}{m(m-1)}\left\{4r - \frac{r^2}{n-1}\right\}, \quad p \text{ even}, \quad r \equiv 0 \pmod 4. \tag{5.16}$$

Case 7: $m \in \mathscr{R}_{31}$ if and only if $m = (q \mp i)(n-1) \pm r$ for some $r \in [0, n/2)$ and either $i = 0$ or 1 where $r \equiv i \pmod 4$. Also, $(q \mp i) \equiv (1-i) \pmod 2$, $i = 0, 1$. Then, using (5.2), the LB $B_{31}/\{m(m-1)\}$ in Theorem 2.1 is equivalent to

$$\frac{g(q) + 2n(n-2) + x}{m(m-1)} = T + \frac{n}{m(m-1)}\left\{2n - (n-1)i^2 + 2ri - 4 + \frac{x}{n} - \frac{r^2}{n-1}\right\}, \tag{5.17}$$

where $x = 32$ if $m \equiv (1+2j) \pmod 4$, $(m+q) \equiv (6-4j) \pmod 8$, for $j = 0$ or 1, else $x = 0$.

Now, using Lemma 2.1, it follows that $m \equiv (1+2j) \pmod 4$ if and only if $m + q = m + 4[(m + (1+2j)(n-1))/4(n-1)] + 2 - (1+2j) = m - (1+2j) + 4[(m + (1+2j)(n-1))/4(n-1)] + 2$. Therefore, $m \equiv (1+2j) \pmod 4$ and $(m+q) \equiv (6-4j) \pmod 8$ is equivalent to saying $m - (1+2j) + 4[(m + (1+2j)(n-1))/4(n-1)] \equiv 4(1-j) \pmod 8$. Thus,

$$\frac{m-(1+2j)}{4} + \left[\frac{m + (1+2j)(n-1)}{4(n-1)}\right] \equiv (1-j) \pmod 2, \quad j = 0, 1$$

are the conditions when $x = 32$. Similarly, it can be verified that

$$\frac{m-(1+2j)}{4} + \left[\frac{m + (1+2j)(n-1)}{4(n-1)}\right] \equiv j \pmod 2, \quad j = 0, 1$$

are the conditions when $x = 0$.

For $n \equiv 2 \pmod 4$ and $m = p(n-1) \pm r$ for some positive $p \equiv (1-i) \pmod 2$ $(0 \leqslant r < n/2)$, substituting $i = 0$ and 1 in (5.17) gives the respective LB as

$$\mathrm{LB} = T + \frac{n}{m(m-1)}\left\{2n - 4 + \frac{x}{n} - \frac{r^2}{n-1}\right\}, \quad p \text{ odd}, \quad r \equiv 0 \pmod 4, \tag{5.18}$$

$$\mathrm{LB} = T + \frac{n}{m(m-1)}\left\{n + 2r - 3 + \frac{x}{n} - \frac{r^2}{n-1}\right\}, \quad p \text{ even}, \quad r \equiv 1 \pmod 4, \tag{5.19}$$

where $x = 32$ if $m \equiv (1+2j) \pmod 4$ and $(m-1-2j)/4 + [(m + (1+2j)(n-1))/4(n-1)] \equiv (1-j) \pmod 2$, $j = 0, 1$, else $x = 0$.

Case 8: $m \in \mathscr{R}_{32}$ if and only if $m = (q \mp i)(n-1) \pm r$ for some $r \in [0, n/2]$ and $i = -1$ where $r \equiv 3 \pmod 4$. Also, $(q \pm 1) \equiv 0 \pmod 2$. Then, with $i = -1$, using (5.1) and (5.2), the LB $B_{32}/\{m(m-1)\}$ in Theorem 2.1 is equivalent to

$$\frac{g(q) - 2n(n-2) + 4n|m - q(n-1)|}{m(m-1)} = T + \frac{n}{m(m-1)}\left\{n + 2r + 1 - \frac{r^2}{n-1}\right\}. \tag{5.20}$$

When $r = n/2$, it can be verified that the expression in (5.20) is the same as the expression that one would get on substituting $r = n/2 - 1$ in (5.22). Therefore in the above range of r, we can take $r < n/2$.

For $n \equiv 2 \pmod 4$ and $m = p(n-1) \pm r$ for some positive $p \equiv 0 \pmod 2$ $(0 \leqslant r < n/2)$, (5.20) gives the LB as

$$\mathrm{LB} = T + \frac{n}{m(m-1)}\left\{n + 2r + 1 - \frac{r^2}{n-1}\right\}, \quad p \text{ even}, \quad r \equiv 3 \pmod 4. \tag{5.21}$$

Case 9: $m \in \mathscr{R}_{33}$ if and only if $m = (q \mp i)(n-1) \pm r$ for some $r \in [0, n/2 - 1)$ and $i = 2$ where $r \equiv 2 \pmod 4$. Also, $(q \mp 2) \equiv 1 \pmod 2$. Then, with $i = 2$, using (5.1) and (5.2), the LB $B_{23}/\{m(m-1)\}$ in Theorem 2.1 is equivalent to

$$\frac{g(q) + 4n(n-3) + 8|m - q(n-1)| + 8}{m(m-1)} = T + \frac{n}{m(m-1)}\left\{4r - \frac{8r}{n} - \frac{8}{n} + 8 - \frac{r^2}{n-1}\right\}. \tag{5.22}$$

As noted in the previous case, when $r = n/2 - 1$, the expression in (5.22) is the same as the expression that one would get on substituting $r = n/2$ in (5.20). Therefore in the above range of r, we can take $r < n/2$.

For $n \equiv 2 \pmod 4$ and $m = p(n-1) \pm r$ for some positive $p \equiv 1 \pmod 2$ $(0 \leqslant r < n/2)$, (5.22) gives the LB as

$$\mathrm{LB} = T + \frac{n}{m(m-1)} \left\{ 4r - \frac{8r}{n} - \frac{8}{n} + 8 - \frac{r^2}{n-1} \right\}, \quad p \text{ odd}, \quad r \equiv 2 \pmod 4. \tag{5.23}$$

Summarizing all the above cases, we have the following:

(i) Eqs. (5.4), (5.5), (5.7) and (5.9) give the LBs for $E(s^2)$ when $n \equiv 0 \pmod 4$, $m = p(n-1) \pm r$, p positive and $0 \leqslant r < n/2$,
(ii) Eqs. (5.11), (5.16), (5.19) and (5.21) give the LBs for $E(s^2)$ (subject to the fact that $E(s^2) \geqslant 4$) when $n \equiv 2 \pmod 4$, $m = p(n-1) \pm r$, p even and $0 \leqslant r < n/2$,
(iii) Eqs. (5.12), (5.14), (5.18) and (5.23) give the LBs for $E(s^2)$ (subject to the fact that $E(s^2) \geqslant 4$) when $n \equiv 2 \pmod 4$, $m = p(n-1) \pm r$, p odd and $0 \leqslant r < n/2$. $\square$

Proof of Theorem 4.1. For $n \equiv 2 \pmod 4$, let H be a Hadamard matrix of order $n+2$ where without loss of generality, the first row and first column of H has all $+1$'s. Delete the first row and first column of H and call the resultant $(n+1) \times (n+1)$ matrix G. Now, there are $n/2$ columns of G, each of which has $+1$ in the first row. Let these columns be labelled as $c_1, c_2, \ldots, c_{n/2}$ and let $\mathscr{C} = \{c_1, c_2, \ldots, c_{n/2}\}$, $\mathscr{C}' = \{1, 2, \ldots, n+1\} - \mathscr{C}$. Delete first row of G and call the resultant $n \times (n+1)$ matrix F.

Consider the $n \times n/2$ sub-matrix E consisting of the columns $c_1, c_2, \ldots, c_{n/2}$ of F. Carry out the following operation: In any row of E replace all -1's by $+1$'s. This would affect only certain columns of E where a -1 was replaced by $+1$. Do the same operation for the remaining 'unaffected' columns of E and carry out this process iteratively till there are no unaffected columns in E. Call the resultant matrix $\bar{E}$.

Let X be the $n \times (n+1)$ matrix obtained by replacing E by $\bar{E}$ in F. Let s_{ij} be the (i,j)th element of $X'X$. Then, $s_{ij} = \pm 2$ for all $i \neq j$, since

(i) for $i, j \in \mathscr{C}'$, $s_{ij} = -2$,
(ii) for $i, j \in \mathscr{C}$, $s_{ij} = \pm 2$,
(iii) for $i \in \mathscr{C}$ and $j \in \mathscr{C}'$, $s_{ij} = \pm 2$.

The rest of the proof follows from the fact that for $n \equiv 2 \pmod 4$, $E(s^2) \geqslant 4$. $\square$

Acknowledgements

The authors wish to thank an Associate Editor and two referees for their constructive comments on an earlier version which have helped greatly to improve the presentation. Part of this work was supported by a CERG research grant from the Research Grants Council of Hong Kong.

References

Booth, K.H.V., Cox, D.R., 1962. Some systematic supersaturated designs. Technometrics 4, 489–495.
Bulutoglu, D.A., Cheng, C.S., 2004. Construction of $E(s^2)$-optimal supersaturated designs. Ann. Statist. 32, 1662–1678.
Butler, N.A., Mead, R., Eskridge, K.M., Gilmour, S.G., 2001. A general method of constructing $E(s^2)$-optimal supersaturated designs. J. Roy. Statist. Soc. B 63, 621–632.
Cheng, C.S., Tang, B., 2001. Upper bounds on the number of columns in supersaturated designs. Biometrika 88, 1169–1174.
Eskridge, K.M., Gilmour, S.G., Mead, R., Butler, N.A., Travnicek, D.A., 2004. Large supersaturated designs. J. Statist. Comput. Simulation 74, 525–542.
Nguyen, N.K., 1996. An algorithmic approach to constructing supersaturated designs. Technometrics 38, 69–73.
Ryan, K.J., Bulutoglu, D.A., 2007. $E(s^2)$-optimal supersaturated designs with good minimax properties. J. Statist. Plann. Inference 137, 2250–2262.
Tang, B., Wu, C.F.J., 1997. A method for constructing supersaturated designs and its $E(s^2)$-optimality. Canad. J. Statist. 25, 191–201.