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Nonuniform Rates of Convergence to Normality

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Abstract

Nonuniform rates of convergence to normality are studied for standardized sample sum of independent random variables in a triangular array when mth moment of the variables is of order $L^m \exp(\gamma m \log m)$, $L > 0, 0 < \gamma < 1$, $\forall m > 1$; equivalently, $\sup_{n\geq 1} n^{-1} \sum_{i=1}^{n} E \exp(s|X_{ni}|^{1/\gamma}) < \infty$, for some $s > 0$. This assumption goes beyond the existence of moment generating functions of individual random variables. As $0 < \gamma < 1$, one gets a clear picture of the role of γ on rates of convergence, while one moves from the assumption of existence of the moment generating functions of the random variables to the boundedness of the random variables, by varying γ. Linnik (1961) considered convergence rates in iid setup with variables having moment generating functions at the most. The general results considered in the present paper reduce to those of Dasgupta (1992) in the special case $\gamma = 1/2$. The nonuniform bounds are used to obtain rates of moment type convergences and L_p version of Berry-Esseen theorem. An upper bound for the tail probability of standardized sample sum being greater than t is computed. For $0 < \gamma < 1/2$ and t large, this probability is shown to have a faster rate of decrease than normal tail probability. The results are extended to general nonlinear statistics and linear process.

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1 Introduction

Let $[X_{ni} : 1 \leq i \leq n, n \geq 1]$ be a triangular array of random variables, where variables in each array are independently distributed. Assume, without loss of generality,

$$EX_{ni} = 0, \quad \forall n \geq 1, \;\; 1 \leq i \leq n. \tag{1.1}$$

Define $S_n = \sum_{i=1}^n X_{ni},\ s_n^2 = \sum_{i=1}^n EX_{ni}^2$ and $F_n(t) = P(s_n^{-1}S_n \leq t)$. Let,

$$\inf_{n\geq 1} n^{-1/2}s_n = C(>0). \tag{1.2}$$

Then $F_n \to \Phi$, weakly under Lindberg-assumption. To study the speed of convergence, one needs to assume the existence of moments slightly higher than two of the random variables X_{ni}. Consider then

$$\sup_{n\geq 1} n^{-1}\sum_{i=1}^n EX_{ni}^2 g(X_{ni}) < \infty, \tag{1.3}$$

where $g(x)$ is a non-negative, even, nondecreasing function on $[0,\infty)$.

Assumption (1.3) gives rise to following three broadly classified cases.

I. Some finite order moment ≥ 2 of the individual random variables exists.

II. Moments of all finite order exist but moment generating functions of the random variables may not exist.

III. Moment generating functions of the random variables exist but the random variables may not be bounded.

The uniform bound $O(n^{-1/2})$ in CLT due to Berry and Esseen was extended by Katz (1963) in the iid set up. The nonuniform rates of convergence of $|F_n(t)-\Phi(t)|$ to zero has been studied under various moment assumptions, see e.g. Petrov (1975), Michel (1976). See also Chen and Shao (2004) for nonuniform bounds under local dependence.

Michel (1976) computed the deviation zone for t_n where $1 - F_n(t_n) \sim \Phi(-t_n),\ t_n \longrightarrow \infty$ in the iid set up, with $g(x) = |x|^c;\ c > 0$. This was generalized for slightly more general g, by Ghosh and Dasgupta (1978), for triangular array of independent random variables.
With $g(x)$ such that, $|x|^k << g(x) << \exp(s|x|),\ \forall\ k > 0$ and some $s > 0$, Dasgupta (1989) computed nonuniform central limit bounds and computed the normal approximation zone of tail probabilities; the necessary and sufficient conditions for such results are shown to be identical for some forms of g. All these results refer to case I and case II.

The remaining case, viz. case III is considered in this paper. We study the nonuniform rates of convergence under the assumption:

$$\sup_{n\geq 1} \; n^{-1}\sum_{i=1}^{n} E|X_{ni}|^m \leq L^m e^{\gamma m \; \log m}, \; \forall \; m > 1, \; \text{where } L > 0, 0 < \gamma < 1; \tag{1.4}$$

or, the following equivalent assumption (vide remark 2.1), where the summands have a $1/\gamma$ th power with finite moment generating function; viz. for some $s > 0$,

$$\sup_{n\geq 1} n^{-1}\sum_{i=1}^{n} E\exp(s|X_{ni}|^{1/\gamma}) < \infty. \tag{1.5}$$

Nonuniform rates of convergence in case III have not attracted much attention except the case when the random variables are *bounded.* However, in Dasgupta (1992), the author considered a special case $\gamma = 1/2$, to compute the nonuniform bounds of $|F_n(t) - \Phi(t)|$. In this paper we cover a broad spectrum of g in a more general situation where γ has a larger range of variation, i.e. $0 < \gamma < 1$ in (1.4). *This provides a clear picture of changes in rates and tail probabilities, as one moves from the assumption of existence of moment generating functions ($\gamma = 1$) to the boundedness of the random variables ($\gamma \to 0$) while γ varies in the range (0,1). See Theorems 2.1–2.3.*

The tail probabilities of the standardized sample sum are expected to decrease fast in a 'continuous' manner if the tail probabilities of individual random variables decrease rapidly. Theorem 2.1 provides a result in this direction. Combining Theorem 2.1 with the results of Ghosh and Dasgupta (1978) and Dasgupta (1989), one obtains a sharp overall nonuniform bound in the CLT as stated in Theorem 2.2. Consequently, results on moment type convergences and L_p version of Berry-Esseen theorem are immediate.

The technique of proof in Theorem 2.1 can be briefly described as follows. The assumption (1.5) yields a moment bound (1.4) for the random variables X_{ni}. Next, a term wise comparison of the series expansion of the moment generating functions of the individual random variables with an appropriate exponential function provides a sharp bound for the moment generating functions of the standardized sample sum of the independent random variables in a triangular array. Theorem 2.1 then follows from Markov inequality.

In this paper we obtain large deviation probabilities under weaker assumptions alongwith moment type convergences and L_p version of Berry-Esseen theorem. *The tail probabilities stated in Theorem 2.1, under weaker* *assumptions go beyond the known results even when specialized to iid bounded random variables, see Remark 2.1.*

The paper is arranged as follows. Section 2 provides results on standardized sum of independent random variables in a triangular array. Examples of random variables, including extreme value distribution satisfying the conditions of Section 2 are given in Section 3. The results are extended to general nonlinear statistics in Section 4. Convergence rates for linear process are considered in Section 5.

2 Results for Standardized Sum of Independent Random Variables in a Triangular Array

We start with the following theorem, stating an upper bound of the tail probability $1 - F_n(t) = P(s_n^{-1} S_n > t)$, for large t.

THEOREM 2.1. *Let* $[X_{ni} : 1 \leq i \leq n,\ n \geq 1]$ *be a triangular array of random variables satisfying* (1.1), (1.2) *and* (1.4). *Then there exist a positive constant* $\lambda = \lambda(L, \gamma)$ *such that, for all* $t > \lambda n^{1/2}$

$$1 - F_n(t) \leq \exp[-\alpha^* n (s_n t/n)^{1/\gamma}],\ 0 < \gamma < 1 \tag{2.1}$$

where $\alpha^* = \gamma[(1-\gamma)e/\alpha]^{(1-\gamma)/\gamma}$; $\alpha = \alpha(L, \gamma) > 0$ *is a constant such that for each* $L > 0$, $\alpha(L, \gamma)$ *remains bounded for* γ *in a neighbourhood of zero.*

PROOF. Write,

$$P(s_n^{-1} S_n > t) \leq \prod_{i=1}^{n} \beta_i \exp(-h s_n t),\ \beta_i = E \exp(h X_{ni}),\quad h > 0;\ i = 1, \cdots, n. \tag{2.2}$$

Then

$$\begin{aligned} \left(\prod_{i=1}^{n} \beta_i \right)^{1/n} &\leq n^{-1} \sum_{i=1}^{n} \beta_i \leq \sum_{m=0}^{\infty} \frac{h^m}{m!} \left(n^{-1} \sum_{i=1}^{n} E|X_{ni}|^m \right) \\ &\leq \sum_{m=0}^{\infty} \frac{h^m}{m!} e^{\gamma m \log m} L^m, \text{ under(1.4)}. \\ &\leq \sum_{m=0}^{\infty} (L^* h)^m e^{-(1-\gamma) m \log m} = \sum_{m=0}^{\infty} a(m), \end{aligned} \tag{2.3}$$

say; by Stirling's approximation, where $L^* > L/e$. Below we write L in place L^*, without any confusion. For $d \geq \alpha/e$, we shall show that the above series is dominated by

$$\exp(dh^{1/(1-\gamma)}) \geq \sum_{r=0}^{\infty} \left[\alpha h^{1/(1-\gamma)}\right]^r e^{-r\log r} = \sum_{r=0}^{\infty} b(r), \tag{2.4}$$

say; for a sufficiently large choice of α, provided $h \not\longrightarrow 0$.
To this end, note that

$$b(r) = \left[\alpha r^{-1} h^{1/(1-\gamma)}\right]^r \geq (Lhm^{\gamma-1})^m = a(m),\ 0 < \gamma < 1; \tag{2.5}$$

if, the following three conditions hold.

$$\begin{aligned} \alpha r^{-1} h^{1/(1-\gamma)} &\geq 1, & (2.6)\\ r &\geq (1-\gamma)m, & (2.7)\\ (\alpha r^{-1})^{(1-\gamma)} &\geq L. & (2.8) \end{aligned}$$

(2.6)–(2.8) are satisfied by taking α to be large enough, since $h \not\longrightarrow 0$ and taking

$$r = m^*, \text{ the smallest integer } \geq (1-\gamma)m + \gamma. \tag{2.9}$$

This particular choice of r will be made more clear later.

So, given a (large) integer m, there exists an integer m^* such that the mth term $a(m)$ of the series in (2.3) is dominated by the m^*th term $b(m^*)$ of the series in (2.4). Observe that $m - m^* \simeq \gamma m$ for large m.

Again, by a large choice of α, the $(m+1)$th term of the series in (2.3) is dominated by (m^*+1)th term of the series in (2.4), and so on. This is so, because

$$\frac{a(m+1)}{a(m)} \leq \frac{b(m^*+1)}{b(m^*)} \tag{2.10}$$

if, to a first degree of approximation

$$h > [\alpha^{-1}(1-\gamma)L(em)^{\gamma}]^{(1-\gamma)/\gamma}. \tag{2.11}$$

Since $h \not\longrightarrow 0$, (2.11) can be ensured by selecting α large. Hence,

$$\sum_{i=m}^{\infty} a(i) \leq \sum_{i=m^*}^{\infty} b(i) \tag{2.12}$$

Again,

$$\sum_{i=0}^{m-1} a(i) \leq (m-1)\left[1 \vee (Lh)^{m-1}\right] \leq \left[\alpha h^{1/(1-\gamma)}/m^*\right]^{m^*-1}, \quad [\leq b(m^*-1)], \tag{2.13}$$

if,

$$\left[(m-1)^{1/(m-1)}\{1 \vee (Lh)\}\right]^{m-1} \leq \left[\alpha^{(1-\gamma)} h (m^*)^{\gamma-1}\right]^{(m^*-1)/(1-\gamma)}. \tag{2.14}$$

Observe that $m^{1/m} \downarrow 1$ as $m \uparrow \infty$. Therefore, (2.14) holds if

$$(m^*)^{(\gamma-1)}\alpha^{(1-\gamma)} h > \{1 \vee (Lh)\}(m-1)^{1/(m-1)}, \text{ i.e., if}$$

$$\alpha > \left[\{L \vee h^{\gamma-1}\}(m-1)^{1/(m-1)}\right]^{1/(1-\gamma)} m^*. \tag{2.15}$$

And if, $(m^*-1)/(1-\gamma) \geq (m-1)$. i.e., if,

$$m^* \geq (1-\gamma)m + \gamma. \tag{2.16}$$

Select α large, so that (2.15) holds and note that (2.16) is fulfilled by the choice of m^* as in (2.9). Then, from (2.12) and (2.13), we get

$$\sum_{i=0}^{\infty} a(i) \leq \sum_{i=0}^{\infty} b(i). \tag{2.17}$$

Hence from (2.3), (2.4) and (2.17), one gets

$$\left(\prod_{i=1}^{n} \beta_i\right)^{1/n} \leq \exp\left[d h^{1/(1-\gamma)}\right]. \tag{2.18}$$

Therefore, from (2.2)

$$1 - F_n(t) \leq \exp\left(d n h^{1/(1-\gamma)} - h s_n t\right). \tag{2.19}$$

The minimum of the r.h.s. of (2.19) with respect to h is attained when

$$h = h_0 = [s_n t(1-\gamma)/(dn)]^{(1-\gamma)/\gamma}. \tag{2.20}$$

This value of h is required to satisfy (2.11). In view of (1.2), this states

$$t > Le^{-1}(em)^{\gamma} n/s_n \geq [LC^{-1}e^{-1}(em)^{\gamma}]n^{1/2}. \tag{2.21}$$

The minimum value of r.h.s. of (2.19) gives

$$1 - F_n(t) \leq \exp\left[-\alpha^* n(s_n t/n)^{1/\gamma}\right], \tag{2.22}$$

where α^* is defined in the theorem. Observe that the conditions (2.6) - (2.8), (2.15), (2.21) are all well behaved for $\gamma \to 0$. The resulting α satisfying these conditions also remains bounded as $\gamma \to 0$, for every fixed $L > 0$. This completes the proof. □

The upper bound of the tail probabilities $1 - F_n(t)$ computed above decreases at a faster rate than the normal tail probability for $\gamma \to 0$; as seen from the following remark. The same cannot be said for $|F_n(t) - \Phi(t)|$, as $\Phi(-t) \sim (2\pi)^{-1/2} t^{-1} e^{-t^2/2}$, $t \to \infty$; and for $\gamma < 1/2$, $\Phi(-t)$ becomes dominant in the difference $|F_n(t) - \Phi(t)| = |1 - F_n(t) - \Phi(-t)|$.

Remark 2.1. From symmetry, a similar bound holds for the tail probability, $F_n(-t), t > 0$. Theorem 2.1 essentially applies to large values of t and for such values the bound is sharper than existing bounds. It is known that an upper bound of the type $n^{-1/2}\, e^{-t^2/2}$ holds for $|F_n(t) - \Phi(t)|$; t lying in a neighbourhood of the origin. In view of this, till now the aim has been to approximate the tail probability $1 - F_n(t)$ by an upper bound of the type $e^{-t^2/2}$, even for large t. See e.g. Pollard (1984), Appendix B. These bounds are not sharp enough, as $1 - F_n(t)$ *may even be zero for large values of* t. For large t, $t > (ln^{1/2})^{1/(1-\delta)}$, $l > 0$, consider the bound (2.1) for small γ:

$$1 - F_n(t) \leq \exp[-\alpha^* n(s_n t/n)^{1/\gamma}] \leq \exp[-\alpha^* n(Cl)^{1/\gamma} t^{\delta/\gamma}],$$

$$t > (ln^{1/2})^{1/(1-\delta)} >> n^{1/2}$$

for $0 < \delta < 1$, as $n^{-1/2} s_n \geq C$. Now,

$$\alpha^*(Cl)^{1/\gamma} = [\gamma^\gamma(1-\gamma)Cle/\alpha]^{(1-\gamma)/\gamma} \to \infty, \text{ if } l > \alpha/(Ce), \text{ as } \gamma \to 0.$$

Hence,

$$1 - F_n(t) \leq e^{-b^*(\gamma) n t^{\delta/\gamma}}, \quad t > [n^{1/2}\alpha/(Ce)]^{1/(1-\delta)} (>> n^{1/2}), \quad 0 < \delta < 1, \tag{2.23}$$

where $b^*(\gamma) \to \infty$ as $\gamma \to 0$. The bound in (2.23) decreases at a faster rate than the normal tail probability, if $\gamma < \delta/2$. Then r.h.s of (2.23) is faster than $\exp(-|t|^\delta)$, $\delta > 2$, thus it is sharper than available results of polynomial decay of t corresponding to case I, or exponential decay of t corresponding to case II; see e.g. Dasgupta (1988), Dasgupta (1989) and

Ghosh and Dasgupta (1978). *Hence we obtain a sharper bound of the type* $\exp(-|t|^{\delta})$, $\delta > 2$ *under weaker assumptions (1.4) or (1.5), which do not require boundedness of the random variables.* Such exponential error bounds are of interest with application to compute the V-C dimension of a class of functions, e.g. see Chapter 2, Sections 2 and 4 of Pollard (1984).

Next, we show that the moment assumption (1.4) can be related to the finite expectation of some exponential type function of the random variables X_{ni}.

PROPOSITION 2.1. *Condition (1.4) implies condition (1.5).*

PROOF. From (1.4), one can write

$$n^{-1}\sum_{i=1}^{n} P(|X_{ni}| > t) \le t^{-m} L^m e^{\gamma m \log m}, \quad m > 1. \tag{2.24}$$

We shall minimize the r.h.s. of (2.24) with respect to m to find an optimal bound. Differentiating the logarithm of the right hand side of (2.24) with respect to m and equating it to zero, we obtain the optimal value of m as $m = e^{-1}(t\, L^{-1})^{1/\gamma}$. The corresponding optimal bound for (2.24) is

$$n^{-1}\sum_{i=1}^{n} P(|X_{ni}| > t) \le \exp(-\gamma e^{-1} L^{-1/\gamma} t^{1/\gamma}). \tag{2.25}$$

It may be mentioned that selecting an optimal value of m was also considered in Dasgupta (1979, page 177), Dasgupta (1988), Dasgupta (1989).

Now, observe that for a random variable Y,

$$Eg(Y) = \int_0^{\infty} g'(t) P(|Y| > t) dt, \tag{2.26}$$

where $g \ge 0$ is an even function; $g(0) = 0$. Therefore, (2.25) implies that,

$$n^{-1}\sum_{i=1}^{n} Eg(X_{ni}) < K, \quad g(x) = \exp(s|x|^{1/\gamma}) - 1, \quad 0 < s < \gamma e^{-1} L^{-1/\gamma}, \quad K > 0, \tag{2.27}$$

that is,

$$\sup_{n\ge 1}\, n^{-1}\sum_{i=1}^{n} E\exp(s|X_{ni}|^{1/\gamma}) < \infty, \tag{2.28}$$

for some $s > 0$. Hence the proposition. □

REMARK 2.2. The reverse implication of proposition 2.1 is shown to hold in Dasgupta (1988, page 449); (the first inequality of page 450 therein should be read in the reverse direction), Thus, conditions (1.4) and (1.5) are equivalent. Observe that when $\gamma = 1$, the moment generating functions of the random variables exist, whereas γ can be taken arbitrarily near to zero, when the variables are bounded.

As a general phenomena note that convergence rate of $|F_n(t) - \Phi(t)| = |P(s_n^{-1} S_n \in (t, \infty)) - P(T \in (t, \infty))|$ to zero is faster for larger t. For example, the error in approximating the probability of the event of hitting a ball in a Hilbert space H by CLT is seen to decrease not only if the number of summands in S_n increases but also if the distance between a bound of the ball and zero in the space H increases, see e.g. Bogatyrev (2002). Next, let

$c^{**} = \min_{0<r<\infty} \sup_{n \geq 1} n^{-1} \sum_{i=1}^{n} [(2r/3) E|X_{ni}|^3 \exp(2r|X_{ni}|) - 1] r$. A bound of the type $|F_n(t) - \Phi(t)| \leq bn^{-1/2} \exp(-kt^2)$ holds for t lying in a neighbourhood of the origin; see, e.g. Theorem 2.5 and (2.23), (2.24) of Dasgupta (1992). The following theorem provides a similar bound for *all* t.

THEOREM 2.2. *Let* $[X_{ni} : 1 \leq i \leq n,\ n \geq 1]$ *be a triangular array of independent random variables where variables in each array are independent and satisfy* (1.1), (1.2) *and* (1.4). *There exist a constant* $b(> 0)$ *and* $k \in (0, 1/2)$ *depending on* L, γ *and* c^{**} *such that for all real* t, *the following holds.*

$$|F_n(t) - \Phi(t)| \leq bn^{-1/2} \exp(-k|t|^{2 \wedge 1/\gamma}).$$

PROOF. The idea of the proof is as follows : It is possible to obtain an upper bound for $|F_n(t) - \Phi(t)|$ of the type $n^{-1/2} e^{-t^2/2}$ for t lying in a large neighbourhood of origin. See e.g. Ghosh and Dasgupta (1978), Dasgupta (1989). On the other hand, for t sufficiently large, one may use Theorem 2.1. Now, $\Phi(-t) \leq bt^{-1} e^{-t^2/2}$, $t > 0$. Write, $|F_n(t) - \Phi(t)| = |1 - F_n(t) - \Phi(-t)|$ to obtain a bound.

Without loss of generality take $t > 0$. The case $t < 0$ is similar. Observe that the moment generating functions of the random variables X_{ni} exist, as $0 < \gamma < 1$ and therefore the computations (2.20)–(2.21) of Dasgupta (1992) hold in the range $t \leq f(p) n^{1/2}$ for some $f(p) > 0$.

Following the steps (2.25)–(2.27) with $g(x) = x^2 \exp(u|x|^{1/\gamma})$, one gets

$$\sup_{n\geq 1} n^{-1} \sum_{i=1}^{n} EX_{ni}^2 \exp(u|X_{ni}|^{1/\gamma}) < \infty,\ 0 < u < \gamma e^{-1} L^{1/\gamma}. \tag{2.29}$$

Therefore, calculation of $\sum_{i=1}^{n} P(|X_{ni}| > rs_n t)$, in Dasgupta (1992, p.204), can be rewritten in the present case as follows.

$$\begin{aligned} \sum_{i=1}^{n} P(|X_{ni}| > rs_n|t|) &\leq bt^{-2} \exp(-u|rs_n t|^{1/\gamma}) \\ &\leq bn^{-1/2} \exp(-u_1|t|^{1/\gamma}), \end{aligned} \tag{2.30}$$

where $b > 0$ denotes a generic constant and $u_1 > 0$ may be taken arbitrary large, as $s_n \geq Cn^{1/2}$ under (1.2). Theorem 2.2 holds in the region $t < f(p)n^{1/2}$, in view of (2.30) above and the calculations (2.20)–(2.21) of Dasgupta (1992).

Next, for $t > \lambda n^{1/2}$, where λ is large enough so as to apply Theorem 2.1, one can write from the said theorem,

$$\begin{aligned} |F_n(t) - \Phi(t)| &\leq |1 - F_n(t)| + \Phi(-t) \\ &\leq \exp[-kt^2] + \Phi(-t), \text{ if } 0 < \gamma < 1/2, \\ &\leq \exp[-kt^{1/\gamma}] + \Phi(-t), \text{ if } 1/2 < \gamma < 1. \end{aligned} \tag{2.31}$$

Also, for $t > 0$,

$$\Phi(-t) \leq bt^{-1} e^{-t^2/2} \leq bn^{-1/2}\, e^{-t^2/2}, \text{ for } t > \lambda n^{1/2}. \tag{2.32}$$

From (2.31) and (2.32), it follows that Theorem 2.2 holds for $t > \lambda n^{1/2}$.

Finally, for the region $f(p)n^{1/2} < t \leq \lambda n^{1/2}$, one may adopt the same procedure used to get (2.23) of Dasgupta (1992), (see also (2.4.77) of Dasgupta, 1979) to obtain

$$|F_n(t) - \Phi(t)| \leq be^{s_n c^{**} t} \leq bn^{-1/2} e^{-kt^2}, \text{ as } t = O_e(n^{1/2}) \tag{2.33}$$

and $c^{**} < 0$. This completes the proof. □

One of the pleasant features of the nonuniform bounds is that, it produces moment type convergences, tail probabilities of standardized sample sum and L_p version of Berry-Esseen theorem as by products. Although very helpful, the uniform rates of convergences of $F_n(t)$ to $\Phi(t)$ or Edgeworth expansion of F_n (see e.g. Bhattacharya and Rao, 1986) fail to provide such results. The following results are immediate from Theorem 2.2; see also Theorem 2.5 and Corollary 2.1 of Dasgupta (1992).

THEOREM 2.3. *Let the assumptions of Theorem* 2.2 *be satisfied. Let* $g : (-\infty, \infty) \to [0, \infty)$ *be an even function,* $g(0) = 0$ *and* $Eg(T) < \infty$, *where* T *is a normal deviate. Suppose,* $g'(x) = O[\exp(k|x|^{2\wedge 1/\gamma})(1 + |x|)^{-q}]$, $q > 1$. *Then,*

$$|Eg(s_n^{-1}S_n) - E\,g(T)| = O(n^{-1/2}).$$

COROLLARY 2.1. *Under the assumption of Theorem* 2.2,

$$\|\exp(k|t|^{2\wedge 1/\gamma})(1 + |x|)^{-q}(F_n(t) - \Phi(t))\|_q = O(n^{-1/2}), \quad \text{for any } q > 1.$$

3 Some Examples

Next we provide a few examples of random variables satisfying the assumptions of Theorem 2.2. Observe that (1.4) is equivalent to (2.25) and (2.28). The condition (2.25) essentially states the tail behaviour of the distribution of the random variables, whereas (2.28) ensures finite expectation of some exponential type functions of the variables. We will check the condition (2.28) for some $s > 0$.

EXAMPLE 1. Let $P(X = i) = A^{-1}e^{-\beta|i|^\alpha}$, $i \in Z = \{0, \pm 1, \pm 2, \pm 3 \cdots\}$ where $A = \sum_{i=-\infty}^{\infty} e^{-\beta|i|^\alpha} < \infty$; $\alpha > 1$, $\beta > 0$. Let X_{ni} be *iid* random variables distributed as X. Condition (2.28) is satisfied for $\gamma > 1/\alpha$.

EXAMPLE 2. Extreme value distribution of second and third types.

Consider a random variable X with distribution function:

$$G_{2,\alpha}(x) = \begin{cases} 1 & \text{for } x \geq 0, \\ \exp(-(-x)^\alpha), & \text{for } x < 0, \end{cases}$$

and let X_{ni} be *iid* copies of X. Then (2.28) is satisfied for $\gamma > 1/\alpha$.

For $G_{3,\alpha}(x) = \exp(-e^{-x})$, $-\infty < x < \infty$; a similar conclusion holds; (2.28) is true for any γ, $0 < \gamma < 1$. *This provides an example where* γ *may be taken arbitrary near to zero, although the variables are not bounded.*

Mean of the above distributions are nonzero. So one should really check (2.28) with $|X_{ni}|$ replaced by $|X_{ni} - \mu|$. However, this does not create any problem since $|X_{ni} - \mu|^{1/\gamma} \leq 2^{(1-\gamma)/\gamma}\left(|X_{ni}|^{1/\gamma} + |\mu|^{1/\gamma}\right)$, $\gamma \in (0, 1)$.

The above distributions explain the limiting behaviour of sample extremes. Average of several such extremes has a much faster rate of convergence to normality according to the results of Section 2.

Example 3. Let X be a random variable with probability density $f(x) = A\exp(-\beta|x|^{\alpha})$, $-\infty < x < \infty$, where $A^{-1} = 2\int_0^{\infty}\exp(-\beta x^{\alpha})dx$, $\beta > 0$, $\alpha > 1$. Let X_{ni} be iid copies of X. Then (2.28) is satisfied for $\gamma > 1/\alpha$.

Example 4. Let X_{ni} be a symmetric random variable taking values $\pm\alpha_{ni}$, each with probability 1/2. Then, one may select any sequence of positive reals $\{\alpha_{ni}\}$, such that $\sup_{n\geq 1} n^{-1}\sum_{i=1}^{n}\exp(s(\alpha_{ni})^{1/\gamma}) < \infty$, e.g., take

$$\alpha_{ni} = \begin{cases} \alpha(\log i)^{\gamma}, & \text{if } 1 \leq i < k_n, \\ \alpha, & \text{if } k_n \leq i \leq n, \end{cases} \tag{3.1}$$

where $\alpha > 1$, $k_n = [n^{\epsilon}]$, the integer part of n^{ϵ}, $0 < \epsilon < 1$. Then,

$$\begin{aligned} \sum_{1\leq i<k_n}\exp(s(\alpha_{ni})^{1/\gamma}) &= \sum_{1\leq i<k_n} i^{\beta}, \text{ where } \beta = s\alpha^{1/\gamma} \\ &\leq \int_0^{k_n} x^{\beta}dx = (\beta+1)^{-1}k_n^{\beta+1} \leq (\beta+1)^{-1}n^{\epsilon(\beta+1)}. \end{aligned} \tag{3.2}$$

Therefore,

$$\begin{aligned} &\sup_{n\geq 1} n^{-1}\sum_{i=1}^{n}\exp\left(s(\alpha_{ni})^{1/\gamma}\right) \\ &\leq \sup_{n\geq 1} n^{-1}\{(\beta+1)^{-1}n^{\epsilon(\beta+1)} + (n-[n^{\epsilon}]+1)\alpha\} < \infty, \\ &\qquad \text{provided,} \quad \epsilon \leq (\beta+1)^{-1} = (s\alpha^{1/\gamma}+1)^{-1}. \end{aligned} \tag{3.3}$$

The calculated bounds of $|F_n(t) - \Phi(t)|$ and $1 - F_n(t)$ decrease fast with small choice of γ and that requires ϵ to be small for small γ.

Example 5. Linear combination of variables satisfying the assumptions of Theorem 2.1. Let $[(X_{ni}, Y_{ni}) : 1 \leq i \leq n,\ n \geq 1]$ be two triangular arrays of independent random variables satisfying condition (1.4) with $L = L_1$ and $L = L_2$, for X and Y arrays respectively; $\gamma \in (0,1)$, being same for both arrays. Also let (1.2) and (1.3) hold for X, Y variables. Then, for the random variables $Z_{ni} = \alpha_1 X_{ni} + \alpha_2 Y_{ni}$, where α_1 and α_2 are any fixed real numbers

$$\begin{aligned} \tfrac{1}{n}\sum_{i=1}^{n} E|Z_{ni}|^m &\leq 2^{m-1}[|\alpha_1|^m \tfrac{1}{n}\sum_{i=1}^{n} E|X_{ni}|^m + |\alpha_2|^m \frac{1}{n}\sum_{i=1}^{n} E|Y_{ni}|^m], \\ &\leq 2^m[|\alpha_1 L_1|^m + |\alpha_2 L_2|^m] e^{\gamma m \log m}, \quad \text{from (1.4)} \\ &\leq L^m e^{\gamma m \log m}, \quad \text{for some } L > 0. \end{aligned} \tag{3.4}$$

So, the assumption (1.4) is fulfilled for $Z_{ni} = \alpha_1 X_{ni} + \alpha_2 Y_{ni}$. Further, $EZ_{ni} = 0$, as $EX_{ni} = EY_{ni} = 0$. Also,

$$\text{Var}\left(\sum_{i=1}^{n} Z_{ni}\right) = \sum_{i=1}^{n} EZ_{ni}^2 = \alpha_1^2 \sum_{i=1}^{n} EX_{ni}^2 + \alpha_2^2 \sum_{i=1}^{n} EY_{ni}^2 > C_1 n,$$

for some $C_1 > 0$, as (1.2) holds for the variables X and Y, when X array is independent of Y array. Therefore (1.2) holds for Z_{ni} and the theorems remain valid. The independence of X_{ni} and Y_{ni} are used only to check the assumption (1.2). By directly checking the condition (1.2), one may relax the assumption of independence of (X_{ni}, Y_{ni}).

4 Rates of Convergence for Nonlinear Statistics

Consider a nonlinear statistics T_n of the form:

$$T_n = s_n^{-1}\, S_n + R_n, \tag{4.1}$$

where, $S_n = \Sigma_{i=1}^n X_{ni}, s_n^2 = \Sigma_{i=1}^n EX_{ni}^2$, $\inf_{n\geq 1}\ n^{-1}\ s_n^2 > 0$. Here, $X_{n1}, X_{n2}, \ldots, X_{nn}$ are independent random variables with zero expectation and R_n is a negligible remainder. A representation of this type is fairly general and is obtainable, e.g., via Hájek's projection lemma. Nonuniform central limit bound for T_n are obtained under different moment assumptions on the remainder in Ghosh and Dasgupta (1978), Dasgupta (1989) and Dasgupta (1992), with applications to probabilities of deviations, moment convergences and allied results. Here we deal with the situation when the variables X_{ni} satisfy (1.4). Assume that, for some $\beta \geq 0$,

$$E|R_n|^m\ \leq c(m) n^{-m/2} (\log\ n)^{\beta m}, m > 1, \tag{4.2}$$

where, $c(m) \leq L_1^m e^{(\gamma+\delta)\ m\ \log\ m}$, for some $\delta \geq 0$ and $L_1 > 0$.

In Section 5, we shall show that these conditions are fulfilled in particular case of linear process. The bound (4.2) implies that, for $(\gamma+\delta)>0$

$$P(|R_n|>a_n(t)) \le \exp\left[-(\gamma+\delta)e^{-1}\{n^{1/2}(\log\ n)^{-\beta}L_1^{-1}a_n(t)\}^{1/(\gamma+\delta)}\right], \tag{4.3}$$

see (2.24) and (2.25). Take $a_n(t)=\epsilon n^{-1/2}(\log\ n)^{\beta+\gamma+\delta}|t|$, $\epsilon>0$. Then (4.3) states, for some $\epsilon^*>0$,

$$P(|R_n|>a_n(t)) \le e^{-\epsilon^*|t|^{1/(\gamma+\delta)}\log\ n} \le bn^{-1/2}\exp\left[-k_1|t|^{1/(\gamma+\delta)}\right], \tag{4.4}$$

where k_1 may be taken large enough for $|t|>t_o$, say.

Due to representation (4.1), one may write

$$\begin{aligned}|P(T_n\le t)-\Phi(t)| \;&\le\; |P(s_n^{-1}S_n\le t\pm a_n(t))-\Phi(t\pm a_n(t))| \\ &\quad +|\Phi(t\pm a_n(t))-\Phi(t)|+P(|R_n|>a_n(t)).\end{aligned} \tag{4.5}$$

The first term in the r.h.s. of (4.5) may be approximated from Theorem 2.2, the second term is less than $ba_n(t)e^{-t^2/2}\le bn^{-1/2}(\log n)^{\beta+\gamma+\delta}|t|e^{-t^2/2}$ and the third term is estimated in (4.4). Combining these, one may obtain a bound like (4.6) below, for $|t|>t_o$ (see also (4.5)–(4.7) of Dasgupta (1992), for similar calculations). Also, observe that an uniform bound $O\left(n^{-1/2}(\log\ n)^{\beta+\gamma+\delta}\right)$ is available for $||P(T_n\le t)-\Phi(t)||$, letting $a_n(t)=n^{-1/2}(\log\ n)^{\beta+\gamma+\delta}$ and using the relation

$$||F(X+Y)-\Phi|| \;\le\; ||F(X)-\Phi||+(2\pi)^{-1/2}a_n+P(|Y|>a_n).$$

Thus (4.6) holds for $|t|\le t_o$. Therefore, one may obtain the following theorem for T_n, providing a nonuniform bound for *all t*.

THEOREM 4.1. *Under the assumptions of Theorem* 2.2 *and* (4.2), *there exist constants* $b(>0)$, *and* $k\in(0,1/2)$ *such that the following holds for the nonlinear statistics* T_n *defined in* (3.1),

$$|P(T_n\le t)-\Phi(t)| \;\le b\, n^{-1/2}(\log\ n)^{\beta+\gamma+\delta}\exp\left(-k|t|^{2\wedge 1/(\gamma+\delta)}\right). \tag{4.6}$$

In view of (4.6), results similar to Theorem 2.3 and Corollary 2.1 hold for T_n where γ is replaced by $(\gamma+\delta)$.

5 Rates of Convergence for Linear Process

Consider $X_n = \Sigma_{i=1}^{\infty} a_i \xi_{n-i+1}$ or, $X_n = \Sigma_{i=1}^{\infty} a_i \xi_{n+i-1}$ where a_i is a sequence of constants with $\Sigma_{i=1}^{\infty} a_i^2 < \infty$ and $\xi_i s$ are pure white noise. Without loss of generality, let $E\,\xi = 0$ and $E\,\xi^2 = 1$. Write,

$$S_n = \Sigma_{i=1}^{n} X_i = \Sigma_{i=1}^{n} X_{ii} + \Sigma_{i=1}^{n} (X_i - X_{ii}); X_{m,n} = \Sigma_{i=1}^{m} a_i\, \xi_{n-i+1}. \quad (5.1)$$

In the above expression of S_n, the first part is the leading term and the second part may be treated as remainder. Assume that, for some γ, $0 < \gamma < \infty$;

$$E|\xi_1|^m \leq L^m e^{\gamma m \ \log \ m}, \forall\, m \geq \ 1. \quad (5.2)$$

By Minkowski's inequality we get,

$$E|\Sigma_{i=1}^{n} (X_i - X_{ii})|^m \ \leq \ (\Sigma_{i=1}^{\infty}\ i|a_i|)^m\, E|\xi_1|^m \leq \ L_1^m\ e^{\gamma m \ \log \ m}, \quad (5.3)$$

where $L_1 = L\ \Sigma_{i=1}^{\infty} i|a_i|$. Then, following the steps of Dasgupta (1992), Section 4; see also Babu and Singh (1978), one may write

$$Y_n := [V(S_n)]^{-1/2}\ S_n = [V(S_n)]^{-1/2}\ \ \Sigma_{i=1}^{n}\ X_{ii} \ + \ R_n, \quad (5.4)$$

where $R_n = [V(S_n)]^{-1/2}\ \ \Sigma_{i=1}^{n}\ (X_i - X_{ii})$ satisfies (4.2) with $\beta \ = \ 0,\ \delta \ = 0$.

Thus, Theorem 4.1 holds for the linear process X_n. We restate below the theorem in this special case. See also (4.6) of Dasgupta (1992).

THEOREM 5.1. *Let* $\Sigma_{i=1}^{\infty}\ i|a_i| < \infty$ *and* $\Sigma_{i=1}^{\infty}\ a_i \ \neq \ 0$ *for a linear process* X_n. Let $E\ \xi = \ 0$, $E\xi^2 = \ 1$ *and* (5.2) *holds. Then there exist constants* $b(> 0)$, *and* $k \in (0, 1/2)$ *such that for the standardized sum* Y_n *defined in* (5.4) *of the linear process* X_n, *one has*

$$|P(Y_n \ \leq \ t) \ - \ \Phi(t)| \ \leq \ b\, n^{-1/2} (\log \ n)^{\gamma} \ \exp\left(-k|t|^{2 \wedge 1/\gamma}\right).$$

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