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REFLECTING BROWNIAN MOTION IN A LIPSCHITZ DOMAIN AND A CONDITIONAL GAUGE THEOREM

By S. RAMASUBRAMANIAN
Indian Statistical Institute, Bangalore

SUMMARY. Let $\{P_x : x \in \overline{D}\}$ denote the reflecting Brownian motion in D with normal reflection at the boundary where D is a bounded Lipschitz domain in $\mathbb{R}^d$. Let $q(x)dx, c(x)d\sigma(x)$ belong to Kato class; consider the third boundary value problem for the operator $\left(\frac{1}{2}\Delta + q\right)$ in D with boundary condition determined by $\left(\frac{\partial}{\partial n} + c\right)$;(here $d\sigma$ denotes the surface area measure on ∂D, and $n(\cdot)$ the inward normal). Let $\{T_t\}$ denote the corresponding Feynman-Kac semigroup and G the gauge function. After indicating a way of getting the integral kernel ζ for $\{T_t\}$, we set $F(x, z) = \int_0^\infty \zeta(t, x, z)dt, \quad x, z \in \overline{D}$. It is proved that if $F(x, z) < \infty$ for some $x, z \in \overline{D}$ then the gauge G is a bounded continuous function on $\overline{D}$, and that $F(\cdot, \cdot)$ is finite and continuous on $\{x \neq z\}$. A connection between F and conditioned Brownian motion is given; a consequence is that if the gauge for the third boundary value problem is finite then so is the gauge for the Dirichlet problem.

1. **Introduction**

Probabilistic treatment of Feynman-Kac semigroups in smooth domains has been considered by many authors. In the context of the Dirichlet problem, following the lead given by Chung and Rao (1981), the literature is now quite extensive; see Ma and Song (1990), Falkner (1983), Zhao (1986) and the references given therein. Following the probabilistic approach to the Neumann problem for the Schrodinger operator initiated by Hsu, third boundary value problem has been studied by Papanicolaou (1990) using reflecting Brownian motion. Gauge theorems and connections with certain spectral properties of the operators have been established in these works. For the Dirichlet problem, Falkner (1983) and Zhao (1986) discuss a conditional gauge theorem making use of the conditioned Brownian motion of Doob; motivated by these, Cranston, Fabes and Zhao (1988) have studied the conditional gauge and potential theory for Schrodinger operator with Dirichlet boundary condition in a bounded Lipschitz domain.

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The aim of this article is to study the Feynman-Kac semigroup $\{T_t\}$ associated with the operator $\frac{1}{2}\Delta + q$ in a bounded Lipschitz domain D, with the boundary operator $\frac{\partial}{\partial n} + c$, using the reflecting Brownian motion in D with normal reflection at the boundary; here the potentials $q(\cdot)$ in the interior and $c(\cdot)$ on the boundary can belong to generalized Kato class $GK_d(\overline{D})$; (see Section 3 for definitions).

Regarding reflecting Brownian motion (with normal reflection at the boundary) in nonsmooth domains, though the seminal paper of Fukushima (1967) has given a very general construction of transition density in an arbitrary open set way back in 1967, the next major development had to wait till 1990/91 when Bass and Hsu (1990), (1991) elaborated on this theme for Lipschitz and Hölder domains; in particular they showed that the Skorokhod equation holds and that the process can be realized on the Euclidean closure of the domain. It is in this set up one is able to extend the analysis of Papanicolaou (1990) to Lipschitz domains.

After briefly recalling the salient features of reflecting Brownian motion in a Lipschitz domain D in Section 2, we indicate in Section 3 how the integral kernel ζ for $\{T_t\}$ can be defined as a continuous function on $(0, \infty) \times \overline{D} \times \overline{D}$; a Gaussian upper bound for ζ is also established. The gauge function for the third boundary value problem is

$$G(x) = \frac{1}{2}\int_0^\infty \int_{\partial D} \zeta(t, x, z) d\sigma(z) dt.$$

In Section 4 after proving the gauge theorem, we define

$$F(x, z) = \int_0^\infty \zeta(t, x, z) dt.$$

It is shown that if $F(x, z) < \infty$ for some x, z then the gauge is a bounded continuous function on $\overline{D}$ and that F is finite and continuous on $\overline{D} \times \overline{D}$ off the diagonal; this result may be called a conditional gauge theorem for the third boundary value problem. A Harnack inequality is an immediate corollary.

A connection between F and the conditioned Brownian motion via Martin kernel is discussed in Section 5 using a representation for positive $\left(\frac{1}{2}\Delta + q\right)$-harmonic functions with Dirichlet boundary condition due to Zhao (1986). As a consequence it is shown that finiteness of the gauge for third boundary value problem implies that of the gauge for the Dirichlet problem.

A few comments may be in order to put things in proper perspective. When $q(\cdot), c(\cdot)$ are smooth, negative and bounded away from zero, and D is a bounded smooth domain, the corresponding third boundary value problem is quite classical and has been studied by analytic methods; in such a case one can also consider more general second order elliptic operators; see Friedman (1983), Ito (1992). The stochastic representation of the solution can be given in terms of reflected diffusions; the probabilistic treatment is facilitated by the exponential rate of convergence of the transition probability of the reflected diffusion to the unique invariant measure; see Brosamler (1976), Freidlin (1985), Ramasubramanian (1992). If $q(\cdot) \le -\beta < 0$, $c(\cdot) \le -\beta < 0$ for some positive constant β, it is easily seen that the Feynman-Kac semigroup decays exponentially fast.

Once $q(\cdot), c(\cdot)$ are not necessarily negative (as is the case in the present article) the functionals $e_q(\cdot), \widehat{e}_c(\cdot)$ (given by (3.4), (3.5)) are no longer bounded; even well-definedness of various quantities is not clear a priori. However, a satisfactory analysis becomes possible if the corresponding Feynman-Kac semigroup decays exponentially fast. This can be ensured if the gauge function is not infinite, when $q(x)dx, c(x)d\sigma(x)$ belong to the generalized Kato class; see Theorem 4.1. A further refinement, which is the main result of this paper, is that the same is ensured once the "Green function / Poisson kernel" $F(x,z)$ for the problem is not infinite; see (the conditional gauge theorem) Theorem 4.4 and Remark 4.2. Also the representation (5.1) for the "Poisson kernel" $F(x,z), x \in D, z \in \partial D$ in terms of the conditioned Brownian motion may be of independent interest.

2. **RBM in a Lipschitz Domain**

Let D be a bounded Lipschitz domain in $\mathbb{R}^d, d \geq 2$. That is, D is a bounded connected open set, and for each $x \in \partial D$ there exist $r > 0$, a Lipschitz continuous function Γ and an orthonormal coordinate system $\mathcal{O}$ (all possibly depending on x) such that $D \cap B(x:r) = \{y = (y_1, \cdots, y_{d-1}, y_d) \text{ in } \mathcal{O} : y_d > \Gamma(y_1, \cdots, y_{d-1})\} \cap B(x : r)$. A good source for information concerning such domains is Bass (1995).

Let $\{P_x : x \in \overline{D}\}$ denote the reflecting Brownian motion in $\overline{D}$ with normal reflection at the boundary; that is, $\{P_x\}$ is a family of probability measures supported on $C([0,\infty) : \overline{D}) \subset C([0,\infty) : \mathbb{R}^d)$ so that under $\{P_x\}$ the canonical process $\{X(t) : t \geq 0\}$ given by coordinate projections is a reflecting Brownian motion with $\overline{D}$ as state space and with normal reflection at the boundary. Let $p(t,x,z), t > 0, x, z \in \overline{D}$ denote its transition probability density function; note that $\frac{dP_x X(t)^{-1}}{dz}(z) = p(t,x,z)$. We make the following observations.

Remark 2.1. For any arbitrary bounded open set D without any smoothness/ regularity assumptions on the boundary, Fukushima (1967) has given a construction of $p(t,x,z)$ for $t > 0, x, z \in D$ by the L^2-method; and by standard interior regularity argument it follows that $p(t,\cdot,\cdot)$ is smooth on $D \times D$. A natural state space for the associated continuous strong Markov process is the so called Kuramochi compactification of D; (this process may be called the RBM in D); for $x \in D$, the measure P_x is the law of the process starting at x. Also p is symmetric in (x,z).

Remark 2.2. When D is a bounded Lipschitz domain Bass and Hsu (1991) have shown that

(i) the Kuramochi compactification of D coincides with the Euclidean compactification $\overline{D}$;

(ii) p can be extended in a continuous fashion to $(0,\infty) \times \overline{D} \times \overline{D}$;

(iii) for any $T > 0$ there exist positive constants C_1, C_2 such that for all $0 < t \leq T, x, z \in \overline{D}$

$$p(t,x,z) \leq C_1 t^{-d/2} \exp\left(-\frac{|x-z|^2}{C_2 t}\right); \tag{2.1}$$

(iv) there exist $T_0 > 0, C_0 > 0$ such that for all $t \geq T_0, x, z \in \overline{D}$

$$\left| p(t,x,z) - \frac{1}{|D|} \right| \leq e^{-C_0 t}. \tag{2.2}$$

By Theorem 2.3, Remark 3.11 of Bass and Hsu (1991), and continuity of p over $(0,\infty) \times \overline{D} \times \overline{D}$ note that (2.1) follows for $0 < t \leq 1, x, z \in \overline{D}$; then apply repeatedly Chapman-Kolmogorov equation for p and the Gaussian kernel to get (2.1) for $0 < t \leq T, x, z \in \overline{D}$, for any $T > 0$. From (iv) it follows that there exists $T_1 > 0$ such that for all $t \geq T_1$,

$$\inf\{p(t,x,z) : x, z \in \overline{D}\} > 0. \tag{2.3}$$

Perhaps (2.3) is true for any $t > 0$, but we do not have a proof. (In case D is a smooth domain (2.3) can be established for any $t > 0$ using (parabolic) maximum principles; see S.Ito (1992)). By (ii) and (iii) it is clear that $\{P_x : x \in \overline{D}\}$ is strong Feller.

Note that the uniform distribution on D is the unique invariant measure for the process $\{P_x : x \in \overline{D}\}$, and (2.2) indicates the exponential rate of convergence to it. One could say that this corresponds to a spectral gap between zero and the rest of the spectrum of the self adjoint generator of the process (with reflection) when the corresponding semigroup is viewed to be acting on $L^2(D, dx)$.

Remark 2.3. Again in the case of a bounded Lipschitz domain D, Bass and Hsu (1990) have proved that under P_x, the Skorokhod equation

$$X(t) = x + W(t) + \int_0^t n(X(s))d\xi(s) \tag{2.4}$$

holds a.s., where $\{W(t)\}$ is a standard d-dimensional Brownian motion, $\{\xi(t)\}$ is the boundary local time of $\{X(t)\}$, $n(\cdot)$ is the unit inward normal vector field on ∂D. Note that $\{\xi(t)\}$ is the continuous additive functional associated with the surface area measure $d\sigma(\cdot)$ on ∂D. For a Lipschitz domain, the inward normal vector field $n(\cdot)$ is defined only a.e.$d\sigma(\cdot)$ on ∂D. However, as $\xi(\cdot)$ does not charge $d\sigma$ null sets, the integral in (2.4) is well defined. In particular $\{X(t)\}$ is a continuous $\overline{D}$ -valued semimartingale.

For any bounded measurable function f on ∂D, proceeding as in the proof of Proposition 1.1 of Papanicolaou (1990) one can show that

$$E_x \left[\int_0^t f(X(s))d\xi(s) \right] = \frac{1}{2} \int_0^t \int_{\partial D} f(z)p(s,x,z)d\sigma(z)ds. \tag{2.5}$$

It may be noted that the semimartingale representation (2.4) above is the same as in Papanicolaou (1990) but it differs from that in Bass and Hsu (1990) by the factor $\frac{1}{2}$ in the integral. It is also clear that (2.5) holds for any nonnegative measurable function f on ∂D. Using (2.1) and a localization argument it can be shown that there are constants K_1, K_2 such that

$$\int_{\partial D} p(t,x,z)d\sigma(z) \leq K_1 t^{-1/2} + K_2 \tag{2.6}$$

for all $t > 0, x \in \overline{D}$. From (2.5) and (2.6) it is clear that

$$E_x(\xi(t)) \leq K_1 t^{1/2} + K_2 t. \tag{2.7}$$

Remark 2.4. Though the proofs given in Bass and Hsu (1991) cover the case $d \geq 3$, the results for $d = 2$ can be obtained as follows. Let D be a bounded Lipschitz domain in $\mathbb{R}^2$. Then $\widehat{D} = D \times (0,1)$ is a bounded Lipschitz domain in $\mathbb{R}^3$ and $\{\widehat{P}_{(x_1,x_2,x_3)} \equiv P_{(x_1,x_2)} \times Q_{x_3} : (x_1,x_2) \in \overline{D}, x_3 \in [0,1]\}$ is the RBM in $\widehat{D}$ with normal reflection at the boundary, where $\{Q_{x_3} : x_3 \in [0,1]\}$ is the RBM in $(0,1)$. Use the results for $\widehat{D}$ to read off the corresponding results for D. (The author thanks R. Bass for this remark.)

3. Feynman-Kac Semigroup $\{T_t\}$

In this section we consider the Feynman-Kac semigroup associated with the third boundary value problem and the corresponding integral kernel.

A signed Radon measure ν on $\overline{D}$ is said to belong to the class $GK_d(\overline{D})$ if

$$\lim_{t \downarrow 0} \sup_{x \in \overline{D}} \int_0^t \int_{\overline{D}} p(s,x,y)|\nu|(dy)ds = 0 \tag{3.1}$$

Let q and c respectively be measurable functions on D and ∂D such that the signed measures $q(x)dx$ and $c(x)d\sigma(x)$ belong to GK_d. By (2.5) it is clear that

$$\lim_{t \downarrow 0} \sup_{x \in \overline{D}} E_x \left[\int_0^t |c(X(s))| d\xi(s) \right] = 0. \tag{3.2}$$

Also

$$\lim_{t \downarrow 0} \sup_{x \in \overline{D}} E_x \left[\int_0^t |q(X(s))| ds \right] = 0. \tag{3.3}$$

Define the functionals

$$e_q(t) = \exp\left(\int_0^t q(X(s))ds \right), \tag{3.4}$$

$$\widehat{e}_c(t) = \exp\left(\int_0^t c(X(s))d\xi(s) \right), \tag{3.5}$$

and the semigroup of operators

$$T_t f(x) = E_x \left[e_q(t) \widehat{e}_c(t) f(X(t)) \right] \tag{3.6}$$

whenever the right side makes sense. By (3.2), (3.3) we can find $t_0 > 0$ such that

$$\sup\{E_x(A(t)) : x \in \overline{D}\} \leq \alpha < 1, \forall t \leq t_0$$

where $A(t) = \int_0^t |q(X(s))|ds + \int_0^t |c(X(s))|d\xi(s)$. Then Khasminskii's lemma (see Papanicolaou (1990) or Simon (1982)) states that

$$\sup\{E_x[e_q(t)\widehat{e}_c(t)] : x \in \overline{D}\} = \sup\{E_x[e^{A(t)}] : x \in \overline{D}\} \leq \frac{1}{(1-\alpha)}$$

for all $t \leq t_0$. By the Markov property it follows that $\{T_t : t \geq 0\}$ forms a semigroup of operators on $L^\infty(\overline{D})$; and by the strong Feller property of the process we get that $T_t f$ is continuous for any $t > 0, f \in L^\infty(\overline{D})$.

We now indicate how a continuous integral kernel for the semigroup $\{T_t\}$ can be obtained. Set $\zeta_0(t,x,z) = p(t,x,z), t > 0, x, z \in \overline{D}$; for $n = 1, 2, 3, \cdots$ define

$$\begin{aligned}\zeta_n(t,x,z) = &\int_0^t \int_D p(s,x,y)q(y)\zeta_{n-1}(t-s,y,z)dyds \\ &+ \frac{1}{2}\int_0^t \int_{\partial D} p(s,x,y)c(y)\zeta_{n-1}(t-s,y,z)d\sigma(y)ds \end{aligned} \tag{3.7}$$

for $t > 0, x, z \in \overline{D}$; define

$$\zeta(t,x,z) = \sum_{n=0}^{\infty} \zeta_n(t,x,z). \tag{3.8}$$

In view of the upper bound (2.1) and continuity of p, proceeding as in the proof of Theorems 3.2, 3.4 of Papanicolaou (1990) one can establish the following.

THEOREM 3.1. *Let D be a bounded Lipschitz domain; let q, c respectively be measurable functions on $D, \partial D$ such that $q(x)dx, c(x)d\sigma(x)$ belong to $GK_d(\overline{D})$. Then*

(i) ζ is a nonnegative continuous function on $(0,\infty) \times \overline{D} \times \overline{D}$; $\zeta(t,x,z)$ is symmetric in x, z; $\zeta(t,x,z)dz \Rightarrow \delta_x$ as $t \downarrow 0$;

(ii) for $f \in L^1(\overline{D}), t > 0, x \in \overline{D}$

$$(T_t f)(x) = \int_{\overline{D}} f(z)\zeta(t,x,z)dz; \tag{3.9}$$

in particular ζ satisfies the Chapman-Kolmogorov equation;

(iii) for $t > 0, T_t$ is a compact operator from $L^1(\overline{D})$ into $C(\overline{D})$;

(iv) there exist positive constants K, β depending only on D, q, c such that

$$\sup_{x\in\overline{D}} |(T_t f)(x)| \leq K t^{-d/2} e^{\beta t} \|f\|_1 \tag{3.10}$$

for any $f \in L^1(\overline{D})$.

It may be noted that in the proofs of Theorems 3.2 and 3.4 of Papanicolaou (1990), smoothness of the domain and of p are not used. For another (but essentially equivalent) way of obtaining the integral kernel see Ramasubramanian (1996).

For $t > 0$ set

$$M(t) = \sup_{x\in\overline{D}} E_x\left[\int_0^t |q(X(s))|ds\right] + \sup_{x\in\overline{D}} E_x\left[\int_0^t |c(X(s))|d\xi(s)\right]. \tag{3.11}$$

In view of (2.5) for nonnegative functions note that

$$M(t) = \sup_{x\in\overline{D}} \int_0^t \int_D |q(y)|p(s,x,y)dyds + \sup_{x\in\overline{D}} \int_0^t \int_{\partial D} |c(y)|p(s,x,y)d\sigma(y)ds. \quad (3.12)$$

These $M(t)$'s have indeed been made use of in the proof of Theorem 3.1. If $q(x)dx, c(x)d\sigma(x)$ belong to $GK_d(\overline{D})$ note that (by Markov property) $M(t) < \infty$ for all t, $M(t)$ is nondecreasing in t and $M(t) \downarrow 0$ as $t \downarrow 0$.

LEMMA 3.2. *Let D, q, c be as in Theorem 3.1; let ζ_n. be defined by (3.7). There exists a constant $K > 0$ such that*

$$\int_0^t \int_{\partial D} |\zeta_n(s,x,z)|d\sigma(z)ds \le K(\sqrt{t}+t)(M(t))^n \quad (3.13)$$

for $t > 0, x \in \overline{D}, n = 0, 1, 2, \cdots$

PROOF. Enough to consider the case when q, c are nonnegative. In such a case $|\zeta_n| = \zeta_n$. For $n = 0$, by (2.5)–(2.7) we get

$$\text{l.h.s. of (3.13)} = \int_0^t \int_{\partial D} p(s,x,z)d\sigma(z)ds \le K(\sqrt{t}+t).$$

Using (3.7), induction on n gives

$$\begin{aligned}
&\int_0^t \int_{\partial D} \zeta_n(s,x,z)d\sigma(z)ds \\
&= \int_0^t \int_{\partial D} \int_0^s \int_D p(r,x,y)q(y)\zeta_{n-1}(s-r,y,z)dydrd\sigma(z)ds \\
&+ \int_0^t \int_{\partial D} \frac{1}{2} \int_0^s \int_{\partial D} p(r,x,y)c(y)\zeta_{n-1}(s-r,y,z)d\sigma(y)drd\sigma(z)ds \\
&= \int_0^t \int_D \left[\int_0^{t-r} \int_{\partial D} \zeta_{n-1}(s,y,z)d\sigma(z)ds\right] q(y)p(r,x,y)dydr \\
&+ \int_0^t \int_{\partial D} \frac{1}{2} \left[\int_0^{t-r} \int_{\partial D} \zeta_{n-1}(s,y,z)d\sigma(z)ds\right] c(y)p(r,x,y)d\sigma(y)dr \\
&\le K(\sqrt{t}+t)(M(t))^{n-1}M(t).
\end{aligned}$$

□

We conclude this section with the following Gaussian upper bound for ζ.

LEMMA 3.3. *For any $t_0 > 0$ there exist constants $k_1, k_2 > 0$ such that*

$$0 \le \zeta(t,x,z) \le k_1 t^{-d/2} \exp\left(-\frac{k_2|x-z|^2}{t}\right) \quad (3.14)$$

for all $0 < t \le t_0, x, z \in \overline{D}$.

PROOF. For any nonnegative continuous function f on $\overline{D}$ with $\|f\|_1 = 1$, by Schwartz inequality, (2.1) and (3.10) we get

$$\int_{\overline{D}} f(z)\zeta(t,x,z)dz \le \left\{E_x[e_{2|q|}(t)\widehat{e}_{2|c|}(t)f(X(t))]\right\}^{\frac{1}{2}} \left\{\int_{\overline{D}} f(z)p(t,x,z)dz\right\}^{\frac{1}{2}}$$
$$\le Kt^{-d/4}e^{\beta t}\left[\int_{\overline{D}} f(z)t^{-d/2}\exp\left(-\frac{k|x-z|^2}{t}\right)dz\right]^{\frac{1}{2}}$$

for all $t \le t_0, x \in \overline{D}$. Letting f approach a δ function we get the required estimate. □

4. The Gauge and a Conditional Gauge Theorem

The gauge for the third boundary value problem is given by

$$G(x) = E_x\left[\int_0^\infty e_q(t)\widehat{e}_c(t)d\xi(t)\right], \qquad x \in \overline{D}. \tag{4.1}$$

An important result concerning the gauge is the following gauge theorem.

THEOREM 4.1. *Let D, q, c be as in Theorem 3.1. If $G(x) < \infty$ for some $x \in \overline{D}$, then G is a bounded continuous function on $\overline{D}$ and $\inf\{G(x) : x \in \overline{D}\} > 0$. In such a case there exist $K, \lambda > 0$ such that $\|T_t\|_{\infty,\infty} \le Ke^{-\lambda t}$; that is, T_t decays exponentially fast.*

We need the following lemma.

LEMMA 4.2. *Let D, q, c be as above. Let $t > 0$ be such that (2.3) holds. Then there is a constant a_t such that for any nonnegative measurable function f on $\overline{D}$,*

$$\|f\|_1 \le a_t \inf\{T_t f(x) : x \in \overline{D}\}. \tag{4.2}$$

PROOF. Proceed as in the proof of Proposition 3.5 of Papanicolaou (1990); (see also Chung and Hsu (1986)). What is really needed to make the proof go through is that (2.3) holds for some $t > 0$. □

PROOF OF THEOREM 4.1. By Markov property, for any $t > 0$

$$G(x) = E_x\left[\int_0^t e_q(s)\widehat{e}_c(s)d\xi(s)\right] + T_tG(x).$$

So, if $G(x) < \infty$ for some $x \in \overline{D}$ then for any $t > 0$ for which (2.3) holds, by the preceding lemma

$$\|G\|_1 \le a_tT_tG(x) \le a_tG(x) < \infty$$

Thus G is integrable if $G(x) < \infty$ for some x. As this is the crucial step in the proof of Theorem 3.6 of Papanicolaou (1990), the first assertion can now be proved analogously. The second assertion can then be proved along the lines of Theorem 3.7 of Papanicolaou (1990); the second assertion basically means that the first eigenvalue for the third boundary value problem is strictly negative if the gauge is finite; that is, β in (3.10) can be taken to be negative. In fact, the second assertion is equivalent to finiteness of the gauge. $\square$

The next result is perhaps known to experts, but an explicit proof has not appeared in the literature to our knowledge.

PROPOSITION 4.3. *Let D, q, c be as before. Then for any nonnegative measurable function f on ∂D and any $x \in \overline{D}$*

$$E_x\left[\int_0^\infty e_q(s)\widehat{e}_c(s)f(X(s))d\xi(s)\right] = \frac{1}{2}\int_0^\infty \int_{\partial D} f(z)\zeta(s,x,z)d\sigma(z)ds \tag{4.3}$$

in the sense that the l.h.s. is finite if and only if the r.h.s. is and equality of the two in case they are finite.

PROOF. By monotone convergence theorem it is enough to show that

$$E_x\left[\int_0^t e_q(s)\widehat{e}_c(s)f(X(s))d\xi(s)\right] = \frac{1}{2}\int_0^t \int_{\partial D} f(z)\zeta(s,x,z)d\sigma(z)ds \tag{4.4}$$

for any $t > 0$ and any *bounded* measurable function f on ∂D. For $n = 0, 1, 2, \cdots$ set

$$V_n(t,x;f) = \frac{1}{2}\int_0^t \int_{\partial D} f(z)\zeta_n(s,x,z)d\sigma(z)ds$$

where ζ_n is defined by (3.7); Lemma 3.2 assures that V_n is well defined. By (3.7), interchanging order of integration and (2.5)

$$\begin{aligned} V_n(t,x;f) &= \int_0^t \int_D p(r,x,y)q(y)V_{n-1}(t-r,y;f)dydr \\ &\quad + \frac{1}{2}\int_0^t \int_{\partial D} p(r,x,y)c(y)V_{n-1}(t-r,y;f)d\sigma(y)dr \\ &= E_x\left[\int_0^t q(X(r))V_{n-1}(t-r,X(r);f)dr\right] \\ &\quad + E_x\left[\int_0^t c(X(r))V_{n-1}(t-r,X(r);f)d\xi(r)\right] \end{aligned} \tag{4.5}$$

To prove (4.4) it suffices to establish

$$V_n(t,x;f) = E_x\left[\int_0^t \frac{1}{n!}\left(\int_0^s q(X(r))dr + \int_0^s c(X(r))d\xi(r)\right)^n f(X(s))d\xi(s)\right] \tag{4.6}$$

for each n. We apply induction on n. For $n = 0$, it is clear from (2.5). For $n \geq 1$ observe that (4.6) is the same as

$$V_n(t,x;f) = E_x\left[\int_0^t \sum\nolimits^{(n)}\left(\int\limits_{0<s_1<\cdots<s_n<s} d\mu(s_1',\cdots s_n')\right) f(X(s))d\xi(s)\right] \tag{4.7}$$

where $d\mu(s'_1,\cdots,s'_n)$ is a signed measure of the form $q(X(s'_1))ds'_1\cdots q(X(s'_k))$ $ds'_k c(X(s'_{k+1}))d\xi(s'_{k+1})\cdots c(X(s'_n))d\xi(s'_n)$, $k=0,1,\cdots,n$ where $(s'_1,\cdots,s'_n)$ is a permutation of $(s_1,\cdots,s_n)$ and $\sum^{(n)}$ denotes summation over 2^n such terms. Therefore by (4.5), Markov property and Fubini's theorem we get

$$\begin{aligned}
&V_n(t,x;f)\\
&= E_x\int_0^t c(X(r))E_{X(r)}\int_0^{t-r}\sum^{(n-1)}\int^{(n-1)} d\mu(s'_1,\ \cdots,s'_{n-1})\ f(X(s))d\xi(s)d\xi(r)\\
&+ E_x\int_0^t q(X(r))E_{X(r)}\int_0^{t-r}\sum^{(n-1)}\int^{(n-1)} d\mu(s'_1,\cdots s'_{n-1})f(X(s))d\xi(s)dr\\
&= E_x\int_0^t c(X(r))\int_r^t\sum^{(n-1)}\int_{r<s_1<\cdots<s_{n-1}<s} d\mu(s'_1,\cdots s'_{n-1})f(X(s))d\xi(s)d\xi(r)\\
&+ E_x\int_0^t q(X(r))\int_r^t\sum^{(n-1)}\int_{r<s_1<\cdots<s_{n-1}<s} d\mu(s'_1,\cdots s'_{n-1})f(X(s))d\xi(s)dr\\
&= E_x\int_0^t\sum^{(n)}\left(\int_{0<s_1<\cdots<s_n<s} d\mu(s'_1,\cdots,s'_n)\right)f(X(s))d\xi(s)
\end{aligned}$$

where $\int^{(n-1)} d\mu(s'_1,\cdots s'_{n-1})$ denotes $\int\limits_{0<s_1<\cdots<s_{n-1}<s} d\mu(s'_1,\cdots s'_{n-1})$. From the above (4.7), and hence (4.6), follows. □

By (4.1), (4.3) we have

$$G(x)=\frac{1}{2}\int_0^\infty\int_{\partial D}\zeta(t,x,z)d\sigma(z)dt,\qquad x\in\overline{D}. \tag{4.8}$$

Now for $x,z\in\overline{D}$, set

$$F(x,z)=\int_0^\infty\zeta(t,x,z)dt. \tag{4.9}$$

The main result of this section is

THEOREM 4.4. *Let D be a bounded Lipschitz domain. Let $q(x)dx, c(x)d\sigma(x)$ belong to $GK_d(\overline{D})$.*

(*a*) *If $F(x,z)<\infty$ for some $x,z\in\overline{D}$, then the gauge G is a bounded continuous function on $\overline{D}$.*

(*b*) *In such a case $(x,z)\mapsto F(x,z)$ is finite and continuous on the set $\{x\neq z\}$.*

REMARK 4.1. If the hypothesis of the theorem holds then by Proposition 4.3

$$F(x,z)=\lim_{\varepsilon\downarrow 0}\frac{1}{\sigma(B_{z,\varepsilon})}E_x\left[\int_0^\infty e_q(t)\widehat{e}_c(t)I_{z,\varepsilon}(X(t))d\xi(t)\right]$$

for $x \in \overline{D}, z \in \partial D$ where $B_{z,\varepsilon} = B(z:\varepsilon) \cap \partial D$ and $I_{z,\varepsilon}$ is the indicator function of $B_{z,\varepsilon}$. Similarly for $x \in \overline{D}, z \in D, x \neq z$ by Theorem 3.1

$$F(x,z) = \lim_{\varepsilon \downarrow 0} \frac{1}{|B(z:\varepsilon)|} E_x \left[\int_0^\infty e_q(t) \widehat{e}_c(t) I_{B(z:\varepsilon)}(X(t)) dt \right].$$

This is a justification for calling the above result a conditional gauge theorem for the third boundary value problem.

For any finite measure μ on $\overline{D}$, write

$$v(x;\mu) = \int_{\overline{D}} \int_0^\infty \zeta(t,x,z) dt d\mu(z) \tag{4.10}$$

By the Chapman-Kolmogorov equation note that for any $t > 0, x \in \overline{D}$

$$v(x;\mu) = \int_{\overline{D}} \int_0^t \zeta(s,x,z) ds d\mu(z) + T_t v(x;\mu). \tag{4.11}$$

We need a few lemmas.

Lemma 4.5. *Let $F(x_0, z_0) < \infty$ for some $x_0, z_0 \in \overline{D}$. Then $x \mapsto F(x, z_0)$ is a well defined continuous function on $\overline{D} \backslash \{z_0\}$.*

Proof. Putting $\mu = \delta_{z_0}$ in (4.11) we get for $t > 0$

$$\infty > F(x_0, z_0) = \int_0^t \zeta(s, x_0, z_0) ds + T_t F(x_0, z_0) \tag{4.12}$$

Let $t > 0$ be such that (2.3) holds; then by (4.12) and Lemma 4.2 we get that $x \mapsto F(x, z_0)$, $x \in \overline{D}$ is an integrable function. Hence by Theorem 3.1, for such a t we have $x \mapsto T_t F(x, z_0)$ is a continuous function on $\overline{D}$. And by Lemma 3.3 it is clear that $x \mapsto \int_0^t \zeta(s, x, z_0) ds$ is a continuous function on $\overline{D} \backslash \{z_0\}$. □

Lemma 4.6. *$F(x,z) < \infty$ for all $x \neq z$ if and only if the gauge G is finite.*

Proof. For any $x \in \overline{D}$ clearly

$$G(x) = \frac{1}{2} \int_{\partial D} F(x,z) d\sigma(z) \tag{4.13}$$

If the gauge is finite, by (4.13), $F(x,z) < \infty$ for a.a. $z \in \partial D$ for all $x \in \overline{D}$. So by the preceding lemma and symmetry of $F(x,z)$ it follows that $F(x,z) < \infty$ for all $x \neq z$.

Conversely let $F(x,z) < \infty$ for all $x \neq z$. In particular $F(x_0, z_0) < \infty$ for some $x_0, z_0 \in D$. So by the preceding lemma $z \mapsto F(x_0, z)$ is bounded on ∂D. Hence by (4.13) and Theorem 4.1 the required result follows. □

Proof of Theorem 4.4. Proof of assertion (a) is contained in the proofs of Lemmas 4.5 and 4.6; the only comment we make is that if $F(x_0, z_0) < \infty$ for some $x_0, z_0 \in \partial D$ then by symmetry and Lemma 4.5, $F(z, x) < \infty$ for all $z \in \partial D$ for some $x \in D$.

It remains to prove (b). Note that continuity in each variable on the set $\{x \neq z\}$ is already established. Let $H \neq \overline{D}$ be a compact subset of $\overline{D}$. We claim that

$$\sup\{\|T_t F(\cdot, z)\|_\infty : z \in H\} < \infty \tag{4.14}$$

for any $t > 0$ for which (2.3) holds. Under our hypothesis, by the proof of Lemma 4.5, $x \mapsto T_t F(x, z)$ is a bounded continuous function on $\overline{D}$ for each z; by continuity in z (4.14) now follows.

Let $(x', z') \to (x, z)$ with $x \neq z$; so we may assume that $(x', z'), (x, z)$ lie in a compact set $K \subset \overline{D} \times \overline{D} \backslash \{\text{diagonal set}\}$. Let $t > 0$ be large enough that (2.3) holds. Note that

$$\begin{aligned} |F(x', z') - F(x, z)| \leq & \int_0^{s+t} |\zeta(r, x', z') - \zeta(r, x, z)| ds \\ & + |T_s[T_t F(x', z') - T_t F(x, z)]|. \end{aligned} \tag{4.15}$$

Since the gauge is finite, by Theorem 4.1 and (4.14) the second term on the r.h.s. of (4.15) decays exponentially fast as $s \to \infty$; and by Lemma 3.3 it is easily seen that the first term on the r.h.s. of (4.15) goes to zero for any $t + s$ as $(x', z') \to (x, z)$. This completes the proof. $\square$

Corollary 4.7. *Let A be a nonempty open subset of ∂D. If*

$$E_x \int_0^\infty e_q(s) \widehat{e}_c(s) I_A(X(s)) d\xi(s) < \infty$$

for some $x \in \overline{D}$, then the gauge is a bounded continuous function on $\overline{D}$.

The above is immediate from Theorem 4.4. The above corollary has been proved in Ramasubramanian (1993) when D is smooth using the conditional gauge theorem for Dirichlet problem.

Remark 4.2. Consider the third boundary value problem

$$\begin{aligned} \frac{1}{2}\Delta u(x) + q(x)u(x) = -\psi(x), \qquad & x \in D \\ \frac{\partial u}{\partial n}(x) + c(x)u(x) = -\varphi(x), \qquad & x \in \partial D \end{aligned} \tag{4.16}$$

where $n(\cdot)$ denotes the inward normal, ψ, φ are bounded measurable functions respectively on $D, \partial D$. A function u on $\overline{D}$ is said to be a *probabilistic solution* to (4.16) if for each $x \in \overline{D}$

$$\begin{aligned} & u(X(t)) - u(x) + \int_0^t q(X(s))u(X(s))ds + \int_0^t \psi(X(s))ds \\ & + \int_0^t c(X(s))u(X(s))d\xi(s) + \int_0^t \varphi(X(s))d\xi(s) \\ & = \text{ a continuous } P_x - \text{martingale} \end{aligned} \tag{4.17}$$

w.r.t. the canonical filtration. By Theorem 4.8 of Papanicolaou (1990) note that the notion of a continuous probabilistic solution coincides with that of a continuous weak solution in the analytic/ classical sense. If the gauge is finite, essentially mimicking the arguments in Section 4 of Papanicolaou (1990) it can be shown that

$$u(x) = \int_D \psi(z)F(x,z)dz + \frac{1}{2}\int_{\partial D} \varphi(z)F(x,z)d\sigma(z), \qquad x \in \overline{D} \tag{4.18}$$

is the continuous probabilistic solution to the inhomogeneous problem (4.16). Therefore the function $(x,z) \mapsto F(x,z), x,z \in D$ may be called the "Green function", and the function $(x,z) \mapsto F(x,z), x \in D, z \in \partial D$ the "Poisson kernel" for the third boundary value problem (4.16). This may be contrasted with the case of the Dirichlet problem where the Poisson kernel is the normal derivative of the corresponding Green function; this is hardly surprising in view of Green's theorem. Of course, the point we make here is that the analogy carries over to "Kato class" potentials (in the interior and on the boundary) and to Lipschitz domains.

Another immediate corollary to Theorem 4.4 is the following *Harnack inequality.*

COROLLARY 4.8. *Let D,q,c be as before; let $F(x,z) < \infty$ for some $x,z \in \overline{D}$. For compact $D_1 \subset D$ there exist positive constants k_1,k_2 such that $k_1 \le F(x,z) \le k_2$ for all $x \in D_1, z \in \partial D$.*

5. **Connection with Conditioned Brownian Motion**

For $x \in D, z \in \partial D$ let $\widetilde{P}_{x:z}$ denote the law of the z-conditioned Brownian motion starting at x; that is, the law of the Brownian motion starting at x and conditioned to exit D at z. Under $\{\widetilde{P}_{x:z}\}$ the canonical process $\{X(t)\}$ is a strong Markov process with transition density given by

$$k^{(z)}(t,x,y) = M(x,z)^{-1}k_D(t,x,y)M(y,z)$$

where k_D denotes the transition density for Brownian motion killed on exit from D, $M : D \times \partial D \to (0,\infty)$ is the Martin kernel (relative to a fixed reference point x_*). This can be considered as the diffusion killed on exit from D, with constant dispersion $I = ((\delta_{ij}))$ and drift function $y \mapsto \frac{1}{M(y,z)}\nabla_y M(y,z)$. As D is a Lipschitz domain it is known that the Euclidean boundary ∂D coincides with the Martin boundary. In case D is a smooth domain the Martin kernel is the same as the Poisson kernel for the Dirichlet problem. See Bass (1995), Falkner (1983), Zhao (1986), Cranston, Fabes and Zhao (1988) for more information. Of course, $\widetilde{E}_{x:z}$ will denote expectation w.r.t. $\widetilde{P}_{x:z}$.

The next result gives a connection between F and $\widetilde{P}_{x:z}$.

THEOREM 5.1. *Let the hypothesis of Theorem 4.4 hold. Let $F(x,z) < \infty$ for some x,z. Then for $x \in D, z \in \partial D$*

$$F(x,z) = E_x \int_0^\infty e_q(s)\widehat{e}_c(s)M(x,X(s))\widetilde{E}_{x:X(s)}(e_q(\tau))d\xi(s) \tag{5.1}$$

where τ is the first hitting time of ∂D.

We need the following lemma.

Lemma 5.2. *Let the hypotheses of the theorem above hold. Let φ be a nonnegative continuous function on ∂D such that $\int_{\partial D} \varphi(z) d\sigma(z) = 1$. Set*

$$u(x) \triangleq u(x;\varphi) = \frac{1}{2}\int_{\partial D} \varphi(z)F(x,z)d\sigma(z), \quad x \in \overline{D}$$

Then $\inf\{u(x) : x \in \overline{D}\} > 0$.

Proof. Clearly u is nonnegative; and proceeding as in the proof of Theorem 4.4 it can be shown that u is continuous. So it is enough to show that $u(x) \neq 0$ for any $x \in \overline{D}$.

Suppose $u(x) = 0$ for some $x \in \overline{D}$. We can find a nonempty open subset $A \subset \partial D$ and $\epsilon > 0$ such that $\varphi(z) \geq \epsilon$ for all $z \in A$. Then by Proposition 4.3

$$\begin{aligned} 0 &= \frac{1}{2}\int_{\partial D} \varphi(z)F(x,z)d\sigma(z) = \frac{1}{2}\int_{\partial D}\int_0^\infty \varphi(z)\zeta(t,x,z)d\sigma(z)dt \\ &\geq \frac{\epsilon}{2}\int_A\int_0^\infty \zeta(t,x,z)dtd\sigma(z) \\ &= \epsilon E_x \int_0^\infty e_q(s)\widehat{e}_c(s)I_A(X(s))d\xi(s) \geq 0 \end{aligned}$$

whence it follows

$$E_x \int_0^\infty e_q(s)\widehat{e}_c(s)I_A(X(s))d\xi(s) = 0.$$

Consequently, as e_q and $\widehat{e}_c$ are strictly positive, we have $E_x \int_0^\infty I_A(X(s))d\xi(s) = 0$ and hence by (2.5) $\int_0^\infty \int_A p(s,x,z)d\sigma(z)ds = 0$. But this contradicts (2.3). Hence the claim. □

Proof of Theorem 5.1. Let φ and u be as in the preceding lemma. By Remark 4.2 u is the continuous probabilistic solution to (4.16) with $\psi \equiv 0$. Put $f(z) \triangleq u(z;\varphi), z \in \partial D$. Note that u is the continuous probabilistic solution to the Dirichlet problem

$$\begin{aligned} \frac{1}{2}\Delta u(x) + q(x)u(x) = 0, &\qquad x \in D \\ u(x) = f(x), &\qquad x \in \partial D. \end{aligned}$$

Also by the preceding lemma u is bounded away from zero. Consequently (recalling that u is also the continuous weak solution in analytic sense) by Theorem A and Proposition B of Zhao (1986) and Theorem 5.5 of Cranston, Fabes and Zhao (1988) we get for any $x \in D$

$$\begin{aligned} u(x;\varphi) &= \int_{\partial D} f(z)\widetilde{E}_{x:z}(e_q(\tau))M(x,z)d\sigma(z) \\ &= \frac{1}{2}\int_{\partial D}\varphi(\eta)\int_{\partial D}\int_0^\infty M(x,z)\widetilde{E}_{x:z}(e_q(\tau))\zeta(t,\eta,z)d\sigma(z)dtd\sigma(\eta) \\ &= \int_{\partial D}\varphi(\eta)E_\eta\left[\int_0^\infty e_q(t)\widehat{e}_c(t)M(x,X(t))\widetilde{E}_{x:X(t)}(e_q(\tau))d\xi(t)\right]d\sigma(\eta) \end{aligned}$$

Now letting φ approach a δ-function (on ∂D) we get (5.1). □

We now indicate a connection with spectral properties. Let $c(x)d\sigma(x) \in GK_d(\overline{D})$ be fixed. For $q(x)dx \in GK_d(\overline{D})$ let $\mu(q)$ denote the principal eigenvalue of $\frac{1}{2}\Delta + q$ with boundary condition $\frac{\partial u}{\partial n} + cu = 0$; and $\lambda(q)$ denote the principal eigenvalue of $\frac{1}{2}\Delta + q$ with Dirichlet boundary condition. See Ma and Song (1990), Papanicolaou (1990) for definitions.

Theorem 5.3. $\lambda(q) \leq \mu(q)$.

Proof. Let $\epsilon > 0$. Write $\mu_0 = \mu(q)$ and $\alpha(x) = q(x) - (\mu_0 + \epsilon)$. Observe that $\mu(\alpha) = -\epsilon < 0$. By the proof of Theorem 4.1 it follows that the gauge function for the third boundary value problem for $\frac{1}{2}\Delta + \alpha$ is bounded. So

$$\int_0^\infty \exp(-(\mu_0+\epsilon)t)\zeta(t,x,z)dt < \infty, \qquad x \neq z,$$

where ζ is as before. Hence Theorem 5.1 applied to $\alpha(x)dx$ and $c(x)d\sigma(x)$ gives $\widetilde{E}_{x:z}(e_\alpha(\tau)) < \infty$ for some $x \in D, z \in \partial D$. This implies that the gauge function for the Dirichlet problem for $\frac{1}{2}\Delta + \alpha$ is bounded, and hence $\lambda(\alpha) < 0$; see Zhao (1986) and Cranston, Fabes and Zhao (1988). Thus $\lambda(q) < \mu_0 + \epsilon$. As ϵ is arbitrary the result follows. □

Remark 5.1: The above result is well known when D is smooth, q, c are smooth functions such that $-\infty < k_1 \leq q(\cdot),\ c(\cdot) \leq k_0 < 0$. See Courant and Hilbert (1975).

Remark 5.2. It is immediately clear from the preceding theorem that finiteness of the gauge for the third boundary value problem implies that of the gauge for the Dirichlet problem. The converse, however, is not true; take $q \equiv 0$, $c \equiv 0$.

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S. Ramasubramanian
Indian Statistical Institute
8th Mile, Mysore Road
Bangalore 560 059
India
E-mail: ram@isibang.ac.in