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SHARP BERRY-ESSEEN BOUND FOR THE MAXIMUM LIKELIHOOD ESTIMATOR IN THE ORNSTEIN-UHLENBECK PROCESS

By J.P.N. BISHWAL
Indian Statistical Institute, Calcutta

SUMMARY. The paper shows that the distribution of the normalized maximum likelihood estimator of the drift parameter in the Ornstein-Uhlenbeck process observed over $[0, T]$ converges to the standard normal distribution with an error bound $O(T^{-1/2})$.

1. **Introduction**

Let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, P)$ be a stochastic basis on which is defined the Ornstein-Uhlenbeck process X_t satisfying the Itô stochastic differential equation

$$dX_t = -\theta X_t dt + dW_t, t \geq 0, \quad X_0 = 0 \qquad \ldots (1.1)$$

where $\{W_t\}$ is a standard Wiener process with respect to the filtration $\{\mathcal{F}_t\}_{t\geq 0}$ and $\theta \in \Theta \subseteq \mathbb{R}^+$ is the unknown parameter to be estimated on the basis of continuous observation of the process $\{X_t\}$ on the time interval $[0, T]$.

Let us denote the realization $\{X_t, 0 \leq t \leq T\}$ by X_0^T. Let P_θ^T be the measure generated on the space (C_T, B_T) of continuous functions on $[0, T]$ with the associated Borel σ-algebra B_T generated under the supremum norm by the process X_0^T and P_0^T be the standard Wiener measure. It is well known that when θ is the true value of the parameter P_θ^T is absolutely continuous with respect to P_0^T and the Radon-Nikodym derivative (likelihood) of P_θ^T with respect to P_0^T based on X_0^T is given by

$$L_T(\theta) := \frac{dP_\theta^T}{dP_0^T}(X_0^T) = \exp\left\{-\theta \int_0^T X_t dX_t - \frac{\theta^2}{2}\int_0^T X_t^2 dt\right\}. \qquad \ldots (1.2)$$

Maximizing the log-likelihood w.r.t θ provides the maximum likelihood estimate (MLE)

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$$\theta_T = -\frac{\int_0^T X_t dX_t}{\int_0^T X_t^2 dt.} \qquad \ldots (1.3)$$

It is well known that θ_T is strongly consistent and $T^{1/2}(\theta_T - \theta)$ asymptotically $\mathcal{N}(0, \frac{1}{2\theta})$ distributed as $T \to \infty$ (see Basawa and Prakasa Rao (1980)). Our aim in this paper is to obtain the Berry-Esseen bound, i.e., the rate of convergence to normality of the MLE.

Note that

$$\left(\frac{T}{2\theta}\right)^{1/2} (\theta_T - \theta) = -\frac{\left(\frac{2\theta}{T}\right)^{1/2} Z_T}{\left(\frac{2\theta}{T}\right) I_T} \qquad \ldots (1.4)$$

where

$$Z_T = \int_0^T X_t dW_t \quad \text{and} \quad I_T = \int_0^T X_t^2 dt.$$

In (1.4), the numerator of the normalized MLE is a normalized martingale which converges to the standard normal variable and the denominator is its corresponding increasing process which converges to one as $T \to \infty$.

One way of obtaining the Berry-Esseen bound in the present context is to use the technique of Michel and Pfanzagl (1971) for the i.i.d case. Lemma-1 in Michel and Pfanzagl (1971) states that the Berry-Esseen rate for the ratio of two processes can be split up into three components : the Berry-Esseen rate for the numerator, the rate of convergence of the denominator to one and a small positive number depending on the second component. For the Berry-Esseen bound of the MLE θ_T, Mishra and Prakasa Rao (1985) used this approach. For the normal approximation of the numerator, they embedded it in a Brownian motion by the Kunita-Watanabe theorem and used Lemma 3.2 of Hall and Heyde (1980) on the Berry-Esseen bound for the Brownian motion with random time. For the convergence of the denominator to one, they used Chebyshev's inequality. They applied these two together and obtained the rate $O(T^{-1/5})$. One can use Burkholder inequality for the convergence of the denominator to one to improve this rate to $O(T^{-1/4+\epsilon})$, $\epsilon > 0$. Note that using Skorohod embedding method one cannot obtain any better rate than $O(T^{-1/4})$ (see Borokov (1973)). Bose (1986) used characteristic function followed by Esseen's lemma for the numerator. The denominator was linked with the numerator via Itô formula. He obtained the rate $O(T^{-1/2}(\log T)^2)$. Theorem 3.4 in Bose (1986) has a misprint and gives the rate as $O(T^{-1/2})$, but by following the proof given there it is clear that the rate is $O(T^{-1/2}(\log T)^2)$. Bose (1985) decomposed the numerator in to two parts using Itô formula. He used normal approximation by Esseen's lemma for one part and rate of convergence to zero for the other part and obtained the rate $O(T^{-1/2} \log T)$. Bishwal and Bose (1995) used the approach in Michel and Pfanzagl (1971). For the normal approximation of the numerator, they used characteristic function followed by Esseen's lemma. For the convergence of the denominator to one, they obtained exponential bound using the moment generating function of the denominator. These two together improve the rate to $O(T^{-1/2}(\log T)^{1/2})$. Bishwal and Bose (1995) also obtained the rate $O(T^{-1/2}(\log T)^{1/2})$ using different random normings which are useful for computation of a confidence interval.

Here we improve the Berry-Esseen bound for θ_T to $O(T^{-1/2})$ using the squeezing technique, due to Pfanzagl (1971) developed for the minimum contrast estimators in the i.i.d. case. This method was adopted by Prakasa Rao (1973, 1975) respectively for the minimum contrast estimators in the independent nonidentically distributed case and discrete time Markov processes, and by Ivanov (1976) for the least squares estimator of the nonlinear regression parameter to obtain a Berry-Esseen bound of the order $O(n^{-1/2})$.

2. **Main Results**

Let $\Phi(\cdot)$ denote the standard normal distribution function. Throughout the paper C denotes a generic constant (perhaps depending on θ, but not on anything else).

We start with some preliminary lemmas. The first lemma gives a useful bound for I_T. Note that

$$I_T := \frac{1}{\theta}\left(\int_0^T X_t dW_t + \frac{T - X_T^2}{2}\right),$$

which is known as the energy of the O-U process.

Lemma 2.1. *For every* $\delta > 0$,

$$P\left\{\left|\frac{2\theta}{T} I_T - 1\right| \geq \delta\right\} \leq CT^{-1}\delta^{-2}.$$

Proof. It is clear that $X_t = \int_0^t e^{-\theta(t-s)} dW_s$. Note that

$$E(X_T^2) = \frac{1 - e^{-2\theta T}}{2\theta}; \quad E(X_T^4) = \frac{3(1 - e^{-2\theta T})^2}{4\theta}; \quad \text{and } E(I_T) = \frac{2\theta T - 1 + e^{-2\theta T}}{4\theta^2}.$$

By Itô formula (see Friedman (1975)), we have

$$2\theta\frac{I_T}{T} - 1 = \frac{2}{T}\int_0^T X_t dW_t - \frac{X_T^2}{T}.$$

Using this and Chebyshev inequality, we have

$$\begin{aligned}
& P\left\{\left|\frac{2\theta}{T} I_T - 1\right| \geq \delta\right\} \\
\leq\ & \frac{1}{\delta^2} E\left|\frac{2\theta}{T} I_T - 1\right|^2 \quad = \quad \frac{1}{\delta^2} E\left|\frac{2}{T}\int_0^T X_t dW_t - \frac{X_T^2}{T}\right|^2 \\
\leq\ & \frac{2}{\delta^2}\left\{\frac{4}{T^2} E(I_T) + E(\frac{X_T^4}{T^2})\right\} \quad = \quad \frac{2}{\delta^2}\left\{\frac{4}{T^2}\frac{2\theta T - 1 + e^{-2\theta T}}{4\theta^2} + \frac{3(1 - e^{-2\theta T})^2}{4\theta^2 T^2}\right\} \\
\leq\ & CT^{-1}\delta^{-2}.
\end{aligned}$$

□

The following lemma gives the characteristic functions of the quantities in the expression for the MLE.

LEMMA 2.2. (a) *Let* $\phi_T(z_1, z_2) := E\exp(z_1 I_T + z_2 X_T^2), z_1, z_2 \in \mathbb{C}$. *Then* $\phi_T(z_1, z_2)$ *exists for* $|z_i| \le \delta$, *1 = 1,2 for some* $\delta > 0$ *and is given by*

$$\phi_T(z_1, z_2) = \exp\left(\frac{\theta T}{2}\right)\left[\frac{2\gamma}{(\gamma-\theta+2z_2)e^{-\gamma T} + (\gamma+\theta-2z_2)e^{\gamma T}}\right]^{1/2}$$

where $\gamma = (\theta^2 - 2z_1)^{1/2}$ *and we choose the principal branch of the square root.*

(b) *Let* $G_{t,x} := -\left(\frac{2\theta}{T}\right)^{1/2} Z_T - \left(\frac{2\theta}{T} I_T - 1\right) x$. *Then for* $|x| \le 2(\log T)^{1/2}$ *and for* $|t| \le \epsilon T^{1/2}$, *where* ϵ *is sufficiently small,*

$$\left|E\exp(itG_{T,x}) - \exp(\frac{-t^2}{2})\right| \le C\exp(\frac{-t^2}{4})(|t| + |t|^3)T^{-1/2}.$$

(c) *For* $|t| \le \epsilon_1 T^{\frac{1}{2}}$, *where* ϵ_1 *is sufficiently small, we have as* $T \to \infty$,

$$\left|E\exp\left\{it\left(\frac{2\theta}{T}\right)^{1/2}\left(\theta I_T - \frac{T}{2}\right)\right\} - \exp(-\frac{t^2}{2})\right| \le C\exp(-\frac{t^2}{4})(|t| + |t|^3)T^{-1/2}.$$

(d) *Statement (c) above holds when* $(\frac{2\theta}{T})^{1/2}(\theta I_T - \frac{T}{2})$ *is replaced by* $(\frac{2\theta}{T})^{1/2} Z_T$.

Part (a) is essentially given in Liptser and Shiryayev (1978) for $z_1 \in \mathbb{R}, z_2 = 0$. A complete proof for the case $z_1, z_2 \in \mathbb{C}$ may be found in Bose (1986). We shall prove part (b) in details. Proof of part (c) is very similar to part (b) and will be omitted. Proof of part (d) is also similar and may be found in Bose (1986).

PROOF. By Itô formula,

$$Z_T = \theta I_T + \frac{X_T^2}{2} - \frac{T}{2}.$$

Note that

$$\begin{aligned}
& E\exp(itG_{T,x}) \\
= \quad & E\exp\left[-it\left(\frac{2\theta}{T}\right)^{1/2} Z_T - it\left(\frac{2\theta}{T} I_T - 1\right)x\right] \\
= \quad & E\exp\left[-it\left(\frac{2\theta}{T}\right)^{1/2}\left\{\theta I_T + \frac{X_T^2}{2} - \frac{T}{2}\right\} - it\left(\frac{2\theta}{T} I_T - 1\right)x\right] \\
= \quad & E\exp(z_1 I_T + z_2 X_T^2 + z_3) \\
= \quad & \exp(z_3)\phi_T(z_1, z_2).
\end{aligned}$$

where $z_1 = -it\theta\delta_{T,x}, z_2 = -\frac{it}{2}\left(\frac{2\theta}{T}\right)^{1/2}$ and $z_3 = \frac{itT}{2}\delta_{T,x}$ with $\delta_{T,x} = \left(\frac{2\theta}{T}\right)^{1/2} + \frac{2x}{T}$. Note that (z_1, z_2) satisfies the conditions of (a) by choosing ϵ sufficiently small. Let $\alpha_{1,T}(t)$, $\alpha_{2,T}(t)$, $\alpha_{3,T}(t)$ and $\alpha_{4,T}(t)$ be functions which are $O(|t|T^{-1/2})$,

$O(|t|^2T^{-1/2})$, $O(|t|^3T^{-3/2})$ and $O(|t|^3T^{-1/2})$ respectively. Note that for the given range of values of x and t, the conditions on z_i for part (a) of Lemma are satisfied. Note also that $z_2 = \alpha_{1,T}(t)$. Further, with $\beta_T(t) = 1 + it\frac{\delta_{T,x}}{\theta} + \frac{t^2\delta_{T,x}^2}{2\theta^2}$,

$$\begin{aligned}\gamma &= (\theta^2 - 2z_1)^{1/2} = \theta\left[1 - \frac{z_1}{\theta^2} - \frac{z_1^2}{2\theta^4} + \frac{z_1^3}{2\theta^8} + \cdots\right] \\ &= \theta\left[1 + it\frac{\delta_{T,x}}{\theta} + \frac{t^2\delta_{T,x}^2}{2\theta^2} + \frac{it^3\delta_{T,x}^3}{2\theta^3} + \cdots\right] = \theta[1 + \alpha_{1,T}(t) + \alpha_{2,T}(t) + \alpha_{3,T}(t)] \\ &= \theta\beta_T(t) + \alpha_{3,T}(t) = \theta[1 + \alpha_{1,T}(t)].\end{aligned}$$

Thus

$$\gamma - \theta = \alpha_{1,T},\ \gamma + \theta = 2\theta + \alpha_{1,T}.$$

Hence the above expectation equals

$$\begin{aligned}&\exp\left(z_3 + \frac{\theta T}{2}\right) \\ \times &\left[\frac{2\theta\beta_T(t) + \alpha_{3,T}(t)}{\alpha_{1,T}\exp\{-\theta T\beta_T(t) + \alpha_{4,T}(t)\} + (2\theta + \alpha_{1,T}(t))\exp\{\theta T\beta_T(t) + \alpha_{4,T}(t)\}}\right]^{1/2} \\ = &\left[\frac{1 + \alpha_{1,T}(t)}{\alpha_{1,T}\exp(\chi_T(t)) + (1 + \alpha_{1,T}(t))\exp(\psi_T(t))}\right]^{1/2}\end{aligned}$$

where

$$\begin{aligned}\chi_T(t) &= -\theta T\beta_T(t) + \alpha_{4,T}(t) - 2z_3 - \theta T \\ &= -2\theta T + \alpha_{1,T}(t) + t^2\alpha_{1,T}(t).\end{aligned}$$

and

$$\begin{aligned}\psi_T(t) &= \theta T\beta_T(t) + \alpha_{4,T}(t) - 2z_3 - \theta T \\ &= \theta T\left[1 + it\frac{\delta_{T,x}}{\theta} + \frac{t^2\delta_{T,x}^2}{2\theta^2}\right] + \alpha_{4,T}(t) - itT\delta_{T,x} - \theta T \\ &= \frac{t^2T}{2\theta}\left[\left(\frac{2\theta}{T}\right)^{1/2} + \frac{2x}{T}\right]^2 \\ &= t^2 + t^2\alpha_{1,T}(t).\end{aligned}$$

Hence, for the given range of values of t, $\chi_T(t) - \psi_T(t) \le -\theta T$.

Hence the above expectation equals

$$\begin{aligned}&\exp(-\frac{t^2}{2})(1 + \alpha_{1,T})^{1/2}\left[\alpha_{1,T}\exp\{-2\theta T + \alpha_{1,T} + t^2\alpha_{1,T}\}\right. \\ &\left. + (1 + \alpha_{1,T}(t))\exp\{t^2\alpha_{1,T}(t)\}\right]^{-1/2} \\ = &\exp(-\frac{t^2}{2})\left[1 + \alpha_{1,T})(1 + \alpha_{1,T}(1 + \alpha_{1,T})\exp\{-\theta T + \alpha_{1,T} + t^2\alpha_{1,T}\}\right] \\ &\exp(t^2\alpha_{1,T}(t)).\end{aligned}$$

□

Parts (c) and (d) of Lemma 2.2 give the Berry-Esseen rate for Z_T and I_T immediately by using the Esseen's lemma.

Corollary 2.3.

$$\text{(a)} \quad \sup_{x\epsilon\mathbb{R}} \left| P\left\{ \left(\frac{2\theta}{T}\right)^{1/2} Z_T \le x \right\} - \Phi(x) \right| \le CT^{-1/2}.$$

$$\text{(b)} \quad \sup_{x\in\mathbb{R}} \left| P\left\{ (\frac{2\theta}{T})^{1/2}(\theta I_T - \frac{T}{2}) \le x \right\} - \Phi(x) \right| \le CT^{-1/2}.$$

Before we prove the results on the Berry-Esseen bound for the MLE with nonrandom norming, we need the following estimate on the tail behaviour of the MLE.

Lemma 2.4.

$$P\left\{ (\frac{T}{2\theta})^{1/2}|\theta_T - \theta| \ge 2(\log T)^{1/2} \right\} \le CT^{-1/2}$$

Proof.

$$\begin{aligned}
& P\left\{ (\frac{T}{2\theta})^{1/2}|\theta_T - \theta| \ge 2(\log T)^{1/2} \right\} \\
= \quad & P\left\{ \left| \frac{(\frac{2\theta}{T})^{1/2} Z_T}{(\frac{2\theta}{T}) I_T} \right| \ge 2(\log T)^{1/2} \right\} \\
\le \quad & P\left\{ \left| (\frac{2\theta}{T})^{1/2} Z_T \right| \ge (\log T)^{1/2} \right\} + P\left\{ \left| \frac{2\theta}{T} I_T \right| \le \frac{1}{2} \right\} \\
\le \quad & \left| P\left\{ (\frac{2\theta}{T})^{1/2} |Z_T| \ge (\log T)^{1/2} \right\} - 2\Phi(-(\log T)^{1/2}) \right| \\
& +2\Phi(-(\log T)^{1/2}) + P\left\{ \left| \frac{2\theta}{T} I_T - 1 \right| \ge \frac{1}{2} \right\} \\
\le \quad & \sup_{x\in\mathbb{R}} \left| P\left\{ (\frac{2\theta}{T})^{1/2} |Z_T| \ge x \right\} - 2\Phi(-x) \right| \\
\le \quad & \sup_{x\epsilon\mathbb{R}} \left| P\left\{ (\frac{2\theta}{T})^{1/2} |Z_T| \ge x \right\} - 2\Phi(-x) \right| \\
& +2\Phi(-(\log T)^{1/2}) + P\left\{ \left| (\frac{2\theta}{T}) I_T - 1 \right| \ge \frac{1}{2} \right\} \\
\le \quad & CT^{-1/2} + C(T \log T)^{-1/2} + CT^{-1} \\
\le \quad & CT^{-1/2}.
\end{aligned}$$

The bounds for the first and the third terms come from Corollary 2.3 (a) and Lemma 2.1 respectively and that for the middle term comes from Feller (1957, p. 166). □

We are now in a position to obtain the Berry-Esseen bound of the order $O(T^{-1/2})$ for the MLE.

THEOREM 2.5.

$$\sup_{x\in\mathbb{R}}\left|P\left\{(\frac{T}{2\theta})^{1/2}(\theta_T-\theta)\le x\right\}-\Phi(x)\right|=O(T^{-1/2}).$$

PROOF. We shall consider two posibilities.

(i) $|x|>2(\log T)^{1/2}$.

We shall give a proof for the case $x>2(\log T)^{1/2}$. The proof for the case $x<-2(\log T)^{1/2}$ runs similarly. Note that

$$\left|P\left\{(\frac{T}{2\theta})^{1/2}(\theta_T-\theta)\le x\right\}-\Phi(x)\right|\le P\left\{(\frac{T}{2\theta})^{1/2}(\theta_T-\theta)\ge x\right\}+\Phi(-x)$$

But $\Phi(-x)\le\Phi(-2(\log T)^{1/2})\le CT^{-2}$. See Feller (1957, p. 166).

Moreover by Lemma 2.4, we have

$$P\left\{(\frac{T}{2\theta})^{1/2}(\theta_T-\theta)\ge 2(\log T)^{1/2}\right\}\le CT^{-1/2}.$$

Hence

$$\left|P\left\{(\frac{T}{2\theta})^{1/2}(\theta_T-\theta)\le x\right\}-\Phi(x)\right|\le CT^{-1/2}.$$

(ii) $|x|\le 2(\log T)^{1/2}$.

Let $A_T=\left\{(\frac{T}{2\theta})^{1/2}|\theta_T-\theta|\le 2(\log T)^{1/2}\right\}$ and $B_T=\left\{\frac{I_T}{T}>c_0\right\}$

where $0<c_0<\frac{1}{2\theta}$. By Lemma 2.4, we have

$$P(A_T^c)\le CT^{-1/2}. \qquad \ldots(2.1)$$

By Lemma 2.1, we have

$$P(B_T^c)=P\left\{\frac{2\theta}{T}I_T-1<2\theta c_0-1\right\}<P\left\{|\frac{2\theta}{T}I_T-1|>1-2\theta c_0\right\}\le CT^{-1}. \qquad \ldots(2.2)$$

Let b_0 be some positive number. For $w\in A_T\cap B_T$ and for all $T>T_0$ with $4b_0(\log T_0)^{1/2}(\frac{2\theta}{T_0})^{1/2}\le c_0$, we have

$$\begin{aligned}
&(\frac{T}{2\theta})^{1/2}(\theta_T-\theta)\le x\\
\Rightarrow\quad & I_T+b_0T(\theta_T-\theta)<I_T+(\frac{T}{2\theta})^{1/2}2b_0\theta x\\
\Rightarrow\quad & (\frac{T}{2\theta})^{1/2}(\theta_T-\theta)[I_T+b_0T(\theta_T-\theta)]<x[I_T+(\frac{T}{2\theta})^{1/2}2b_0\theta x]\\
\Rightarrow\quad & (\theta_T-\theta)I_T+b_0T(\theta_T-\theta)^2<(\frac{2\theta}{T})^{1/2}I_Tx+2b_0\theta x^2\\
\Rightarrow\quad & Z_T+(\theta_T-\theta)I_T+b_0T(\theta_T-\theta)^2<Z_T+(\frac{2\theta}{T})^{1/2}I_Tx+2b_0\theta x^2\\
\Rightarrow\quad & 0<Z_T+(\frac{2\theta}{T})^{1/2}I_Tx+2b_0\theta x^2
\end{aligned}$$

since

$$\begin{aligned} I_T + b_0 T(\theta_T - \theta) &> Tc_0 + b_0 T(\theta_T - \theta) \\ > 4b_0(\log T)^{1/2}(\frac{2\theta}{T})^{1/2} - 2b_0(\log T)^{1/2}(\frac{2\theta}{T})^{1/2} &= 2b_0(\log T)^{1/2}(\frac{2\theta}{T})^{1/2} > 0. \end{aligned}$$

On the other hand, for $w \in A_T \cap B_T$ and for all $T > T_0$ with $4b_0(\log T_0)^{1/2}(\frac{2\theta}{T_0})^{1/2} \leq c_0$, we have

$$\begin{aligned} &(\frac{T}{2\theta})^{1/2}(\theta_T - \theta) > x \\ \Rightarrow\ & I_T - b_0 T(\theta_T - \theta) < I_T - (\frac{T}{2\theta})^{1/2} 2b_0\theta x \\ \Rightarrow\ & (\frac{T}{2\theta})^{1/2}(\theta_T - \theta)[I_T - b_0 T(\theta_T - \theta)] > x[I_T - (\frac{T}{2\theta})^{1/2} 2b_0\theta x] \\ \Rightarrow\ & (\theta_T - \theta) I_T - b_0 T(\theta_T - \theta)^2 > (\frac{2\theta}{T})^{1/2} I_T x - 2b_0\theta x^2 \\ \Rightarrow\ & Z_T + (\theta_T - \theta) I_T - b_0 T(\theta_T - \theta)^2 > Z_T + (\frac{2\theta}{T})^{1/2} I_T x - 2b_0\theta x^2 \\ \Rightarrow\ & 0 > Z_T + (\frac{2\theta}{T})^{1/2} I_T x - 2b_0\theta x^2 \end{aligned}$$

since

$$\begin{aligned} I_T - b_0 T(\theta_T - \theta) &> Tc_0 - b_0 T(\theta_T - \theta) \\ > 4b_0(\log T)^{1/2}(\frac{2\theta}{T})^{1/2} - 2b_0(\log T)^{1/2}(\frac{2\theta}{T})^{1/2} &= 2b_0(\log T)^{1/2}(\frac{2\theta}{T})^{1/2} > 0. \end{aligned}$$

Hence

$$0 < Z_T + (\frac{2\theta}{T})^{1/2} I_T x - 2b_0\theta x^2 \Rightarrow (\frac{T}{2\theta})^{1/2}(\theta_T - \theta) \leq x.$$

Letting $D^{\pm}_{T,x} := \left\{ Z_T + (\frac{2\theta}{T})^{1/2} I_T x \pm 2b_0\theta x^2 > 0 \right\}$, we obtain

$$D^{-}_{T,x} \cap A_T \cap B_T \subseteq A_T \cap B_T \cap \left\{ (\frac{T}{2\theta})^{1/2}(\theta_T - \theta) \leq x \right\} \subseteq D^{+}_{T,x} \cap A_T \cap B_T. \quad \ldots\ldots (2.3)$$

If it is shown that

$$\left| P\left\{ D^{\pm}_{T,x} \right\} - \Phi(x) \right| \leq C T^{-1/2} \quad \ldots (2.4)$$

for all $T > T_0$ and $|x| \leq 2(\log T)^{1/2}$, then the theorem would follow from (2.1) - (2.3).

We shall prove (2.4) for $D^{+}_{T,x}$. The proof for $D^{-}_{T,x}$ is analogous. Note that

$$\begin{aligned} &\left| P\left\{ D^{+}_{T,x} \right\} - \Phi(x) \right| \\ = &\left| P\left\{ -(\frac{2\theta}{T})^{1/2} Z_T - \left(\frac{2\theta}{T} I_T - 1 \right) x < x + 2(\frac{2\theta}{T})^{1/2} b_0\theta x^2 \right\} - \Phi(x) \right| \end{aligned}$$

$$\begin{aligned}
&\leq \sup_{y\in\mathbb{R}} \left| P\left\{ -(\frac{2\theta}{T})^{1/2} Z_T - \left(\frac{2\theta}{T} I_T - 1\right) x \leq y \right\} - \Phi(y) \right| \\
&\quad + \left| \Phi\left(x + (\frac{2\theta}{T})^{1/2} b_0 \theta x^2 \right) - \Phi(x) \right| \\
&=: \Delta_1 + \Delta_2. \qquad \ldots (2.5)
\end{aligned}$$

Lemma 2.2 (b) and Esseen's lemma immediately yield

$$\Delta_1 \leq CT^{-1/2}. \qquad \ldots (2.6)$$

On the other hand, for all $T > T_0$,

$$\Delta_2 \leq 2(\frac{2\theta}{T})^{1/2} b_0 \theta x^2 (2\pi)^{-1/2} \exp(-\overline{x}^2/2)$$

where

$$|\overline{x} - x| \leq 2(\frac{2\theta}{T})^{1/2} b_0 \theta x^2.$$

Since $|x| \leq 2(\log T)^{1/2}$, it follows that $|\bar{x}| > |x|/2$ for all $T > T_0$ and consequently

$$\Delta_2 \;\leq\; 2(\frac{2\theta}{T})^{1/2} b_0 \theta x^2 (2\pi)^{-1/2} x^2 \exp(-x^2/8) \;\leq\; CT^{-1/2}. \qquad \ldots (2.7)$$

From (2.5) - (2.7), we obtain

$$\left| P\left\{ D^+_{T,x} \right\} - \Phi(x) \right| \leq CT^{-1/2}.$$

This completes the proof of the theorem. □

Remarks. (1) The bounds in Theorem 2.5 are uniform over compact subsets of the parameter space Θ.

(2) The bounds in Theorem 2.5 cannot be improved further.

(3) Here we have studied the properties of the MLE in the ergodic case, i.e., when $\theta > 0$. Note that in the nonergodic case i.e., when $\theta < 0$, when normalized by a nonrandom norming $e^{\theta T}$, converges to a Cauchy distribution (see Kutoyants (1994)) and when normalized by a random norming (by I_T), converges to a normal distribution. It remains to obtain the rate of convergence in this case. Note that in the critical case, i.e., when $\theta = 0$, the MlE has a distribution concentrated on a half line, precisely the distribution of the ratio of a noncentral chisquare to the to the sum of chisquares.

(4) In a Bayesian framework, the rates of convergence of the posterior distributions and the Bayes estimators has been studied in Bishwal (1998).

(5) The rates of convergence of the conditional least squares estimator and an approximate maximum likelihood estimator when the O-U process is observed at discrete time points in $[0, T]$ has been studied in Bishwal and Bose (1998).

(6) The rates of convergence of the MLE in the nonlinear homogeneous and nonhomogeneous equations remains open.

(7) Extention to multidimensional process and and to multiparameter case remains to be investigated.

(8) It remains to investigate the nonuniform rates of convergence to normality which are more useful.

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J. P. N. Bishwal
Stat-Math Unit
Indian Statistical Institute
203 B.T.Road, Calcutta-700 035
India
e-mail : tsmv9705@isical.ac.in