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A NOTE ON THE MULTIPLIERS AND PROJECTIVE REPRESENTATIONS OF SEMI-SIMPLE LIE GROUPS*

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SUMMARY. We show that, for any connected semi-simple Lie group G, there is a natural isomorphism between the Galois cohomology $H^2(G, \mathbb{T})$ (with respect to the trivial action of G on the circle group T) and the Pontryagin dual of the homology group $H_1(G)$ (with integer coefficients) of G as a manifold. As an application, we find that there is a natural correspondence between the projective representations of any such group and a class of ordinary representations of its universal cover. We illustrate these ideas with the example of the group of bi-holomorphic automorphisms of the unit disc.

1. **Representations and Multipliers**

Let G be a locally compact second countable topological group. For any separable complex Hilbert space $\mathcal{H}$, the group of all unitary operators on $\mathcal{H}$ will be denoted by $\mathcal{U}(\mathcal{H})$. All the usual topologies on the space of bounded linear operators on $\mathcal{H}$ induce the same Borel structure on $\mathcal{U}(\mathcal{H})$. We shall think of $\mathcal{U}(\mathcal{H})$ as a Borel group equipped with this Borel structure. $\mathbb{T}$ and $\mathbb{Z}$ will denote the circle group and the additive group of integers, respectively. $\mathbb{R}$ and $\mathbb{C}$ will denote the real line and the complex plane, as usual. $\mathbb{D}$ will stand for the unit disc in $\mathbb{C}$.

Recall that a measurable function $\pi : G \to \mathcal{U}(\mathcal{H})$ is called a *projective representation* of G on the Hilbert space $\mathcal{H}$ if there is a function (necessarily Borel) $m : G \times G \to \mathbb{T}$ such that

$$\pi(1) = I, \ \pi(g_1 g_2) = m(g_1, g_2)\pi(g_1)\pi(g_2) \tag{1.1}$$

for all g_1, g_2 in G. (More precisely, such a function π is called a projective unitary representation of G; however, we shall often drop the adjective unitary since all representations considered in this paper are unitary.) The projective representation π is called an ordinary representation (and we drop the adjective "projective") if m is the constant function 1. The function m associated with the projective

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representation π via (1.1) is called the multiplier of π. The ordinary representation of G which sends every element of G to the identity operator on a one dimensional Hilbert space is called the identity (or trivial) representation of G. It is surprising that although projective representations have been with us for a long time (particularly in the Physics literature), no suitable notion of equivalence of projective representations seems to be available. We offer the following

DEFINITION 1.1. Two projective representations $\pi_1,\ \pi_2$ of G on the Hilbert spaces $\mathcal{H}_1$, $\mathcal{H}_2$ (respectively) will be called equivalent if there exists a unitary operator $U : \mathcal{H}_1 \to \mathcal{H}_2$ and a function (necessarily Borel) $f : G \to \mathbb{T}$ such that $\pi_2(\varphi)U = f(\varphi)U\pi_1(\varphi)$ for all $\varphi \in G$.

We shall identify two projective representations if they are equivalent. This has the some what unfortunate consequence that any two one dimensional projective representations are identified. But this is of no importance if the group G has no ordinary one dimensional representation other than the identity representation (as is the case for all semi-simple Lie groups G). In fact, this notion of equivalence (and the resulting identifications) save us from the following disastrous consequence of the above (commonly accepted) definition of projective representations : any Borel function from G into $\mathbb{T}$ is a (one dimensional) projective representation of the group !!

1.1 *Multipliers and Cohomology.* Notice that the requirement (1.1) on a projective representation implies that its associated multiplier m satisfies

$$m(\varphi, 1) = 1 = m(1, \varphi),\ m(\varphi_1, \varphi_2)m(\varphi_1\varphi_2, \varphi_3) = m(\varphi_1, \varphi_2\varphi_3)m(\varphi_2, \varphi_3) \qquad (1.2)$$

for all elements $\varphi, \varphi_1, \varphi_2, \varphi_3$ of G. Any Borel function $m : G \times G \to \mathbb{T}$ satisfying (1.2) is called a *multiplier* of G. The set of all multipliers on G form an abelian group $M(G)$, called the multiplier group of G. If $m \in M(G)$, then taking $\mathcal{H} = L^2(G)$ (with respect to Haar measure on G), define $\pi : G \to \mathcal{U}(\mathcal{H})$ by

$$\Big(\pi(\varphi)f\Big)(\psi) = \bar{m}(\psi^{-1}, \varphi)f(\varphi^{-1}\psi) \qquad (1.3)$$

for φ, ψ in G, f in $L^2(G)$. Then one readily verifies that π is a projective representation of G with associated multiplier m. Thus each element of $M(G)$ actually occurs as the multiplier associated with a projective representation. A multiplier $m \in M(G)$ is called *exact* if there is a Borel function $f : G \to \mathbb{T}$ such that $m(\varphi_1, \varphi_2) = \frac{f(\varphi_1)f(\varphi_2)}{f(\varphi_1\varphi_2)}$ for φ_1, φ_2 in G. Equivalently, m is exact if any projective representation with multiplier m is equivalent to an ordinary representation. The set $M_0(G)$ of all exact multipliers on G form a subgroup of $M(G)$. Two multipliers m_1, m_2 are said to be equivalent if they belong to the same coset of $M_0(G)$. In other words, m_1 and m_2 are equivalent if there exists equivalent projective representations $\pi_1,\ \pi_2$ whose multipliers are m_1 and m_2 respectively. The quotient $M(G)/M_0(G)$ is denoted by $H^2(G, \mathbb{T})$ and is called the second cohomology group of G with respect to the trivial action of G on $\mathbb{T}$ (see Moore, 1964 for the relevant group cohomology theory). For $m \in M(G)$, $[m] \in H^2(G, \mathbb{T})$ will denote the cohomology class containing m, i.e., $[\ \ ] : M(G) \to H^2(G, \mathbb{T})$ is the canonical homomorphism.

The connected semi-simple Lie groups arise geometrically as the connected component of the identity in the full group of bi-holomorphic automorphisms of irreducible bounded symmetric domains. Our interest in the multipliers and projective representations of these groups arose from a study of the homogeneous tuples of Hilbert space operators modelled on these domains. For a discussion of this interesting connection, (see the papers of Bagchi and Misra (1995, 1996) and Misra and Sastry (1990)).

The following theorem (and its proof) provides an explicit description of $H^2(G, \mathbb{T})$ for any connected semi-simple Lie group G.

THEOREM 1.1. *Let G be a connected semi-simple Lie group. Then $H^2(G, \mathbb{T})$ is naturally isomorphic to the Pontryagin dual $\widehat{\pi^1(G)}$ of the fundamental group $\pi^1(G)$ of G.*

PROOF. Let $\tilde{G}$ be the universal cover of G and let $\pi : \tilde{G} \to G$ be the covering map. The fundamental group $\pi^1(G)$ is naturally identified with the kernel Z of π. Note that Z is contained in the centre of $\tilde{G}$ (any discrete normal subgroup of a connected topological group is central). Once for all, fix a Borel section $s : G \to \tilde{G}$ for the covering map (i.e., s is a Borel function such that $\pi \circ s$ is the identity on G, and $s(1) = 1$).

For $\chi \in \widehat{Z}$, define $m_\chi : G \times G \to \mathbb{T}$ by :

$$m_\chi(x, y) = \chi(s(y)^{-1} s(x)^{-1} s(xy)), \quad x, y \in G. \tag{4}$$

Claim: $\chi \mapsto [m_\chi]$ is an isomorphism from $\widehat{Z}$ onto $H^2(G, \mathbb{T})$. (We shall see that this isomorphism is independent of our choice of the section s.)

To see that m_χ is in $M(G)$, define $f_\chi : \tilde{G} \to \mathbb{T}$ by

$$f_\chi(x) = \chi(x^{-1} \cdot s \circ \pi(x)), \quad x \in \tilde{G}. \tag{1.5}$$

Then (using the fact that Z is central in $\tilde{G}$ and hence the element $x^{-1} \cdot s \circ \pi(x)$ of Z commutes with the element y of $\tilde{G}$), one readily verifies that

$$m_\chi(\pi(x), \pi(y)) = \frac{f_\chi(xy)}{f_\chi(x) f_\chi(y)}, \quad x, y \in \tilde{G}. \tag{1.6}$$

Since the right hand side of equation (1.6) is clearly a multiplier on $\tilde{G}$, so is the left hand side. Since π is a group homomorphism of $\tilde{G}$ onto G, it follows that m_χ is a multiplier on G.

Thus, $\chi \mapsto [m_\chi]$ is a group homomorphism from $\widehat{Z}$ into $H^2(G, \mathbb{T})$. To see that its kernel is trivial, let m_χ be an exact multiplier. So, there is a Borel function $f : G \to \mathbb{T}$ such that $m_\chi(x, y) = \frac{f(xy)}{f(x)f(y)}$ for x, y in G. Combined with Equation (1.6), this shows that $f \circ \pi / f_\chi$ is a group homomorphism from $\tilde{G}$ into $\mathbb{T}$. Since G (and hence also $\tilde{G}$) is a semi-simple Lie group, the only such homomorphism is the trivial one. (A semi-simple Lie group is its own commutator, so that there is no non-trivial homomorphism from such a group into any abelian group.) So

$f \circ \pi = f_\chi$. But $f \circ \pi$ is a constant on the kernel Z of π, while the restriction of f_χ to Z is χ^{-1}. So χ is the trivial character of Z.

Now, to show that $\chi \mapsto [m_\chi]$ is onto, let m be any multiplier on G. We must show that m is equivalent to m_χ for some (necessarily unique) character χ of Z. Define $\tilde{m} : \tilde{G} \times \tilde{G} \to \mathbb{T}$ by $\tilde{m}(x, y) = m(\pi(x), \pi(y))$. Clearly, $\tilde{m}$ is a multiplier on $\tilde{G}$. But, since $\tilde{G}$ is a connected and simply connected semi-simple Lie group, Theorem 7.37 in Varadarajan (1985) (in conjunction with the Levy-Malcev theorem) shows that $H^2(\tilde{G}, \mathrm{T})$ is trivial.

Hence $\tilde{m}$ is exact, i.e., there is a Borel function $f : \tilde{G} \to \mathbb{T}$ such that

$$m(\pi(x), \pi(y)) = \tilde{m}(x, y) = \frac{f(xy)}{f(x)f(y)}, \quad x, y \in \tilde{G}. \tag{1.7}$$

This shows that the restriction of f to Z is a character of Z, say it is χ^{-1}. (Note that χ is Borel and hence continuous : any Borel homomorphism between locally compact second countable groups is automatically continuous). Hence (1.7) also shows that

$$f(xy) = f(x)\chi^{-1}(y), \ x \in \tilde{G}, \ y \in Z. \tag{1.8}$$

Define the Borel function $g : G \to \mathbb{T}$ by $g = f \circ s$. By (1.8), we get $f(s(x)z) = g(x)\chi^{-1}(z), \quad x \in G, \ z \in Z$. Thus we find that, for $x, y \in G$,

$$\begin{aligned} m(x, y) &= m(\pi(s(x)), \pi(s(y))) \\ &= \frac{f(s(x)s(y))}{f(s(x))f(s(y))} \\ &= \frac{f(s(xy)s(xy)^{-1}s(x)s(y))}{f(s(x))f(s(y))} \\ &= \chi^{-1}(s(xy)^{-1}s(x)s(y)) \cdot \frac{g(xy)}{g(x)g(y)} \\ &= m_\chi(x, y) \cdot \frac{g(xy)}{g(x)g(y)}, \end{aligned}$$

which shows that m is equivalent to m_χ.

Finally, to show that the isomorphism $\chi \mapsto [m_\chi]$ does not depend on the section s, let $t : G \to \tilde{G}$ be any other Borel section for π. Fix $\chi \in \widehat{Z}$ and let m_χ^* be the multiplier on G obtained by replacing s by t in the formula 1.4. Since both s and t are Borel sectoins for π, there is a Borel function $u : G \to Z$ such that $t(x) = s(x)u(x), \quad x \in G$. Define the Borel function $v : G \to \mathbb{T}$ by $v(x) = \chi(u(x)), \quad x \in G$. Then it is easy to verify that $m_\chi^*(x, y) = m_\chi(x, y)v(xy)/(v(x)v(y)), \quad x, y \in G$. Hence $[m_\chi^*] = [m_\chi]$. $\square$

Remark / Question. Since $\pi^1(G) = Z$ is abelian, we have $\pi^1(G) = H_1(G)$ (singular homology with integer coefficients). (In general, the fundamental group $\pi^1(G)$ is the abelianisation of the homology group $H_1(G)$.) Therefore Theorem 1.1 may be written as

$$H^2(G, \mathrm{T}) = \mathrm{Hom}(H_1(G), \mathrm{T})$$

where the left hand side refers to group cohomology and the right hand side refers to singular homology of G as a manifold. Therefore it is natural to ask if this theorem is a special case of a strange duality relating the entire group cohomology sequence with the entire manifold homology sequence fo a semi-simple Lie group.

The following companion theorem shows that to find all the irreducible projective representations of a group satisfying the hypotheses of Theorem 1.1, it suffices to find the ordinary irreducible representations of its universal cover. To state this result, we need :

Definition 1.2. Let $\tilde{G}$ and its central subgroup Z be as in the proof of Theorem 1.1. Let β be an ordinary unitary representation of $\tilde{G}$. Then we shall say that β is of *pure type* if there is a character χ of Z such that $\beta(z) = \chi(z)I$ for all z in Z. If we wish to emphasize the particular character which occurs here, we may also say that β is pure of type χ. Notice that, if β is irreducible then (as Z is central) by Schur's Lemma β is necessarily of pure type.

Theorem 1.2. *Let G be a connected semi-simple Lie group and let $\tilde{G}$ be its universal cover. Then there is a natural bijection between (the equivalence classes of) projective unitary representations of G and (the equivalence classes of) ordinary unitary representations of pure type of $\tilde{G}$. Under this bijection, for each χ the projective representations of G with multiplier m_χ correspond to the representations of $\tilde{G}$ of pure type χ, and vice versa. Further, the irreducible projective representations of G correspond to the irreducible representations of $\tilde{G}$, and vice versa.*

Proof. We shall continue to to use the notations introduced in the proof of the previous theorem. In particular, for any multiplier m on G, $\tilde{m}$ will denote the multiplier on $\tilde{G}$ obtained by lifting m. Let α be any projective representation of G, say with associated multiplier m. In view of Theorem 1.1, we may assume without loss (by replacing α by a suitable equivalent representation, if necessary) that $m = m_\chi$ for some character χ of Z. Let $\tilde{\alpha} := \alpha \circ \pi$ be the projective representation of $\tilde{G}$ obtained by lifting α. Then the multiplier associated with $\tilde{\alpha}$ is $\tilde{m}$. But, by Equation (1.6), we have $\tilde{m}(x,y) = f_\chi(xy)/(f_\chi(x)f_\chi(y))$, where f_χ is as in Equation (1.5). Therefore, if we define β on $\tilde{G}$ by $\beta(x) = f_\chi(x)^{-1}\tilde{\alpha}(x), \quad x \in \tilde{G}$, then β is an ordinary representation which is of pure type χ. We claim that $\alpha \mapsto \beta$ is the required bijection. Clearly if α is irreducible then so is β, and vice versa. Equally clearly, this map respects equivalence and is one to one. To see that it is onto, fix any ordinary representation β of $\tilde{G}$, which is of pure type χ. Thust $\beta(z) = \chi(z)I$ for all z in Z. Define $\tilde{\alpha}$ on $\tilde{G}$ by $\tilde{\alpha}(x) = f_\chi(x)\beta(x)$. Then $\tilde{\alpha}$ is a projective representation of $\tilde{G}$ which is trivial on Z. Therefore there is a well defined (and uniquely determined) projective representation α of G such that $\tilde{\alpha} = \alpha \circ \pi$. Clearly the map $\beta \mapsto \alpha$ is the inverse of the map defined before. □

1.2 Example. the Möbius group. As an illusration of these two theorems, let's work out the details for the Möbius group of all biholomorphic automorphisms of $\mathbb{D}$. For brevity, we denote this group by Möb . Recall that Möb $= \{\varphi_{\alpha,\beta} : \alpha \in \mathbb{T}, \beta \in \mathbb{D}\}$, where

$$\varphi_{\alpha,\beta}(z) = \alpha\frac{\beta - z}{1 - \bar{\beta}z}, \quad z \in \mathbb{D} \tag{1.9}$$

Möb is topologised via the obvious identification with $\mathbb{T} \times \mathbb{D}$. With this topology, Möb becomes a connected semi-simple Lie group. Abstractly, it is isomorphic to $PSL(2, \mathbb{R})$ and to $PSU(1,1)$.

Define the Borel function n : Möb $\times$ Möb $\to \mathbb{Z}$ by

$$n(\varphi_1^{-1}, \varphi_2^{-1}) = \frac{1}{2\pi}\big\{\arg((\varphi_2\varphi_1)'(0)) - \arg(\varphi_1'(0)) - \arg(\varphi_2'(\varphi_1(0)))\big\}.$$

(Here $\arg(\varphi'(z)) := \mathrm{Im}\,\log(\varphi'(z))$, while $(z, \varphi) \mapsto \log(\varphi'(z))$ is a Borel function on Möb $\times\, \mathbb{D}$ which determines an analytic branch of the logarithm of φ' for each fixed $\varphi \in$ Möb. For any $w \in \mathbb{T}$, define m_w : Möb $\times$ Möb $\to \mathbb{T}$ by

$$m_w(\varphi_1, \varphi_2) = w^{n(\varphi_1,\varphi_2)}.$$

Proposition 1.1. *For $w \in \mathbb{T}$, m_w is a multiplier of Möb. It is trivial if and only if $w = 1$. Every multiplier on Möb is equivalent to m_w for a uniquely determined w in $\mathbb{T}$. In other words, $w \mapsto [m_w]$ is a group isomorphism between the circle group $\mathbb{T}$ and the second cohomology group $H^2(Möb, \mathbb{T})$.*

Proof. Topologically, Möb may be identified with $\mathbb{T}\times\mathbb{D}$ via $(\alpha, \beta) \mapsto \varphi_{\alpha,\beta}$, with the notation as in Equation 1.9. Accordingly, the universal cover of Möb is identified with $\mathbb{R} \times \mathbb{D}$, and the covering map π is given by $(t, \beta) \mapsto (e^{2\pi it}, \beta)$. Thus the kernel of π is naturally identified with the integer group $\mathbb{Z}$, and its Pontryagin dual is $\mathbb{T}$. A section $s : \mathbb{T} \times \mathbb{D} \to \mathbb{R} \times \mathbb{D}$ may be chosen to be given by $(\alpha, \beta) \mapsto (\frac{1}{2\pi}\arg(\alpha), \beta)$. Now, if one keeps all these identifications in mind, then a simple calculation shows that this theorem is just the specialisation of Theorem 1.1 to Möb. □

Every projective representation of Möb is a direct integral of irreducible projective representations. Hence, for many purposes, it suffices to have a complete list of these irreducible representations. A complete list of the (ordinary) irreducible unitary representations of the universal cover was obtained by Bargmann (see Sally, 1967 for instance). Since Möb is a semi-simple and connected Lie group, one may manufacture all the irreducible projective representations of Möb (with Bargmann's list as the starting point) via Theorem 1.2. We proceed to describe the result. (*Warning* : Our parametrisation of these representations differs somewhat from the one used by Bargmann and Sally. We have changed the parametrisation in order to produce a unified description.)

For $n \in \mathbb{Z}$, let $f_n : \mathbb{T} \to \mathbb{T}$ be defined by $f_n(z) = z^n$. In all of the following examples, the Hilbert space H is spanned by an orthogonal set $\{f_n : n \in I\}$ where I is some subset of Z. Thus the Hilbert space is specified by the set I and the sequence $\{\|f_n\|, n \in I\}$. In each case, $\|f_n\|$ behaves at worst like a polynomial in $|n|$ as $|n| \to \infty$, so that this really defines a space of function on $\mathbb{T}$. For complex parameters λ and μ, define the representation R_λ, μ by the formula

$$(R_{\lambda,\mu}(\varphi^{-1})f)(z) = \varphi'(z)^{\lambda/2}|\varphi'(z)|^{\mu}(f(\varphi(z))), \quad z \in \mathbb{T},\ f \in \mathcal{F},\ \varphi \in \text{Möb}.$$

Thus the description of the representation is complete if we specify I, $\{\|f_n\|^2, n \in I\}$ and the two parameters λ, μ. Of course, there is no a priori guarantee that

$R_{\lambda,\mu}(\varphi)$ is a unitary operator for φ in Möb. However, when it is, it is easy to see that the associated multiplier is m_w, where $w = e^{i\pi\lambda}$.

In terms of these notations, here is the complete list of the irreducible projective unitary representations of Möb. (However, see the concluding remark of this section.)

• "Discrete series" representations D_λ^+: Here $\lambda > 0,\ \mu = 0,\ I = \{n \in Z : n \geq 0\}$ and $\|f_n\|^2 = \frac{\Gamma(n+1)\Gamma(\lambda)}{\Gamma(n+\lambda)}$ for $n \geq 0$. For each f in the representation space there is an $\tilde{f}$, analytic in $\mathbb{D}$, such that f is the non-tangential boundary value of $\tilde{f}$. By the identification $f \leftrightarrow \tilde{f}$, the representation space may be identified with the functional Hilbert space $\mathcal{H}^{(\lambda)}$ of analytic functions on $\mathbb{D}$ with the reproducing kernel $(1 - z\bar{w})^{-\lambda},\ z, w \in \mathbb{D}$.

• "Discrete series" representations $D_\lambda^-,\ \lambda > 0$. There is an outer automorphism $*$ of order two on Möb given by $\varphi^*(z) = \overline{\varphi(\bar{z})},\ z \in \mathbb{D}$. D_λ^- may be defined as the composition of D_λ^+ with this automorphism : $D_\lambda^-(\varphi) = D_\lambda^+(\varphi^*),\ \varphi$ in Möb. This may be realized on a functional Hilbert space of anti-holomorphic functions on $\mathbb{D}$, in a natural way.

• "Principal series" representations $P_{\lambda,s},\ -1 < \lambda \leq 1,\ s$ purely imaginary. Here $\lambda = \lambda,\ \mu = \frac{1-\lambda}{2} + s,\ I = Z,\ \|f_n\|^2 = 1$ for all n. (so the space is $L^2(\mathbb{T})$).

• "Complementary series" representation $C_{\lambda,\sigma},\ -1 < \lambda < 1,\ 0 < \sigma < \frac{1}{2}(1-|\lambda|)$. Here $\lambda = \lambda,\ \mu = \frac{1}{2}(1 - \lambda) + \sigma,\ I = Z$, and

$$\|f_n\|^2 = \prod_{k=0}^{|n|-1} \frac{k \pm \frac{\lambda}{2} + \frac{1}{2} - \sigma}{k \pm \frac{\lambda}{2} + \frac{1}{2} + \sigma},\ n \in Z,$$

where one takes the upper or lower sign according as n is positive or negative.

Remark 1.1. All these projective representations of Möb are irreducible with the sole exception of $P_{1,0}$ for which we have the decomposition $P_{1,0} = D_1^+ \oplus D_1^-$.

Remark 1.2. According to the referee, Theorem 1.1 is folk-lore in the area. This may well be so but the authors have not been able to locate a precise statement of this result anywhere in the literature despite repeated attempts over last several years. Nor could many experts consulted shed any light on this matter. However, we need this result and its companion Theorem 1.2 in this precise form in order to complete the classification of the homogeneous weighted shifts obtained in Bagchi and Misra (2000). Technical details which are omitted here are available in Bagchi and Misra (2000).

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