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# CHARACTERISATION OF CORRELATION MATRICES OF SPIN VARIABLES

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*SUMMARY.* Necessary and sufficient conditions are given for a correlation matrix of order $n \geq 2$ to be the correlation matrix of spin variables in the classical sense. It is shown that Bell's inequalities (1964) are not sufficient for matrices of order $n \geq 5$.

## 1. **Introduction**

A random variable $\xi$ is called a spin variable if $P(\xi = 1) = P(\xi = -1) = \frac{1}{2}$. If $\{\xi_i, 1 \leq i \leq n\}$ is a family of spin variables and $\mathbb{E}(\xi_i \ \xi_j) = \sigma_{ij}$ so that $\sigma_{ii} = 1$ for every $i$, then the well-known Bell's inequalities (see Bell, 1964 and Parthasarathy, 1992) are

$$1 + \epsilon_i\epsilon_j\sigma_{ij} + \epsilon_i\epsilon_k\sigma_{ik} + \epsilon_j\epsilon_k\sigma_{jk} \geq 0 \quad \forall\, 1 \leq i < j < k \leq n. \qquad ...(1.1)$$

where $\epsilon_1, \epsilon_2, \cdots, \epsilon_n$ are $\pm 1$.

In Balasubramanian, Gupta and Parthasarathy (1998) it is shown that conditions (1.1) are necessary and sufficient when $n = 3$ or 4. Here we give necessary and sufficient conditions for *any* $n \geq 2$.

## 2. **Preliminaries**

Let $\Omega_n = \{-1, 1\}^n$ be the space of all $n$-length sequences $\omega = (\omega_1, \omega_2, \cdots, \omega_n)$ of $\pm 1$. For $T \subset \{1, 2, \cdots, n\}$ we denote by $\omega^T$ that sequence $\omega$ for which $\omega_i = -1$ for all $i$ in $T$ and $\omega_j = 1$ for all $j$ in $T'$ where $T'$ denotes the complement of $T$. Let $\xi_1, \xi_2, \cdots, \xi_n$ denote the coordinate variables on $\Omega_n : \xi_i(\omega) := \omega_i$, $1 \leq i \leq n$. For each $T \subset \{1, 2 \cdots, n\}$ we introduce probability $P^T$ as follows:

$$P^T(\{\omega^T\}) = P^T(\{\omega^{T'}\}) = \frac{1}{2}. \qquad \ldots(2.1)$$

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Under $P^T$, $\xi_1, \xi_2, \cdots, \xi_n$ are spin variables with correlation matrix

$$\Sigma^T = ((\sigma_{ij}^T)), \quad \sigma_{ii}^T = 1 \text{ for all } i \text{ and } \sigma_{ij}^T = (-1)^{|T \cap \{i,j\}|} \text{ for } i \neq j \quad \ldots (2.2)$$

We note that $P^T = P^{T'}$ and $\{ \Sigma^T : T \in \mathcal{T} \}$ where

$$\mathcal{T} = \{T \subset \{1, 2, \cdots, n\} : 1 \in T \} \quad \ldots (2.3)$$

gives an enumeration of $2^{n-1}$ distinct realisable correlation matrices of $n$ spin variables.

## 3. Realisable Correlation Matrices of Spin Variables

THEOREM 3.1. *The class of realisable correlation matrices of spin variables is given by*

$$\mathcal{C}_n = \text{Convex Hull}\{\Sigma^T : T \in \mathcal{T}\} \quad \ldots (3.1)$$

*where* $\Sigma^T$ *and* $\mathcal{T}$ *are given by* (2.2) *and* (2.3).

PROOF. *Sufficiency* : Clearly the matrix $\sum \lambda_T \Sigma^T$, $\lambda_T \geq 0$, $\sum \lambda_T = 1$, is the correlation matrix of $\xi_1, \xi_2, \cdots, \xi_n$ under the probability $P = \sum \lambda_T P^T$ on $\Omega_n$.

*Necessity* : Let $\Lambda$ be the correlation matrix of $n$ spin variables $\eta_1, \eta_2, \cdots, \eta_n$ defined, say, on the probability space $(\Omega, \mathcal{F}, Q)$. On $\Omega_n$ define $P$ as follows:

$$P(\{\omega^T\}) = P(\{\omega^{T'}) = \frac{1}{2} Q\{ \eta_i = -1 \; \forall \, i \in T, \; \eta_j = 1 \; \forall \, j \notin T \}$$

$$+\frac{1}{2} Q\{ \eta_i = 1 \; \forall \, i \in T, \; \eta_j = -1 \; \forall \, j \notin T \}, \; T \in \mathcal{T}.$$

Under $P$, $\xi_1, \xi_2, \cdots, \xi_n$ are spin variables with correlation matrix $\Lambda$. Clearly $P = \sum_T \lambda_T P^T$ with $\lambda_T = P(\{\omega^T\}) + P(\{\omega^{T'}\})$ so that $\Lambda = \sum \lambda_T \Sigma^T \in \mathcal{C}_n$. □

REMARK 3.1. The extreme points of $\mathcal{C}_n$ are given by $\{\Sigma^T : T \in \mathcal{T}\}$. Suppose

$$\Sigma^T = \alpha \Sigma_1 + (1 - \alpha) \Sigma_2, \quad 0 < \alpha < 1, \quad \Sigma_1, \Sigma_2 \in \mathcal{C}_n.$$

Restricting this to any $2 \times 2$ principal submatrix we get an identity of the form

$$\begin{pmatrix} 1 & \pm 1 \\ \pm 1 & 1 \end{pmatrix} = \alpha \begin{pmatrix} 1 & p_1 \\ p_1 & 1 \end{pmatrix} + (1 - \alpha) \begin{pmatrix} 1 & p_2 \\ p_2 & 1 \end{pmatrix}$$

where $1 - p_i^2 \geq 0$, $i = 1, 2$. If $1 - p_i^2 > 0$ then LHS is singular and RHS is nonsingular. So $p_i = \pm 1$, $i = 1, 2$. Varying the $2 \times 2$ principal submatrices we conclude that $\Sigma_1 = \Sigma_2 = \Sigma^T$ and hence $\Sigma^T$ is extremal.

REMARK 3.2. Parthasarathy (1998) has recently characterised the set $\mathcal{E}_n$ of extremal points of *all* correlation matrices of order $n$. He shows that

$$\mathcal{E}_n = \bigcup_{k:k\geq 1,\ k(k+1)\leq 2n} \mathcal{E}_{n,k} \quad \ldots (3.2)$$

where $\mathcal{E}_{n,k}$ is the set of extremal correlation matrices of order $n$ and rank $k$; he has a complete description of $\mathcal{E}_{n,k}$.

It is easily seen that

$$\partial\mathcal{C}_n = \mathcal{E}_{n,1}, \quad \ldots (3.3)$$

where $\partial\mathcal{C}_n$ stands for the set of extremal points of $\mathcal{C}_n$.

## 4. Examples

Let $n = 5$, $\sigma_{ii} = 1$ for all $i$, $\sigma_{1j} = \theta$ for $j = 2, 3, 4, 5$ and $\sigma_{ij} = \rho$ for $i \neq j = 2, 3, 4, 5$. Then it is easily seen that Bell's inequalities reduce to

$$-\frac{1}{3} \leq \rho \leq 1,\ 1 + 2\theta + \rho \geq 0 \ \text{ and } \ 1 - 2\theta + \rho \geq 0. \quad \ldots (4.1)$$

Also $\sum = ((\sigma_{ij}))$ is positive semi-definite if and only if

$$\theta^2 \leq \frac{1}{4}(1 + 3\rho). \quad \ldots (4.2)$$

EXAMPLE 1. Take $\theta = \frac{1}{4}$ and $\rho = -\frac{7}{24}$. Then $\sum$ satisfies Bell's inequalities but it is not positive semi-definite.

EXAMPLE 2. Take $\theta = \frac{1}{4}$ and $\rho = -\frac{5}{24}$. Then $\sum$ is positive semi-definite and satisfies Bell's inequalities. It is not difficult to check that this $\sum$ is not in $C$ given by (3.1) so that it is not realisable as a correlation matrix of spin variables.

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