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REMARKS ON BELL'S INEQUALITY FOR SPIN CORRELATIONS

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SUMMARY. It is shown that Bell's inequalities (1964) for spin correlations are sufficient for a correlation matrix of order ≤ 4 to be the correlation matrix of spin random variables in the classical sense. However, they are not sufficient for matrices of order ≥ 5. Every correlation matrix is realized as a quantum correlation matrix of spin variables.

1. **Introduction**

A random variable ξ on a probability space is called a *spin variable* if $P(\xi = 1) = P(\xi = -1) = \frac{1}{2}$. If $\{\xi_i,\ 1 \leq i \leq n\}$ is a family of spin variables and $\mathbb{E}\ \xi_i\xi_j = \sigma_{ij}$ so that $\sigma_{ii} = 1$, for every i, then the well-known Bell's inequalities (See Bell, 1964; Parthasarathy, 1992 and Varadarajan, 1985) can be expressed in the form

$$1 + \epsilon_i\epsilon_j\sigma_{ij} + \epsilon_j\epsilon_k\sigma_{jk} + \epsilon_k\epsilon_i\sigma_{ki} \geq 0 \ \ \forall i < j < k \leq n \qquad \ldots (1.1)$$

where $\epsilon_1, \epsilon_2, \ldots, \epsilon_n$ are ± 1.

It is not difficult to construct real non-negative definite matrices $\Sigma = ((\sigma_{ij}))$, $1 \leq i, j \leq n$, satisfying $\sigma_{ii} = 1$ for every i but not (1.1). Thus there arises the natural problem of finding simple verifiable conditions on Σ in order to ensure that it is the correlation matrix of n spin variables. Here we shall find such necessary and sufficient conditions when $n = 3$ or 4. We also obtain a characterisation of all correlation matrices of n exchangeable spin variables.

By using the observables of a fermion field obeying the canonical anticommutation relations it is possible to realize any non-negative matrix with diagonal entries unity as the correlation matrix of spin observables.

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2. **Preliminaries**

Let $\{\xi_i,\ 1 \leq i \leq n\}$ be spin observables. For any $S \subset \{1, 2, \ldots, n\}$ define

$$p_S = P\{\xi_i = -1 \quad \forall\, i \in S, \quad \xi_j = 1 \quad \forall\ j \notin S\}$$

$$\sigma_S = \begin{cases} \mathbb{E} \prod_{i \in S} \xi_i & \text{if } S \neq \phi, \\ 1 & \text{otherwise.} \end{cases}$$

Denote by $|S|$ the cardinality of S. We call the map $\sigma :\ S \to \sigma_S$ from subsets of $\{1, 2, \ldots, n\}$ into $[-1, 1]$ the *multiple correlation map* of the spin variables $\{\xi_i,\ 1 \leq i \leq n\}$.

By definition

$$\sigma_S = \sum_{T \subset \{1,2,\ldots,n\}} (-1)^{|S \cap T|} p_T. \qquad \ldots (2.1)$$

Since $((\ (-1)^{|S\cap T|}\ )), \quad S, T\ \subset \{1, 2, \ldots, n\}$ is a $2^n \times 2^n$ orthogonal matrix it follows, in particular, that

$$p_T = 2^{-n} \sum_{S \subset \{1,2,\ldots,n\}} (-1)^{|S \cap T|} \sigma_S. \qquad \ldots (2.2)$$

From (2.1) and (2.2) we conclude immediately the following:

Proposition 2.1. *Let $\sigma :\ S \to \sigma_S$ be a mapping from the space of all subsets of $\{1, 2, \ldots, n\}$ into the closed interval $[-1, 1]$ such that $\sigma_\phi = 1, \quad \sigma_{\{i\}} = 0 \quad \forall\, i$. Then σ is a multiple correlation map of n spin variables if and only if*

$$\sum_{S \subset \{1,2,\ldots,n\}} (-1)^{|S \cap T|} \sigma_S \geq 0 \quad \forall \quad T \subset \{1, 2, \ldots, n\}. \qquad \ldots (2.3)$$

We shall now prove the following:

Theorem 2.2. *Let $\Sigma = ((\sigma_{ij})), 1 \leq i, j \leq 3$ be a real symmetric matrix with $\sigma_{ii} = 1 \quad \forall\, i$. Then Σ is the correlation matrix of three spin variables if and only if*

$$\begin{aligned} &\min\{1 + \sigma_{12} + \sigma_{23} + \sigma_{31}, 1 - \sigma_{12} + \sigma_{23} - \sigma_{31}, \\ &1 - \sigma_{12} - \sigma_{23} + \sigma_{31}, 1 + \sigma_{12} - \sigma_{23} - \sigma_{31}\} \geq 0. \end{aligned} \qquad \ldots (2.4)$$

Proof. *Necessity*: This is immediate from (1.1).

Sufficiency: Choose any real number δ such that $|\delta|$ does not exceed the left hand side of the inequality (2.4). Define the map $S \to \sigma_S$ on the space of subsets of $\{1, 2, 3\}$ by

$$\sigma_\phi = 1, \sigma_{\{i\}} = 0, \sigma_{\{i,j\}} = \sigma_{ij},\ i \neq j, \quad \sigma_{\{1,2,3\}} = \delta.$$

Let $q_T := \sum_{S \subset \{1,2,3\}} (-1)^{|S \cap T|} \sigma_S$. Then by the choice of δ we have

$$q_\phi = 1 + \sigma_{12} + \sigma_{23} + \sigma_{31} + \delta \geq 0,$$

$$\begin{aligned} q_{\{1\}} &= 1 - \sigma_{12} - \sigma_{13} + \sigma_{23} - \delta \geq 0, \\ q_{\{2\}} &= 1 - \sigma_{12} - \sigma_{23} + \sigma_{31} - \delta \geq 0, \\ q_{\{3\}} &= 1 + \sigma_{12} - \sigma_{13} - \sigma_{23} - \delta \geq 0, \\ q_{\{1,2\}} &= 1 + \sigma_{12} - \sigma_{13} - \sigma_{23} + \delta \geq 0, \\ q_{\{2,3\}} &= 1 - \sigma_{12} - \sigma_{13} + \sigma_{23} + \delta \geq 0, \\ q_{\{1,3\}} &= 1 - \sigma_{12} + \sigma_{13} - \sigma_{23} + \delta \geq 0, \\ q_{\{1,2,3\}} &= 1 + \sigma_{12} + \sigma_{13} + \sigma_{23} - \delta \geq 0, \end{aligned}$$

Now sufficiency is immediate from Proposition (2.1). □

PROPOSITION 2.3. *Let $\sigma : S \to \sigma_S$, $S \subset \{1, 2, \ldots, n\}$ be a multiple correlation map for a family of n spin variables. Define the map $\tilde{\sigma} : S \to \tilde{\sigma}_S$ by*

$$\tilde{\sigma}_S = \begin{cases} \sigma_S & \text{if} \quad |S| \quad \text{is even} \\ 0 & \text{if} \quad |S| \quad \text{is odd.} \end{cases}$$

Then $\tilde{\sigma}$ is also a multiple correlation map for a family of n spin variables.

PROOF. For any $T \subset \{1, 2, \ldots, n\}$ denote by T' its complement. By Proposition 2.1 we have

$$\sum_{|S| \text{ odd}} (-1)^{|S \cap T|} \sigma_S + \sum_{|S| \text{ even}} (-1)^{|S \cap T|} \sigma_S \geq 0,$$

$$\sum_{|S| \text{ odd}} (-1)^{|S \cap T'|} \sigma_S + \sum_{|S| \text{ even}} (-1)^{|S \cap T'|} \sigma_S \geq 0.$$

Adding the two and using the relation $|S| = |S \cap T| + |S \cap T'|$ we get

$$\sum_{|S| \text{even}} (-1)^{|S \cap T|} \sigma_S \geq 0$$

or, equivalently,

$$\sum_S (-1)^{|S \cap T|} \tilde{\sigma}_T \geq 0.$$

The required result is now immediate from Proposition 2.1. □

THEOREM 2.4. *Let $\Sigma = ((\sigma_{ij}))$, $1 \leq i, j \leq 4$, $\sigma_{ii} = 1 \quad \forall\, i$. In order that Σ may be the correlation matrix of 4 spin variables it is necessary and sufficient that for any $1 \leq i < j < k \leq 4$*

$$\min\{1+\sigma_{ij}+\sigma_{jk}+\sigma_{ki}, 1-\sigma_{ij}+\sigma_{jk}-\sigma_{ki},$$
$$1-\sigma_{ij}-\sigma_{jk}+\sigma_{ki}, 1+\sigma_{ij}-\sigma_{jk}-\sigma_{ki}\} \geq 0 \qquad \ldots (2.5)$$

PROOF. Necessity is immediate from Theorem 2.2. To prove sufficiency it is enough to show the existence of a multiple correlation map $\sigma : S \to \sigma_S, \quad S \subset \{1, 2, \ldots, n\}$ for which $\sigma_\phi = 1, \quad \sigma_S = 0$ when $|S|$ is odd, $\sigma_{\{i,j\}} = \sigma_{ij}$ for $i \neq j$, and $\sigma_{\{1,2,3,4\}} = \delta$ is a suitable value. For such a map σ, the conditions $\sum_S (-1)^{|S\cap T|}\sigma_S \geq 0$ and $\sum_S (-1)^{|S\cap T'|}\sigma_S \geq 0$ are equivalent. In view of Proposition 2.1 it is enough to choose δ so that (2.3) holds for $T = \phi, \ \{i\}, \ 1 \leq i \leq 4, \{1,2\}, \{1,3\}, \{1,4\}$. This reduces to the following eight inequalities:

$$1+\sigma_{12}+\sigma_{13}+\sigma_{14}+\sigma_{23}+\sigma_{24}+\sigma_{34}+\delta \geq 0,$$
$$1-\sigma_{12}-\sigma_{13}-\sigma_{14}+\sigma_{23}+\sigma_{24}+\sigma_{34}-\delta \geq 0,$$
$$1-\sigma_{12}+\sigma_{13}+\sigma_{14}-\sigma_{23}-\sigma_{24}+\sigma_{34}-\delta \geq 0,$$
$$1+\sigma_{12}-\sigma_{13}+\sigma_{14}-\sigma_{23}+\sigma_{24}-\sigma_{34}-\delta \geq 0,$$
$$1+\sigma_{12}+\sigma_{13}-\sigma_{14}+\sigma_{23}-\sigma_{24}-\sigma_{34}-\delta \geq 0,$$
$$1+\sigma_{12}-\sigma_{13}-\sigma_{14}-\sigma_{23}-\sigma_{24}+\sigma_{34}+\delta \geq 0,$$
$$1-\sigma_{12}+\sigma_{13}-\sigma_{14}-\sigma_{23}+\sigma_{24}-\sigma_{34}+\delta \geq 0,$$
$$1-\sigma_{12}-\sigma_{13}+\sigma_{14}+\sigma_{23}-\sigma_{24}-\sigma_{34}+\delta \geq 0,$$

A detailed examination of these inequalities shows that this is equivalent to $|\delta| \leq \rho_{ijk}$ where ρ_{ijk} denotes the left hand side of (2.5) for each $1 \leq i < j < k \leq 4$. This completes the proof of sufficiency. □

3. Exchangeable Spin Variables

A family $\{\xi_i, \ 1 \leq i \leq n\}$ of spin variables is said to be *exchangeable* if the joint distributions of $(\xi_1, \xi_2, \ldots, \xi_n)$ and $(\xi_{\pi(1)}, \xi_{\pi(2)}, \ldots, \xi_{\pi(n)})$ are same for any permutation π of $\{1, 2, \ldots, n\}$.

PROPOSITION 3.1. *Let $(\xi_1, \xi_2, \ldots, \xi_n)$ be an exchangeable sequence of spin variables for which $\mathbb{E} \ \xi_i\xi_j = \sigma$ for $i \neq j$. Then*

$$1 \geq \sigma \geq \begin{cases} -\frac{1}{n-1} & \text{if } n \text{ is even,} \\ -\frac{1}{n} & \text{if } n \text{ is odd.} \end{cases} \qquad \ldots (3.1)$$

Conversely, for every σ satisfying (3.1) *there exists an exchangeable sequence $(\xi_1, \xi_2, \ldots, \xi_n)$ of spin variables such that $\mathbb{E} \ \xi_i\xi_j = \sigma$ for all $i \neq j$.*

PROOF. *Necessity*: The non-negative definiteness of the correlation matrix

$$\begin{pmatrix} 1 & \sigma & \sigma & \cdots & \sigma \\ \sigma & 1 & \sigma & \cdots & \sigma \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \sigma & \sigma & \ldots & \sigma & 1 \end{pmatrix}$$

implies that its determinant $(1-\sigma)^{n-1}(1+\overline{n-1}\ \sigma)$ is non-negative. Since $\sigma \leq 1$ it follows that $1+\overline{n-1}\,\sigma \geq 0$ and therefore $\sigma \geq -\frac{1}{n-1}$. If n is odd then $|\xi_1+\cdots+\xi_n| \geq 1$. Thus

$$1 \leq \mathbb{E}\ \ (\xi_1+\cdots+\xi_n)^2 = n+(n-1)n\sigma$$

which implies $\sigma \geq -\frac{1}{n}$, completing the proof of necessity.

To prove sufficiency, consider the uniform distribution of $(\xi_1, \xi_2, \ldots, \xi_n)$ with support in the set of all $n-$length sequences of ± 1 with exactly $[\frac{n}{2}] = m$ terms of one sign and the remaining $(n-m)$ terms of the opposite sign. Then $(\xi_1, \xi_2, \ldots, \xi_n)$ is an exchangeable sequence of spin variables with $\mathbb{E}\ \ \xi_i\xi_j = -\frac{1}{n-1}$ if n is even and $-\frac{1}{n}$ if n is odd for $i \neq j$.

If $\xi_1 = \xi_2 = \cdots = \xi_n$ and $P(\xi_1 = 1) = P(\xi_1 = -1) = \frac{1}{2}$ then $(\xi_1, \xi_2, \ldots, \xi_n)$ is an exchangeable sequence of spin variables with $\mathbb{E}\ \ \xi_i\xi_j = 1$.

The space of all permutation invariant probability distributions in the set of all $n-$length sequences of ± 1's is a convex set and the correlation $\mathbb{E}\ \ \xi_1\xi_2 = \sigma$ is a continuous function on this convex set which assumes the values $-\frac{1}{n-1}$ and 1 when n is even and $-\frac{1}{n}$ and 1 when n is odd. Thus σ assumes every value in between and the proof of sufficiency becomes complete. □

REMARK. Proposition 3.1 shows that when $n \geq 5$, non-negative definiteness together with (2.5) is not sufficient for Σ to be the correlation matrix of n spin variables.

Spin observables with a pre-assigned correlation matrix. Till now we dealt with only classical spin variables. We shall now take a look at the case of spin observables in the sense of quantum mechanics. By a *spin observable* we mean a selfadjoint operator X in a Hilbert space with spectrum $\{-1, 1\}$ together with a state ϱ such that X assumes the values -1 and 1 with probability $\frac{1}{2}$ each in the state ϱ.

Consider an arbitrary non-negative definite matrix $\Sigma = ((\sigma_{ij})),\quad 1 \leq i, j \leq n$ with $\sigma_{ii} = 1$ for every i and probably complex entries. Then it is possible to construct a Hilbert space $\mathcal{H}$ of dimension at most n and unit vectors $u_i \in \mathcal{H}, 1 \leq i \leq n$ such that $\sigma_{ij} =< u_i, u_j >$. (This is nothing but a special case of the GNS principle.) Consider the fermion Fock space $\Gamma(\mathcal{H})$ over $\mathcal{H}$ with vacuum vector Φ and fermion annihilation operators $\{a(u),\ u \in \mathcal{H}\}$ so that the canonical anticommutation relations hold:

$a(u)\Phi = o$
$a(u)a(v) + a(v)a(u) = o$
$a(u)a^\dagger(v) + a^\dagger(v)a(u) = < u, v >$

for all $u, v \in \mathcal{H}$, where $a^\dagger(u)$ is the adjoint of $a(u)$ and called the fermion creation operator associated with u and a scalar times the identity operator is denoted by the scalar itself. (See Parthasarathy, 1992). Define the selfadjoint operator $F(u) = a(u) + a^\dagger(u)$.

Then

$$F(u)F(v) + F(v)F(u) = 2 \, Re < u, v > .$$

In particular, $F(u)^2 = ||u||^2$. Since $< \Phi, \; F(u)\Phi >= 0$ it follows that $F(u)$ assumes the values $||u||$ and $-||u||$ with probability $\frac{1}{2}$ each in the vacuum state Φ. If $||u|| = 1, F(u)$ is a spin observable. In particular, $\{F(u_i), \; 1 \leq i \leq n\}$ is a family of spin observables satisfying

$$\begin{aligned} < \Phi, \; F(u_i) \; F(u_j) \; \Phi > &= < a^\dagger(u_i)\Phi, \; a^\dagger(u_j)\Phi > \\ &= < u_i, u_j > \\ &= \sigma_{ij}. \end{aligned}$$

Thus we have realized $\Sigma = ((\sigma_{ij}))$ as a quantum correlation matrix of n spin observables.

For any unitary operator U in $\mathcal{H}$ its second quantization $\Gamma(U)$ is a unitary operator in $\Gamma(\mathcal{H})$ satisfying $\Gamma(U)\Phi = \Phi, \Gamma(U)F(u)\Gamma(U)^{-1} = F(Uu)$. Now consider the case $\sigma_{ij} = \sigma$ for all $i \neq j$. A permutation of $\{u_i, \; 1 \leq i \leq n\}$ yields the unitary operator U in $\mathcal{H}$ and its second quantization $\Gamma(U)$ permutes the operators $\{F(u_i), \; 1 \leq i \leq n\}$ by conjugation. This shows that for any distinct $i_1, i_2, \ldots, i_k, \; < \Phi, \; F(u_{i_1})F(u_{i_2}) \cdots F(u_{i_k})\Phi >$ is independent of the set $\{i_1, i_2, \ldots, i_k\}$ when k is fixed between 1 and n. In fact, it is not difficult to see that for distinct $i_1, i_2, \ldots, i_k$ one has

$$< \Phi, \; F(u_{i_1})F(u_{i_2}) \cdots F(u_{i_k})\Phi >= \begin{cases} \sigma^{k/2} & \text{if } k \text{ is even} \\ 0 & \text{if } k \text{ is odd.} \end{cases}$$

Thus $\{F(u_i), \; 1 \leq i \leq n\}$ may be considered to be exchangeable in a quantum probability sense.

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