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THE RELIABILITY FUNCTION OF CONSECUTIVE-k-OUT-OF-n SYSTEMS FOR THE I.I.D. CASE

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SUMMARY. In this paper we derive expressions for the reliability functions of consecutive-k-out-of-n systems for the i.i.d. case, Application of these results for the computation of indices of structural importance of components is also considered.

1. **Introduction**

A consecutive-k-out-of-n : G (consecutive-k-out-of-n : F) system consists of n linearly ordered components and the system functions (fails) if and only if at least k consecutive components function (fail). We write $C(k, n : G)(C(k, n : F))$ as an abbreviated form of 'consecutive-k-out-of-n : G' (consecutive-k-out-of-n :F). When we are referring to both $C(k, n : G)$ and $C(k, n : F)$ systems we just write $C(k, n)$ systems. A $C(k, n : F)(C(k, n : G))$ system is the dual of a $C(k, n : G)(C(k, n : F))$ system (Chao *et al.* (1995), p. 123).

We denote by $g_k(p_1, p_2, \ldots, p_n : n)(f_k(p_1, p_2, \ldots, p_n : n))$ the reliability function of a $C(k, n : G)(C(k, n : F))$ system where p_i is the reliability of the component i and the components are s-independent. It is known that (Kuo *et al.* (1990), Lemma 1)

$$f_k(p_1, p_2, \ldots, p_n : n) = 1 - g_k(1 - p_1, 1 - p_2, \ldots, 1 - p_n : n).$$

For the case where $p_1 = p_2 = \ldots = p_n = p$, we write $g_k(p : n)(f_k(p : n))$ instead of $g_k(p, p, \ldots, p : n)(f_k(p, p, \ldots, p : n))$.

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In this paper, we consider the problem of evaluating $g_k(p:n)$ (or equivalently $f_k(p:n)$) for any given p. Several authors have obtained recursive formulae for $g_k(p:n)$ or $f_k(p:n)$ for this purpose. As an example, we have (Kuo *et al.* (1990), Lemma 2)

$$g_k(p:n) = g_k(p:n-1) + (1-p)p^k(1 - g_k(p:n-k-1))$$

for $n > k$. A $C(k,n:F)$ system can be modelled as a $(k+1)$ - state discrete time homogeneous Markov Chain with one absorbing state and k transient stages with $f_k(p:n)$ being interpreted as the probability that the number of steps to absorption is greater than n (Chao *et al.* (1995, p. 121)). Let $h_k(l:n)$ denote the number of path sets of size l of a $C(k,n:F)$ system. It is known that (Chao *et al.* (1995, p. 122) or Seth (1990, p. 129)

$$F_{k,n} = \sum_{r=0}^{n} h_k(r:n)$$

where $F_{k,n}$ is the n-th Fibonacci number of order k specified by $F_{k,r} = 2^r$ for $0 \le r \le k-1$. It is easy to see that

$$f_k(p:n) = \sum_{\ell=0}^{n} h_k(l:n)p^l(1-p)^{n-l}$$

and the summation is trivial when $p = \frac{1}{2}$. The $h_k(l:n)$ have a special combinatorial meaning and their computation involves either recursive relations or direct sums (Chao *et al.* (1995), p. 121).

In this paper we show that

$$g_k(p:n) = \sum_{r=1}^{\overline{k}(n)} (-1)^{r-1}(1-p)^{r-1}p^{rk}\left\{\binom{n-rk+1}{r}(1-p) + \binom{n-rk}{r-1}p\right\}$$

where $\overline{k}(n)$ is the degree of complexity of a $C(k,n)$ system which is by definition the integral part of $(n+1)/(k+1)$. A direct consequence of this are the following expressions for $F_{k,n}$ and $h_k(l:n)$:

$$F_{k,n} = 2^n - \sum_{r=1}^{\overline{k}(n)} (-1)^{r-1}\binom{n-rk}{r-1}\frac{n-r(k-1)+1}{r}2^{n-r(k+1)}$$

$$h_t(l:n) = \binom{n}{l} - (l+1)\sum_{r=1}^{\overline{k}(n)} \frac{(-1)^{(r-1)}}{n-rk+1}\binom{n-rk+1}{r, l-r+1, n-l-rk}.$$

The author believes that both the above expressions are new.

In Section 2 we introduce the notation and briefly discuss the required preliminary concepts. In Section 3 we derive an expression for $g_k(p : n)$ which is the main result of this paper. As a consequence we also derive expressions for $F_{k,n}$ and $h_k(l : n)$. In Section 4 we consider the use of the expression for $g_k(p : n)$ in the computation of indices of structural importance of components of $C(k, n)$ systems. In fact we derive an expression for the Barlow-Proschan index of structural importance.

2. Notation and Preliminary Concepts

The following notation is used throughout this paper and shall not be introduced again.

Z : the set of all integers,

N : the set of nonnegative integers,

N_+ : the set of positive integers,

$I(r, s) = \{j : r \leq j \leq s \text{ and } j \in Z\}$ for $r, s \in Z$,

$[x]$: integral part of the real number x,

$F_{k,r}$: Fibonacci number of order k defined by $F_{k,r} = 2^r$, $0 \leq r \leq k-1$ and

$$F_{k,r} = \sum_{1}^{k} F_{k,r-j}, r \geq k,$$

k : minimum number of consecutive components required to function (fail) for the system to function (fail), it is assumed that $k \geq 2$,

n : number of components,

$I(1, n)$: component set,

$C(k, n)$: consecutive-k-out-of-n,

$C(k, n : G)$: consecutive-k-out-of-n : G,

$C(k, n : F)$: consecutive-k-out-of-n : F,

$\overline{k}(n)$; degree of complexity of a $C(k, n)$ system which is by definition $\left[\frac{n+1}{k+1}\right]$,

p_i : reliability of component i,

p : a real number such that $0 \leq p \leq 1$,

$g_k(p_1, p_2, \ldots, p_n : n)$: reliability function of a $C(k, n : G)$ system, taken to be identically equal to zero for $n < k$,

$f_k(p_1, p_2, \ldots, p_n : n)$: reliability function of a $C(k, n : F)$ system, taken to be identically equal to one for $n < k$,

$g_k(p : n) = g_k(p, p, \ldots, p : n)$, $f_k(p : n) = f_k(p, p, \ldots, p : n)$,

$h_k(l : n)$: the number of path sets of size l of a $C(k, n : F)$ system,

$L_k(x : n) = \sum_{\ell=0}^{\infty} h_k(l : n)x^{\ell}$: the generating function of $h_k(\ell : n)$,

$g_k^j(p : n) = \frac{\partial g_k(p_1, p_2, \ldots, p_n : n)}{\partial p_i} |p_1 = p_2 = \ldots = p_n = p$,

$f_k^j(p : n) = \frac{\partial f_k(p_1, p_2, \ldots, p_n : n)}{\partial p_i} |p_1 = p_2 = \ldots = p_n = p$,

$\psi_k(j : n)$: Birnbaum index of structural importance of component j in a $C(k, n)$ system,

$\phi_k(j : n)$: Barlow-Proschan index of structural importance of component j in a $C(k, n)$ system,

$\beta(.,.)$: beta function,

$\lambda_k(r : x) = (x + 1)\frac{(r-1)!(rk-1)!}{(r(k+1))!}$ for $r \in N_+$ and $x \in R$,

$\delta(r, s : x, y) = \frac{(r+s-2)!((r+s)k-1)!(r+s)}{rs((r+s)(k+1))!}((x + 1)(y + 1)(r + s - 1) + rsk(k + 1))$ for $r, s \in N_+$ and $x, y \in R$.

We find it convenient to use the extended definition of the binomial coefficients as in Feller (1968, p. 50) though not in its full strength. For $a \in N$ and $b \in Z$ we define

$$\binom{a}{b} = \begin{cases} \frac{a!}{b!(a-b)!} & \text{when } b \geq 0 \text{ and } a - b \geq 0, \\ 0 & \text{otherwise} \end{cases}$$

We define the trinomial coefficients in a similar way.

Let $a \in N$ and $b, c, d \in Z$ in such that $b + c + d = a$. We define

$$\binom{a}{b, c, d} = \begin{cases} \frac{a!}{b!c!d!} & \text{when } b \geq 0, \ c \geq 0 \text{ and } d \geq 0, \\ 0 & \text{otherwise.} \end{cases}$$

Finally let $r, s \in Z, I(r, s) \subset A \subset Z$ and $\alpha : A \to R$. We use the notation $\sum_{l=r}^{s} \alpha(l)$ to denote the summation of $\alpha(l)$ over the set $I(r, s)$. When $r > s$, we note that $I(r, s) = \emptyset$ and hence

$$\sum_{l=r}^{s} \alpha(l) = 0.$$

We refer to Barlow and Proschan (1975) or Kaufaman *et al.* (1977) or Ramamurthy (1990) for the required preliminary concepts about coherent structures, path sets etc. We make the usual assumptions regarding the binary nature of the component and system states. We also make the assumption of s-independency of the components.

Recall that a $C(k,n)$ system consists of a totally ordered set on n components which we take without loss of generality to be $I(1,n)$. A $C(k,n:G)(C(k,n:F))$ functions (fails) if and only if at least k consecutive components function (fail). It is easy to see that a $C(k,n:F)(C(k,n:G))$ is the dual of a $C(k,n:G)(C(k,n:F))$ system (Chao *et al.* (1995, p. 123)). We also have (Kuo *et al.* (1990, Lemma 1)) that

$$f_k(p_1,p_2,\ldots,p_n:n) = 1 - g_k(1-p_1,1-p_2,\ldots,1-p_n:n)$$

for all $(p_1,p_2,\ldots,p_n) \in [0,1]^n$.

We note that (Chao *et al.* (1995, p. 121)) $h_k(l:n)$ is the number of ways in which $n-l$ identical balls can be distributed amongst $l+1$ distinct urns subject to at most $k-1$ balls being placed in any one urn. It is known (Ramamurthy (1990, p. 9) or Seth (1990, p. 107 and p. 146)) that the number of path sets of a $C(k,n:F)$ system is $F_{k,n}$. It follows (see also Chao *et al.* 1995, p. 120)) that

$$F_{k,n} = \sum_{\ell=0}^{n} h_k(l:n).$$

It is also easy to see (Chao *et al.* (1990, p. 121)) that

$$f_k(p:n) = \sum_{\ell=0}^{n} h_k(l:n)p^l(1-p)^{n-l}.$$

It is known (Kuo *et al.* 1990, p. 246) that for all $p \in (0,1)$

$$g_k^j(p:n) = \frac{1}{p}(g_k(p:n) - g_k(p:j-1) - g_k(p:n-j) + g_k(p:j-1)g_k(p:n-j))$$

$$f_k^j(p:n) = \frac{1}{1-p}(f_k(p:j-1)f_k(p:n-j) - f_k(p:n)).$$

Making use of the fact that $f_k(p:n) = 1 - g_k(1-p:n)$, it is easy to verify that $f_k^j(p:n) = g_k^j(1-p:n)$. It is known that the Birnbaum and Barlow-Proschan indices of structural importance for component $j \in I(1,n)$ are same both in

$C(k,n:G)$ and $C(k,n:F)$ systems (Ramamurthy (1990, p. 78)). In fact we have (Ramamurthy (1990, p. 75 and 76))

$$\psi_k(j:n) = g_k^j(\frac{1}{2}:n) = f_k^j(\frac{1}{2}:n)$$

$$\phi_k(j:n) = \int_0^1 g_k^j(p:n)dp = \int_0^1 f_k^j(p:n)dp.$$

3. The Reliability Function

In Theorem 1 we derive a fairly simple expression for $g_k(p:n)$ the reliability function of a $C(k,n:G)$ system for the i.i.d. case. This is the main result of this paper. As a consequence we obtain expressions for $f_k(p:n)$ the reliability function of a $C(k,n:F)$ system, the Fibonacci number $F_{k,n}$ and $h_k(l:n)$ the number of path sets of size l of a $C(k,n:F)$ system. We require the simple results of Lemma 1 in the sequel.

LEMMA 1 . *For $n \in N$, we have*
(i) $\overline{k}(n) = 0$ *for* $n < k$ *and* $\overline{k}(n) \geq 1$ *for* $n \geq k$

(ii) $0 \leq n+1-\overline{k}(n)(k+1) \leq k$

(iii) $\overline{k}(n-k-1) = \overline{k}(n) - 1$ *and* $\overline{k}(n-k) = \overline{k}(n+1) - 1$

(iv) $n+1-\overline{k}(n)(k+1) < k \Rightarrow \overline{k}(n+1) = \overline{k}(n)$

$$n+1-\overline{k}(n)(k+1) = k \Rightarrow \overline{k}(n+1) = \overline{k}(n)+1$$

(v) *For $n \geq k$ and $r \in I(1,\overline{k}(n))$, we have $n - r(k+1)+1 \geq 0$ and the equality holds true if and only if $n+1-\overline{k}(n)(k+1) = 0$ and also $r = \overline{k}(n)$.*

(vi) *For $n \geq k$ and $r \in I(1,\overline{k}(n))$*

$$\binom{n-rk+1}{r} = \binom{n-rk}{r-1}\frac{n-rk+1}{r}.$$

PROOF. Recall that by definition $\overline{k}(n)$ is the integral part of $(n+1)/(k+1)$. The validity of (i) and (ii) is immediate.

We have

$$\overline{k}(n-k-1) = \left[\frac{n-k}{k+1}\right] = \left[\frac{n+1-(k+1)}{k+1}\right] = \left[\frac{n+1}{k+1}\right] - 1 = \overline{k}(n) - 1$$

$$\overline{k}(n-k) = \left[\frac{n-k+1}{k+1}\right] = \left[\frac{n+2-(k+1)}{k+1}\right] = \left[\frac{n+2}{n+1}\right] - 1 = \overline{k}(n+1) - 1.$$

This proves (iii). Let $l = n + 1 - \overline{k}(n)(k+1)$ and note that $l \in I(0,k)$.

$$\overline{k}(n+1) = \left[\frac{n+2}{k+1}\right] = \left[\frac{\overline{k}(n)(k+1)+l+1}{k+1}\right] = \overline{k}(n) + \left[\frac{l+1}{k+1}\right]$$

The validity of (iv) is immediate. The validity of (v) is trivial whereas the validity of (vi) follows easily from the definition of the binomial coefficients.

THEOREM 1. *For all $n \in N$, we have*

$$g_k(p:n) = \sum_{r=1}^{\overline{k}(n)} (-1)^{r-1}(1-p)^{r-1}p^{rk}\left\{\binom{n-rk+1}{r}(1-p) + \binom{n-rk}{r-1}p\right\}.$$

PROOF. When $n < k$, we note that $\overline{k}(n) = 0$ and $g_k(p:n) \equiv 0$ and therefore the theorem vacuously holds true. Consider now the case $k \le n \le 2k$ and note that $\overline{k}(n) = 1$. It is known that (Zuo and Kuo (1990, Lemma 1) that for $k \le n \le 2k$

$$\begin{aligned} g_k(p:n) &= (n-k+1)p^k - (n-k)p^{k+1} \\ &= (n-k+1)p^k(1-p) + p^{k+1} \end{aligned}$$

and hence the theorem holds. Consider now the case where $n = 2k+1$ and note that $\overline{k}(n) = 2$. Using the well known recursive relation (Kuo *et al.* (1990, Lemma 2))

$$g_k(p:n+1) = g_k(p:n) + (1-p)p^k(1 - g_k(p:n-k)) \quad \text{for} \quad n \ge k$$

we verify that the theorem holds true for $n = 2k+1$. We shall use induction to prove the result for the general case. Let $n \in N$ be such that $n \ge 2k+1$ and note that $\overline{k}(n) \ge 2$. Suppose the theorem is true for all $m \in I(0,n)$. It is therefore enough to show that the result holds true for $n+1$. The recursive equation of Kuo *et al.* states that

$$g_k(p:n+1) = g_k(p:n) + (1-p)p^k(1 - g_k(p:n-k)).$$

Using the induction hypothesis and (iii) of Lemma 1, we get

$$\begin{aligned} &g_k(p:n+1) \\ &= \sum_{r=1}^{\overline{k}(n)} (-1)^{r-1}(1-p)^{r-1}p^{rk}\left\{\binom{n-rk+1}{r}(1-p) + \binom{n-rk}{r-1}p\right\} \end{aligned}$$

$$+(1-p)p^k+\sum_{r=1}^{\overline{k}(n-k)}(p-1)^r p^{(r+1)k}\left\{\binom{n-(r+1)k+1}{r}(1-p)+\binom{n-(r+1)k}{r-1}p\right\}$$

$$=p^k((n+2-k)(1-p)+p)+\sum_{r=2}^{\overline{k}(n)}(p-1)^{r-1}p^{rk}\left\{\binom{n-rk+1}{r}(1-p)\right.$$

$$\left.+\binom{n-rk}{r-1}p\right\}+\sum_{r=2}^{\overline{k}(n+1)}(p-1)^{r-1}p^{rk}\left\{\binom{n-rk+1}{r-1}(1-p)+\binom{n-rk}{r-2}p\right\}.$$

When $n+1-\overline{k}(n)(k+1)<k$, we have $\overline{k}(n+1)=\overline{k}(n)$ and it follows that

$$g_k(p:n+1)$$

$$=p^k\left\{\binom{n+1-k+1}{1}(1-p)+\binom{n+1-k}{0}p\right\}$$

$$+\sum_{r=2}^{\overline{k}(n+1)}(p-1)^{r-1}p^{rk}\left\{\binom{n+1-rk+1}{r}(1-p)+\binom{n+1-rk}{r-1}p\right\}$$

$$=\sum_{r=1}^{\overline{k}(n+1)}(p-1)^{r-1}p^{rk}\left\{\binom{n+1-rk+1}{r}(1-p)+\binom{n+1-rk}{r-1}p\right\}.$$

When $n+1-\overline{k}(n)(k+1)=k$, we have $\overline{k}(n+1)=\overline{k}(n)+1$ and it follows that

$$g_k(p:n+1)$$

$$=\sum_{r=1}^{\overline{k}(n)}(p-1)^{r-1}p^{rk}\left\{\binom{n+1-rk+1}{r}(1-p)+\binom{n+1-rk}{r-1}p\right\}$$

$$+(p-1)^{\overline{k}(n)}p^{(\overline{k}(n)+1)k}\left\{\binom{n-(\overline{k}(n)+1)k+1}{\overline{k}(n)}(1-p)+\binom{n-(\overline{k}(n)+1)k}{\overline{k}(n)-1}p\right\}$$

$$=\sum_{r=1}^{\overline{k}(n)}(p-1)^{r-1}p^{rk}\left\{\binom{n+1-rk+1}{r}(1-p)+\binom{n+1-rk}{r-1}p\right\}$$

$$+(p-1)^{\overline{k}(n)}p^{(\overline{k}(n)+1)k}\{(1-p)+p\}$$

$$=\sum_{r=1}^{\overline{k}(n+1)}(p-1)^{r-1}p^{rk}\left\{\binom{n+1-rk+1}{r}(1-p)+\binom{n+1-rk}{r-1}p\right\}.$$

Hence the theorem is true for $n+1$ also. This completes the proof.

COROLLARY 1. *For all $n \in N$, we have*

$$g_k(p:n) = \sum_{r=1}^{\overline{k}(n)} (-1)^{r-1}(1-p)^{r-1}p^{rk}\binom{n-rk}{r-1}\left\{\frac{n-rk+1}{r}(1-p)+p\right\}.$$

PROOF. Simple consequence of Theorem 1 and Lemma 1.

COROLLARY 2. *For $n \in N$, the degree of the polynomial $g_k(p:n)$ is given by*

$$\text{degree of } g_k(p:n) \begin{cases} = & \overline{k}(n)(k+1)-1 \quad \text{when } n+1 = \overline{k}(n)(k+1) \\ & \overline{k}(n)(k+1) \quad \text{when } n+1 \neq \overline{k}(n)(k+1). \end{cases}$$

PROOF . We note that

$$n+1 = \overline{k}(n)(k+1) \Leftrightarrow \frac{n-\overline{k}(n)k+1}{\overline{k}(n)} = 1.$$

The required result is immediate.

COROLLARY 3. *For all $n \in N$, we have*

$$f_k(p:n) = 1 - \sum_{r=1}^{\overline{k}(n)} (-p)^{r-1}(1-p)^{rk}\left\{\binom{n-rk+1}{r}p + \binom{n-rk}{r-1}(1-p)\right\}.$$

PROOF . The required result follows from Theorem 1 by making use of the fact that $f_k(p:n) = 1 - g_k(1-p:n)$ for all $p \in [0,1]$.

REMARK 1. Chen and Hwang (1988) have shown that

$$f_k(p:n) = 1 - \sum_{j=k}^{n} \sum_{A(k,j-k)} \binom{n_1+n_2+\ldots+n_k}{n_1,n_2,\ldots,n_k} p^{\sum_{i=1}^{k} n_i} (1-p)^{j-\sum_{i=1}^{k} n_i}$$

where $A(k,j-k) = \{(n_1,n_2,\ldots,n_k) \in N^k$ and $\sum_{i=1}^{k_i} in_i = j-k\}$. We note that the formula given in Corollary 3 is much simpler and easier to compute.

THEOREM 2. *The generating function $L_k(x:n)$ of $h_k(r:n)$ for all $x > 0$ and $n \in N$ is given by*

$$L_k(x:n) = (1+x)^n - \sum_{r=1}^{\overline{k}(n)} (-1)^{r-1}x^{r-1}(1+x)^{n-r(k+1)}\left\{\binom{n-rk+1}{r}x + \binom{n-rk}{r-1}\right\}.$$

PROOF. Recall that $h_k(r:n)$ denotes the number of path sets of size r of a $C(k,n:F)$ system and $L_k(x:n)$ is its generating function. Suppose $0 \le n < k$ and note that $\overline{k}(n) = 0$ and also $h_k(r:n) = \binom{n}{r}$ for all $r \in I(0,n)$. It follows that

$$L_k(x:n) = \sum_{r=0}^{n} h_k(r:n)x^n = (1+x)^n$$

$$= (1+x)^n - \sum_{r=1}^{\overline{k}(n)} (-1)^{r-1}x^{r-1}(1+x)^{n-r(k+1)}\left\{\binom{n-rk+1}{r}x + \binom{n-rk}{r-1}\right\}.$$

Consider now the case where $n \ge k$ and note that $\overline{k}(n) \ge 1$. Let $p \in (0,1)$ and recall that

$$f_k(p:n) = \sum_{r=0}^{n} h_k(r:n)p^r(1-p)^{n-r} = (1-p)^n \sum_{r=0}^{n} h_k(r:n)\left(\frac{p}{1-p}\right)^r$$

$$= (1-p)^n L_k\left(\frac{p}{1-p}:n\right).$$

It follows that

$$g_k(p:n) = 1 - f_k(1-p:n) = 1 - p^n L_k\left(\frac{1-p}{p}:n\right).$$

We now have

$$L_k\left(\tfrac{1-p}{p}:n\right)$$

$$= \frac{1}{p^n} - \sum_{r=1}^{\overline{k}(n)} (-1)^{r-1}\left(\frac{1-p}{p}\right)^{r-1}\frac{1}{p^{n-r(k+1)}}\left\{\binom{n-rk+1}{r}\frac{1-p}{p} + \binom{n-rk}{r-1}\right\}.$$

We now put $x = \frac{1-p}{p}$ and note that $1+x = \frac{1}{p}$. It follows that

$$L_k(x:n) = (1+x)^n - \sum_{r=1}^{\overline{k}(n)} (-1)^{r-1}x^{r-1}(1+x)^{n-r(k+1)}\left\{\binom{n-rk+1}{r}x + \binom{n-rk}{r-1}\right\}.$$

COROLLARY . *For all $n \in N$, we have*

$$F_{k,n} = 2^n - \sum_{r=1}^{\bar{k}(n)} (-1)^{r-1} \binom{n-rk}{r-1} \frac{n-r(k-1)+1}{r} \; 2^{n-r(k+1)}.$$

PROOF. Recall that $h_k(r:n)$ denotes the number of path sets of size r and the Fibonacci number $F_{k,n}$ is the total number of path sets of a $C(k,n:F)$ system. In view of (vi) of Lemma 1 we have

$$\begin{aligned} F_{k,n} &= \sum_{r=0}^{n} h_k(r:n) = L_k(1:n) \\ &= 2^n - \sum_{r=1}^{\bar{k}(n)} (-1)^{r-1} \binom{n-rk}{r-1} \frac{n-r(k-1)+1}{r} \; 2^{n-r(k+1)}. \end{aligned}$$

LEMMA 2. *For $n \in N$ and $l \in I(0,n)$ we have*

$$\min(l+1, \left[\frac{n-l}{k}\right]) \leq \bar{k}(n).$$

PROOF. When $n < k$, we have $n - l < k$ for all $l \in I(0,n)$ and the required inequality holds true trivially. Consider now the case when $n \geq k$ and note that $\bar{k}(n) \geq 1$. We have in view of (ii) of Lemma 1.

$$n + 1 - (\bar{k}(n)+1)k = n + 1 - \bar{k}(n)(k+1) + \bar{k}(n) - k \leq \bar{k}(n).$$

It now follows that for $l \in I(0,n)$

$$\left[\frac{n-l}{k}\right] \geq \bar{k}(n)+1 \Rightarrow n-l \geq ((\bar{k}(n)+1)k \Rightarrow l+1 \leq n+1-(\bar{k}(n)+1)k \leq \bar{k}(n).$$

The required result now follows.

THEOREM 3. *For all $n \in N$ and $l \in I(0,n)$ we have*

$$h_k(l:n) = \binom{n}{l} - (l+1)\sum_{r=1}^{s_l} \frac{(-1)^{r-1}}{n-rk+1} \binom{n-rk+1}{r, l-r+1, n-l-rk}$$

where $s_\ell = \min\{l+1, \left[\frac{n-l}{k}\right]\}$

PROOF. When $n < k$, we have $\bar{k}(n) = 0$ and $f_k(p:n) \equiv 1$. In view of Lemma 2 we note that $s_l = 0$. It follows that

$$h_k(l:n) = \binom{n}{l} = \binom{n}{l} - (l+1)\sum_{r=1}^{s_l} \frac{(-1)^{r-1}}{n-rk+1}\binom{n-rk+1}{r, l-r+1, n-l-rk}$$

for all $l \in I(0,n)$. Suppose now $n \geq k$ and note that $\overline{k}(n) \geq 1$. First we consider the case where $n+1-\overline{k}(n)(k+1) \neq 0$. In view of Lemma 1 we have $n - r(k+1) \geq 0$ for all $r \in I(0, \overline{k}(n))$.

We have

$$L_k(x:n) = (1+x)^n - \sum_{r=1}^{\overline{k}(n)} (-1)^{r-1} x^{r-1} (1+x)^{n-r(k+1)} \left\{ \binom{n-rk+1}{r} x + \binom{n-rk}{r-1} \right\}$$

$$= (1+x)^n - \sum_{r=1}^{\overline{k}(n)} (-1)^{r-1} x^{r-1} (1+x)^{n-r(k+1)} \left\{ \binom{n-rk+1}{r} (1+x) - \binom{n-rk}{r} \right\}$$

$$= \sum_{t=0}^{n} \binom{n}{l} x^l - \sum_{r=1}^{\overline{k}(n)} (-1)^{r-1} x^{r-1} \left\{ \binom{n-rk+1}{r} \sum_{j=0}^{n-r(k+1)+1} \binom{n-r(k+1)+1}{j} x^j \right.$$

$$\left. - \binom{n-rk}{r} \sum_{j=0}^{n-r(k+1)} \binom{n-r(k+1)}{j} x^j \right\}$$

$$= \sum_{t=0}^{n} \binom{n}{l} x^l - \sum_{r=1}^{\overline{k}(n)} (-1)^{r-1} \left\{ \binom{n-rk+1}{r} \sum_{l=r-1}^{n-rk} \binom{n-r(k+1)+1}{l-r+1} x^j \right.$$

$$\left. - \binom{n-rk}{r} \sum_{l=r-1}^{n-rk-1} \binom{n-r(k+1)}{l-r+1} x^l \right\}$$

$$= \sum_{t=0}^{n} \binom{n}{l} x^l - \sum_{r=1}^{\overline{k}(n)} (-1)^{r-1} \left\{ \sum_{l=r-1}^{n-rk} \binom{n-rk+1}{r, l-r+1, n-l-rk} x^l \right.$$

$$\left. - \sum_{l=r-1}^{n-rk-1} \left(\binom{n-rk}{r, l-r+1, n-l-rk-1} x^l \right) \right\}$$

$$= \sum_{t=0}^{n} \binom{n}{l} x^l - \sum_{r=1}^{\overline{k}(n)} (-1)^{r-1} \sum_{l=r-1}^{n-rk} \frac{l+1}{n-rk+1} \binom{n-rk+1}{r, l-r+1, n-l-rk} x^l.$$

Consider now the case where $n+1-\overline{k}(n)(k+1)=0$ and note that $n-\overline{k}(n)k+1=\overline{k}(n)$ and also $n-r(k+1)\geq 0$ for $r\in I(0,\overline{k}(n)-1)$. We now have

$$L_k(x:n)$$

$$= (1+x)^n - \sum_{r=1}^{\overline{k}(n)-1}(-1)^{r-1}x^{r-1}(1+x)^{n-r(k+1)}\left\{\binom{n-rk+1}{r}x+\binom{n-rk}{r-1}\right\}$$

$$+(-1)^{\overline{k}(n)-1}x^{\overline{k}(n)-1}(1+x)^{-1}(x+1).$$

$$=\sum_{l=0}^{n}\binom{n}{l}x^l-\sum_{r=1}^{\overline{k}(n)}(-1)^{r-1}\sum_{l=r-1}^{n-rk}\frac{l+1}{n-rk+1}\binom{n-rk+1}{r,l-r+1,n-l-rk}x^l.$$

Therefore the above representation for $L_k(x:n)$ holds true for all $n\geq k$. In the second summation for a given $r\in I(1,\overline{k}(n))$, we note that l is required to satisfy $r-1\leq l\leq n-rk$. Conversely for a given $l\in I(0,n-k)$, we see that r is required to satisfy $r\leq l+1$ and $r\leq \frac{n-l}{k}$. Obviously $1\leq r\leq \overline{k}(n)$. In view of Lemma 2, we have

$$s_l=\min\{l+1,\left[\frac{n-l}{k}\right]\}\leq\overline{k}(n).$$

It now follows that

$$L_k(x:n)$$

$$=\sum_{l=0}^{n}\binom{n}{l}x^l-\sum_{l=0}^{n-k}\left\{\sum_{r=1}^{s_l}(-1)^{r-1}\frac{l+1}{n-rk+1}\binom{n-rk+1}{r,l-r+1,n-l-rk}\right\}x^l.$$

We note that $s_l=0$ and also the trinomial coefficients vanish for $l>n-k$. It follows that

$$L_k(x:n)=\sum_{l=0}^{n}\left\{\binom{n}{l}-(l+1)\sum_{r=1}^{s_l}\frac{(-1)^{r-1}}{n-rk+1}\binom{n-rk+1}{r,l-r+1,n-l-rk}\right\}x^l.$$

Since $h_k(l:n)$ is the coefficient of x^l in $L_k(x:n)$, we have

$$h_k(l:n)=\binom{n}{l}-(l+1)\sum_{r=1}^{s_l}\frac{(-1)^{r-1}}{n-rk+1}\binom{n-rk+1}{r,l-r+1,n-l-rk}$$

for all $l\in I(0,n)$. This completes the proof.

COROLLARY. *For all $n\in N$ and $l\in I(0,n)$ we have*

$$h_k(l:n) = \binom{n}{l} - (l+1)\sum_{r=1}^{\overline{k}(n)} \frac{(-1)^{r-1}}{n-rk+1}\binom{n-rk+1}{r, l-r+1, n-l-rk}.$$

PROOF. We need only show the validity of the result for $n \geq k$.
For each $l \in I(0,n)$ let

$$s_l = \min\left\{l+1, \left[\frac{n-l}{k}\right]\right\}.$$

As in Theorem 3, suppose $l \in I(0,n)$ be such that $s_t < \overline{k}(n)$. Then it should be true that either $r-1 > l$ or $l > n-rk$ for each $r \in I(s_l+1, \overline{k}(n))$. It follows that trinomial coefficient

$$\binom{n-rk+1}{r, l-r+1, n-l-rk} = 0$$

for each $r \in I(s_l+1, \overline{k}(n))$. The required result is immediate.

4. Application

In this section, we discuss an application of the result of Theorem 1 in the computation of indices of structural importance of components of $C(k,n)$ systems. Seth(1990) has computed both the Birnbanm and Barlow-Proschan indices of structural importance in $C(k,n)$ systems for $k = 2,3$ and $k \leq n \leq 20$. Kuo *et al.* (1990) have computed the Birnbaum index for $C(k,14)$ systems for $k = 2,7$ and 14. The Birnbaum index is relatively easier to compute. In fact we have for$j \in I(1,n)$

$$\begin{aligned}\psi_k(j:n) &= g_n^j(\frac{1}{2}:n) = f_n^j(\frac{1}{2}:n) \\ &= 2(f_k(\frac{1}{2}:j-1)f_k(\frac{1}{2}:n-j) - f_k(\frac{1}{2}:n)) \\ &= \frac{2F_{k,j-1}F_{k,n-j} - F_{k,n}}{2^{n-1}}\end{aligned}$$

an elegant result due to Seth (1990, p. 124 and 146). We shall not dwell any further on the Birnbaum index and concern ourselves only with the Barlow-Proschan index.

Recall that the Barlow-Proschan index $\phi_k(j:n)$ for component j is given by

$$\phi_k(j:n) = \int_0^1 g_k^j(p:n)dp = \int_0^1 f_k^j(p:n)dp$$

$$= \int_0^1 \frac{1}{1-p}(f_k(p:j-1)f_k(p:n-j) - f_k(p:n))dp$$

$$= \int_0^1 \frac{1}{p}(g_k(p:n) - g_k(p:j-1) - g_k(p:n-j) + g_k(p:j-1)g_k(p:n-j))dp.$$

Hence to compute the Barlow-Proschan index, we have to carry out integration involving reliability functions and their products. Recall that

$$f_k(p:n) = p^n L_k\left(\frac{1-p}{p}:n\right).$$

For the particular cases $k = 2$ and 3, Seth (1990, p. 129) uses the recursive equation

$$L_k(x:n) = \sum_{j=1}^{k} L_k(x:n-j)x^{j-1}$$

to find $L_k(x:n)$ for the computation of the Barlow-Proschan index. We now derive an exact expression for the Barlow-Proschan index. For this purpose we require the results of Lemmas 3 and 4.

LEMMA 3. *For any positive integer r and real x we have*

$$\lambda_k(r:x) = \int_0^1 (1-p)^{r-1}p^{rk-1}\left(\frac{x-rk+1}{r}(1-p)+p\right)dp.$$

PROOF. Recall that by definition

$$\lambda_k(r:x) = (x+1)\frac{(r-1)!(rk-1)!}{(r(k+1))!}.$$

Let

$$L = \int_0^1 (1-p)^{r-1}p^{rk-1}\left(\frac{x-rk+1}{r}(1-p)+p\right)dp.$$

We now have

$$\begin{aligned} L &= \frac{x-rk+1}{r}\beta(r+1,rk) + \beta(r,rk+1) \\ &= \frac{(r-1)!(rk-1)!}{(r(k+1))!}(x-rk+1+rk) = \lambda_k(r:x). \end{aligned}$$

This completes the proof.

LEMMA 4. *For any positive integers r and s and real x and y we have*

$$\delta(r,s:x,y)$$

$$= \int_0^1 (1-p)^{r+s-2} p^{k(r+s)-1} \left(\frac{x-rk+1}{r}(1-p)+p\right)\left(\frac{y-sk+1}{s}(1-p)+p\right) dp.$$

PROOF. Recall that by definition

$$\delta(r,s:x,y) = \frac{(r+s-2)!((r+s)k-1)!(r+s)}{rs((r+s)(k+1))!}((x+1)(y+1)(r+s-1)+rsk(k+1)).$$

$$L = \int_0^1 (1-p)^{r+s-2} p^{k(r+s)-1} \left(\frac{x-rk+1}{r}(1-p)+p\right)\left(\frac{y-sk+1}{s}(1-p)+p\right) dp.$$

We now have

$$\begin{aligned}
L &= \frac{x-rk+1}{r}.\frac{y-sk+1}{s}\beta(r+s+1,(r+s)k) \\
&+ \left(\frac{x-rk+1}{r}+\frac{y-sk+1}{s}\right)\beta(r+s,(r+s)k+1)+\beta(r+s-1,(r+s)k+2) \\
&= \frac{(r+s-2)!((r+s)k-1)!(r+s)}{rs((r+s)((k+1))!}\{(x-rk+1)(y-sk+1)(r+s-1) \\
&+ (s(x-rk+1)+r(y-sk+1))(r+s-1)k+rsk((r+s)k+1)\} \\
&= \frac{(r+s-2)!((r+s)k-1)!(r+s)}{rs((r+s)(k+1))!}((x+1)(y+1)(r+s-1)+rsk(k+1)) \\
&= \delta_k(r,s:x,y).
\end{aligned}$$

This completes the proof.

LEMMA 5. *For $l, m \in N$ we have*

(i) $\displaystyle\int_0^1 \frac{1}{p} g_k(p:l)dp = \sum_{r=1}^{\overline{k}(l)} (-1)^{r-1}\binom{l-rk}{r-1}\lambda_k(r:l)$

(ii) $\displaystyle\int_0^1 \frac{1}{p} g_k(p:l)g_k(p:m)dp = \sum_{r=1}^{\overline{k}(l)}\sum_{s=1}^{\overline{k}(m)} (-1)^{r+s-2}\binom{l-rk}{r-1}\binom{m-sk}{s-1}$

$\cdot\delta_k(r,s:l,m).$

PROOF. When $l < k$ we have $\overline{k}(l) = 0$ and $g_k(p : n) \equiv 0$. Therefore (i) vacuously holds true. When $l \geq k$, the required result follows from Lemma 3 and Corollary 1 of Theorem 1. The proof of (ii) is similar.

THEOREM 4. *For all $n \in N$ such that $n \geq k$ and $j \in I(1,n)$ we have*

$$\phi_k(j : n) = \sum_{r=1}^{\overline{k}(n)}(-1)^{r-1}\binom{n-rk}{r-1}\lambda_k(r : n) - \sum_{r=1}^{\overline{k}(j-1)}(-1)^{r-1}\binom{j-1-rk}{r-1}$$

$$\lambda_k(r : j-1) - \sum_{r=1}^{\overline{k}(n-j)}(-1)^{r-1}\binom{n-j-rk}{r-1}\lambda_k(r : r-j)$$

$$+\sum_{r=1}^{\overline{k}(j-1)}\sum_{s=1}^{\overline{k}(n-j)}(-1)^{r+s-2}\binom{j-1-rk}{r-1}\binom{n-j-sk}{s-1}\delta(r,s : j-1, n-j).$$

PROOF. Recall that

$$\begin{aligned}\phi_k(j : n) &= \int_0^1 \frac{1}{p}(g_k(p : n) - g_k(p : j-1) - g_k(p : n-j) \\ &+ g_k(p : j-1)g_k(p : n-j))dp\end{aligned}$$

The required result follows from Lemma 5.

EXAMPLE. As an example we calculate $\phi_3(j : 18)$ using the above formula for $j = 3$ and 4. Note that $\overline{k}(n) = \overline{k}(18) = 4$. For $j = 3$ we have $\overline{k}(j-1) = \overline{k}(2) = 0$ and $\overline{k}(n-j) = \overline{k}(18-3) = \overline{k}(15) = 4$. It now follows that

$$\phi(3 : 18) = \sum_{r=1}^{4}(-1)^{r-1}\binom{18-3r}{r-1}\lambda_3(r : 18) - \sum_{r=1}^{4}(-1)^{r-1}\binom{15-3r}{r-1}\lambda_3(r : 15)$$

$$\begin{aligned}&\frac{19}{12} - \frac{12\times 19}{336} + \frac{36\times 38}{11880} - \frac{20\times 114}{524160} - \left(\frac{16}{12} - \frac{9\times 16}{336} + \frac{15\times 32}{11880} - \frac{96}{524160}\right) \\ &= 1.015563603 - 0.944982795 = 0.070580808\end{aligned}$$

For $j = 4$, we have $\overline{k}(j-1) = \overline{k}(3) = 1$ and $\overline{k}(n-j) = \overline{k}(18-4) = \overline{k}(14) = 3$. It now follows

$$\begin{aligned}
\phi_3(4:18) &= \sum_{r=1}^{4}(-1)^{r-1}\binom{18-3r}{r-1}\lambda_3(r:18)-\lambda_3(1:3)\\
&-\sum_{r=1}^{3}(-1)^{r-1}\binom{14-3r}{r-1}\lambda_3(r:14)\\
&+\sum_{s=1}^{3}(-1)^{s-1}\binom{14-3s}{s-1}\delta(1,s:3,14).\\
&= 1.015563603-0.333333333-\left(\frac{15}{12}-\frac{8\times 15}{336}+\frac{10\times 30}{11880}\right)\\
&+\left(\frac{3}{7}-\frac{8\times 1}{55}+\frac{10\times 1}{910}\right)\\
&= 0.015563603-0.333333333-0.918109668+0.294105894\\
&= 0.058226496.
\end{aligned}$$

These are in agreement with the numerical values computed by Seth (1990, Table 5.2).

Appendix

In this appendix, we discuss briefly the close relationship between the probabilistic aspects of $C(k,n)$ systems and the theory of success runs in Bernoulli trials as developed by Feller (1968, Section 7, Chapter XIII).

Let $a_k(p:n)$ denote the probability that the first success run of length k occurs at n-th trial and p is the probability of success. Feller (1968, p. 323) has shown that the generating function

$$A_k(x:p)=\sum_{n=0}^{\infty}a_k(p:n)x^n$$

is given by

$$A_k(x:p)=\frac{p^k x^k(1-px)}{1-x+(1-p)p^k x^{k+1}}.$$

He also suggests a very good approximation for $a_k(p:n)$. We refer to Feller (1968, Section 7, Chapter XIII) for further details. It is easy to see that

$$g_k(p:n)=\sum_{r=0}^{n}a_k(p:r)$$

$$f_k(1-p:n)=\sum_{r=n+1}^{\infty}a_k(p:r).$$

It follows that the generating function $G_k(x:p)$ of $g_k(p:n)$ is given by

$$\begin{aligned} G_k(x:p) &= \frac{A_k(x:p)}{1-x} \\ &= \frac{p^k x^k(1-px)}{(1-x)(1-x+(1-p)p^k x^{k+1})}. \end{aligned}$$

We note that

$$\begin{aligned} a_k(p:n) &= 0 \quad \text{for } n<k \\ &= p^k \quad \text{for } n=k. \end{aligned}$$

For $n \geq k+1$ we have

$$\begin{aligned} &a_k(p:n) \\ &= g_k(p:n) - g_k(p:n-1) \\ &= p^k(1-p)(1-g_k(p:n-k-1)) \\ &= p^k(1-p)\left\{1 - \sum_{r=1}^{\bar{k}(n-k-1)} (-1)^{r-1}(1-p)^{r-1}p^{rk}\left\{\binom{n-k(r+1)}{r}(1-p)\right.\right. \\ &\quad \left.\left. + \binom{n-k(r+1)-1}{r-1}p\right\}\right\} \\ &= p^k(1-p) + \sum_{r=1}^{\bar{k}(n-k-1)} (p-1)^r p^{k(r+1)}\left\{\binom{n-k(r+1)}{r}(1-p)\right. \\ &\quad \left. + \binom{n-k(r+1)-1}{r-1}p\right\} \\ &= p^k(1-p) + \sum_{r=2}^{\bar{k}(n-k-1)+1} (p-1)^{r-1}p^{rk}\left\{\binom{n-rk}{r-1}(1-p)\right. \\ &\quad \left. + \binom{n-rk-1}{r-2}p\right\} \\ &= p^k(1-p) + \sum_{r=2}^{\bar{k}(n)} (p-1)^{r-1}p^{rk}\left\{\binom{n-rk}{r-1}(1-p)\right. \\ &\quad \left. + \binom{n-rk-1}{r-2}p\right\} \\ &= \sum_{r=1}^{\bar{k}(n)} (p-1)^{r-1}p^{rk}\left\{\binom{n-rk}{r-1}(1-p) + \binom{n-rk-1}{r-2}p\right\}. \end{aligned}$$

Thus we have provided an explicit formula for $a_k(p:n)$.

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