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ON OPTIMALITY OF SOME PARTIAL DIALLEL CROSS DESIGNS

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SUMMARY. Various forms of diallel crosses play an important role in evaluating the breeding potential of genetic material in plant and animal breeding. In this paper we consider partial diallel crosses in incomplete block or completely randomized designs. Optimal designs in both the unblocked and blocked situations are characterized. Two methods of construction of MS-optimal designs are proposed leading to design families which have very high A- and D-efficiencies.

1. **Introduction**

Genetic properties of inbred lines in plant breeding experiments are investigated by carrying out diallel crosses. Let p denote the number of lines and let a cross between lines i and i' be denoted by $(i, i'), i < i' = 1, 2, \ldots, p$. Let n denote the total number of crosses observed in the experiment. Our interest lies in comparing the lines with respect to their general combining ability effects.

Complete diallel cross designs involve equal numbers of occurrences of each of the $p(p-1)/2$ distinct crosses among p inbred lines. If r denotes the number of times that each cross occurs in a complete diallel, then such an experiment requires $rp(p-1)/2$ experimental units (or crosses). When p is large, it becomes impractical to carry out a complete diallel cross even for $r = 1$. In such situations, we consider designs having no requirement that the distinct crosses appear equally often. This leads us to what we call Partial Diallel Cross (PDC) designs. In the literature PDC designs have been discussed for $n = ps/2$ $(s < p-1)$ distinct crosses each appearing an equal number $r \geq 1$ times, where $s = 2n/p$

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is an integer. Several methods of obtaining PDC completely randomized designs have been given, together with their efficiency factors, by Kempthorne and Curnow (1961), Curnow (1963), Hinkelmann and Kempthorne (1963), Singh and Hinkelmann (1988, 1990), among others.

PDC designs can, themselves, be quite large and it is sometimes desirable to use a block design for the experiment rather than a completely randomized design. Singh and Hinkelmann (1995) used conventional partially balanced incomplete block designs both to select the diallel crosses to be observed and to arrange them into blocks. Their resulting designs have distinct crosses appearing either $r(\geq 2)$ times in the design or not at all. The numbers of occurrences of the lines within the blocks tend to be uneven, which decreases their design efficiencies. Gupta, Das and Kageyama (1995) and Mukerjee (1997) provide orthogonal blocking schemes for PDC designs (see Section 3 for the definition). Mukerjee (1997) proved the E-optimality of his designs (which are based on group divisible plans with $\lambda_1 = 1$ and $\lambda_2 = 0$). Mukerjee's designs also perform well under the A- and D-optimality criteria, but exist only for a restricted number of parameter values.

Other than the results of Mukerjee (1997), the only work on optimality of diallel crosses has been for the blocked complete diallel, see, for example, Singh and Hinkelmann (1988), Gupta and Kageyama (1994), Dey and Midha (1996), and Das, Dey and Dean (1998).

The purpose of this communication is to investigate the construction of optimal designs for PDC experiments. The optimality criterion chosen is the MS-optimality criterion of Eccleston and Hedayat (1974). Roughly, the idea behind such a criterion is to limit attention to designs which have maximum information in terms of trace of the information matrix; and then from this class to select a design that is close in the Euclidean sense to a symmetric (variance balanced) design. The symmetric design is selected for comparison purposes since, when such a design exists, it is best according to any of the usual optimality criteria, for example, see Proposition 1 in Kiefer (1975). Unlike the standard criteria of A-, D- and E-optimality, MS-optimal designs have no primary statistical interpretation. However, the MS-optimality criterion is much easier to work with than the standard criteria in that it allows for algebraic results in a field where other criteria can often only be examined numerically, design by design.

In this paper, we characterize MS-optimal unblocked and blocked designs for PDC experiments with $s = 2n/p$ an integer, where n is the number of crosses observed in the design. We list two families of MS-optimal PDC designs which are also efficient under the A-criterion (of minimizing the average variance of all pairwise comparisons of lines) and the D-criterion (of minimizing the volume of the confidence ellipsoid for all pairwise comparisons). For p even, our series A designs are based on the orthogonally blocked series 2 complete diallels of Gupta and Kageyama (1994). For p odd, our series B designs are based on the orthogonally blocked Family 5 complete diallels of Das, Dey and Dean (1998) and, equivalently, the orthogonally blocked series 1 plans listed by Gupta, Das and Kageyama (1995). We compare the A- and D- efficiencies of the series A and B designs with series C designs which we obtain from conventional resolvable and 2-resolvable partially balanced incomplete block designs with treatment concurrences 0 and 1. We also compare the series A and B designs with the block designs of Singh and Hinkelmann (1995) and some of the designs of Mukerjee (1997).

2. **Preliminaries**

We consider diallel cross experiments involving p inbred lines, giving rise to a total of $n_c = p(p-1)/2$ possible distinct crosses. Let r_{di} denote the number of times the ith cross appears in a design $d, (i = 1, 2, \ldots, p(p-1)/2)$ and, similarly, let s_{dj} denote the total number of times that the jth line occurs among the crosses in the design d, $(j = 1, 2, \ldots, p)$. Further, define $\boldsymbol{r}_d$ and $\boldsymbol{s}_d$ to be $\boldsymbol{r}_d = (r_{d1}, \ldots, r_{dn_c})', \boldsymbol{s}_d = (s_{d1}, \ldots, s_{dp})'$, and let n denote the number of crosses (observations) in the design d. Then $\mathbf{1}'_{n_c}\boldsymbol{r}_d = n = \frac{1}{2}\mathbf{1}'_p\boldsymbol{s}_d$, where $'$ denotes transpose of a matrix and $\mathbf{1}_t$ denotes a t-component column vector of all ones. We use the following model for an unblocked (completely randomized) diallel cross experiment:

$$\text{Model M1}: \qquad \boldsymbol{Y} = \mu\ \mathbf{1}_n + \Delta_1\boldsymbol{g} + \boldsymbol{\epsilon},$$

and the following model for a blocked diallel cross experiment:

$$\text{Model M2}: \qquad \boldsymbol{Y} = \mu\ \mathbf{1}_n + \Delta_1\boldsymbol{g} + \Delta_2\boldsymbol{\beta} + \boldsymbol{\epsilon},$$

where $\boldsymbol{Y}$ is the $n \times 1$ vector of observed responses, μ is a general mean effect, $\boldsymbol{g}$ and $\boldsymbol{\beta}$ are vectors of p general combining ability effects and b block effects respectively, Δ_1, Δ_2 are the corresponding design matrices, that is, the (h, l)th element of Δ_1 (respectively, of Δ_2) is 1 if the hth observation pertains to the lth line (respectively, to the lth block), and is zero otherwise; $\boldsymbol{\epsilon}$ is the vector of random error components, these components being distributed with mean zero and constant variance σ^2. As is usual for the analysis of PDC experiments, it is assumed that the genetic effect of the cross (i, j) is represented sufficiently well by the general combining ability of the two parental lines (see Singh and Hinkelmann, 1995, for a detailed comment on such a model).

Let $D_0(p, n)$ denote the class of all completely randomized designs with p lines and n crosses. For a design, $d_0 \in D_0(p, n)$, under model M1, it can be shown that the information matrix of the reduced normal equations for estimating linear functions of general combining ability effects $\mathbf{g}$ is

$$C_{d_0} = G_{d_0} - \frac{1}{n}\boldsymbol{s}_{d_0}\ \boldsymbol{s}'_{d_0} \qquad \ldots(1)$$

where $G_{d_0} = (g_{d_0ii'})$, $g_{d_0ii} = s_{d_0i}$, and for $i \neq i', g_{d_0ii'}$ is the number of times the cross (i, i') appears in d_0. Also, $\sum_{i\,<}\sum_{i'} g_{d_0ii'} = n$.

Similarly, let $D(p, b, k)$ denote the class of all block designs with p lines, and b blocks each with k crosses. For a block design $d \in D(p, b, k)$ under model M2, the information matrix for $\boldsymbol{g}$ is given by

$$C_d = G_d - \frac{1}{k} N_d\ N_d' \qquad \ldots (2)$$

where $N_d = (n_{dij})$; n_{dij} is the number of times that line i occurs in block j of d; and $G_d = G_{d_0}$ is defined below (1). For such a block design, $n = bk$ and $N_d \mathbf{1}_b = \boldsymbol{s}_d = \boldsymbol{s}_{d_0}$.

A design d will be called connected if and only if the rank of its information matrix is $p - 1$. Equivalently, d is connected if and only if all elementary comparisons among the general combining ability effects are estimable. A connected design $d_0^* \in D_0(p, n)$ is said to be MS-optimal if

$$\max_{d_0 \in D_0(p,n)} tr(C_{d_0}) = tr(C_{d_0^*}) \quad \text{and} \quad \min_{d_0 \in D_0^*(p,n)} tr(C_{d_0}^2) = tr(C_{d_0^*}^2)\,,$$

where $D_0^*(p, n)$ is the sub-class of all designs $d_0 \in D_0(p, n)$ for which $tr(C_{d_0})$ is maximum.

Let $z_{d_01} \leq z_{d_02} \leq \cdots \leq z_{d_0,p-1}$ be the non-zero eigenvalues of C_{d_0}. Then, design d_0^* is said to be A-optimal if $\min_{d_0 \in D_0(p,n)} tr(C_{d_0}^-) = tr(C_{d_0^*}^-)$, is said to be D-optimal if $\max_{d_0 \in D_0(p,n)} \prod_{i=1}^{p-1} z_{d_0i} = \prod_{i=1}^{p-1} z_{d_0^*i}$ and said to be E-optimal if $\max_{d_0 \in D_0(p,n)} z_{d_01} = z_{d_0^*1}$. Similarly, MS-, A-, D- and E-optimality are defined for connected block designs $d \in D(p, b, k)$.

3. MS-Optimality of PDC Designs

In Theorem 3.1, below, we characterize MS-optimal designs in the class of completely randomized designs $D_0(p, n)$, with $s = 2n/p$ an integer. We need the following well known lemma which is easy to prove.

Lemma 3.1. *For given positive integers α and β, the minimum of $\sum_{i=1}^{\alpha} m_i^2$ subject to $\sum_{i=1}^{\alpha} m_i = \beta$, where the m_i's are non-negative integers, is obtained when $\beta - \alpha[\beta/\alpha]$ of the m_i's are equal to $[\beta/\alpha]+1$ and $\alpha - \beta + \alpha[\beta/\alpha]$ are equal to $[\beta/\alpha]$, where $[z]$ denotes the largest integer not exceeding z. The corresponding minimum of $\sum_{i=1}^{\alpha} m_i^2$ is $\beta(2[\beta/\alpha] + 1) - \alpha[\beta/\alpha]([\beta/\alpha] + 1)$.*

Theorem 3.1. *A design d_0^* with p lines is MS-optimal in $D_0(p, n)$ if and only if*

(i) every line occurs $s = 2n/p$ times in d_0^*, and

(ii) the number of times $g_{d_0^* ii'}$ that cross (i, i') occurs in d_0^* satisfies

$$|g_{d_0^* ii'} - s/(p-1)| < 1 \quad for \quad i \neq i' \ , \quad i, i' = 1, \ldots, p \, .$$

PROOF. From (1), for any design $d_0 \in D_0(p, n)$

$$\mathrm{tr}(C_{d_0}) \quad = \quad \sum_{i=1}^{p} s_{d_0 i} - \frac{1}{n} \sum_{i=1}^{p} s_{d_0 i}^2.$$

Now, since $\sum_{i=1}^{p} s_{d_0 i} = 2n$ and $2n/p = s$, using Lemma 3.1,

$$\sum_{i=1}^{p} s_{d_0 i}^2 \geq 4n^2/p.$$

Hence,

$$\mathrm{tr}(C_{d_0}) \leq 2n - 4n/p = 2n(p-2)/p. \qquad \ldots (3)$$

By Lemma 3.1, equality above is attained if and only if $s_{d_0 i} = 2n/p = s$ for $i = 1, \ldots, p$. Let $D_0^*(p, n)$ be the sub-class of designs for which $s_{d_0 i} = s$ for $i = 1, \ldots, p$. Then for a design $d_0 \in D_0^*(p, n)$

$$C_{d_0} = G_{d_0} - \frac{2s}{p} \mathbf{1}_p \mathbf{1}_p'.$$

and, using the fact that $\sum_{i \, < \, i'} \sum g_{d_0 ii'} = n$,

$$\begin{aligned}
\mathrm{tr}(C_{d_0}^2) \quad &= \quad \sum \sum \ g_{d_0 ii'}^2 - \frac{4s}{p} \sum \sum \ g_{d_0 ii'} + 4s^2 \\
&= \quad 2 \sum_{i \, < \, i'} \sum \ g_{d_0 ii'}^2 + s^2 p - \frac{8s}{p} \ \sum_{i \, < \, i'} \sum \ g_{d_0 ii'} \\
&= \quad s^2 p - 8sn/p + 2 \sum_{i \, < \, i'} \sum \ g_{d_0 ii'}^2 \\
&= \quad s^2 (p-4) + 2 \sum_{i \, < \, i'} \sum \ g_{d_0 ii'}^2.
\end{aligned}$$

But, from Lemma 3.1, with $\alpha = p(p-1)/2$ and $\beta = n$, we have

$$\sum_{i \, < \, i'} \sum g_{d_0 ii'}^2 \geq n(2[s/(p-1)] + 1) - \frac{p(p-1)}{2} [s/(p-1)]([s/(p-1)] + 1).$$

Hence,

$$\mathrm{tr}(C_{d_0}^2) \geq s^2(p-4) + n(2[s/(p-1)] + 1) - \frac{p(p-1)}{2} [s/(p-1)]([s/(p-1)] + 1).$$

By Lemma 3.1, equality above is attained if and only if $g_{d_0 ii'} = [s/(p-1)]$ or $[s/(p-1)]+1$, for $i \neq i'$. □

From Theorem 3.1, PDC designs in which every line appears the same number $s = 2n/p$ of times and in which each cross appears either $\lambda = [s/(p-1)]$ or $\lambda+1$ times are MS-optimal. A common way to construct a PDC design is to take a conventional binary incomplete block design with p treatments each occurring s times, n distinct blocks of size 2 and treatment concurrences λ and $\lambda+1$ (called the auxiliary design by Singh and Hinkelmann, 1995) and to form crosses between the two treatments in each block. Any such PDC satisfies the conditions of Theorem 3.1 and is MS-optimal. Among others, this includes the M-designs of Singh and Hinkelmann (1995), the first series of PDCs of Mukerjee (1997), and the PDCs formed from the basic plans listed by Gupta, Das and Kageyama (1995). We discuss other such PDCs in Section 4.

We consider now the class $D(p,b,k)$ of block designs with $n = bk$ crosses among the p lines, divided into b blocks of size k crosses. Ignoring the division into blocks, the set of $n = bk$ crosses involved in a design $d \in D(p,b,k)$ forms a PDC completely randomized design $d_0 \in D_0(p,bk)$. Thus to every block design d in $D(p,b,k)$, there corresponds a completely randomized design d_0 in $D_0(p,bk)$. Following Gupta, Das and Kageyama (1995), we define a block design $d \in D(p,b,k)$ to be an *orthogonal block design* if the ith line occurs in every block s_i/b times for $i = 1,\dots p$ where s_i is the replication of the ith line in the design, that is

$$N_d = b^{-1}\boldsymbol{s}_d\mathbf{1}_b',$$

where N_d is the line-block incidence matrix of the design d. ¿From (1) and (2) and the fact that $N_d\mathbf{1}_b = \boldsymbol{s}_d$, it follows that

$$C_d = G_d - \frac{1}{k}N_dN_d' = C_{d_0} - \frac{1}{k}N_d(I_b - \frac{1}{b}\mathbf{1}_b\mathbf{1}_b')N_d'. \qquad \dots(4)$$

Thus, $C_d \leq C_{d_0}$,where for a pair of nonnegative definite matrices A and B, $A \leq B$ implies that $B-A$ is non-negative definite. Equality is achieved if and only if $N_d = b^{-1}\boldsymbol{s}_d\mathbf{1}_b'$, which is the condition for an orthogonal block design.

Now, consider a non-increasing optimality criterion ϕ. (An optimality criterion ϕ is non-increasing if $\phi(B) \leq \phi(A)$ whenever $B-A$ is nonnegative definite). If the unblocked PDC design $d_0^* \in D_0(p,bk)$ corresponding to an orthogonal block design $d^* \in D(p,b,k)$ is ϕ-optimal, then d^* is also ϕ-optimal since $\phi(C_{d^*}) = \phi(C_{d_0^*}) \leq \phi(C_{d_0}) \leq \phi(C_d)$ for any $d \in D(p,b,k)$ and corresponding $d_0 \in D_0(p,n)$. The MS-, A-, D- and E-criteria are included in the ϕ-criterion. Thus, in particular, we have the following theorem.

THEOREM 3.2. *An orthogonal block design $d^* \in D(p,b,k)$ is MS-optimal in $D(p,b,k)$ if the corresponding design $d_0^* \in D_0(p,bk)$ satisfies the conditions of Theorem 3.1.*

4. Classes of MS-optimal Designs

Orthogonally blocked MS-optimal PDC designs for p lines with each cross occurring λ or $\lambda+1$ times can be constructed from resolvable or 2-resolvable auxiliary incomplete block designs with p treatments each occurring s times, n blocks of size 2 and treatment concurrences λ or $\lambda+1$. The PDC design is obtained, as described earlier, by forming a cross from the pair of treatments in each of the n blocks. Each resolvable (or 2-resolvable) set of blocks in the conventional design partitions the crosses of the PDC into orthogonal blocks. We call such designs *Series C designs*.

An alternative construction is as follows. If p is even (odd), divide the $n_c = p(p-1)/2$ crosses of the complete diallel cross into $p-1$ $((p-1)/2)$ blocks of size $k = p/2$ $(k = p)$ in such a way that every line appears once (twice) per block, i.e. is orthogonally blocked. Select any subset of $b = n/k$ blocks. Orthogonally blocked complete diallel crosses are given by the series 2 complete diallels of Gupta and Kageyama (1994) and by the Family 5 complete diallels of Das, Dey and Dean (1998), (or, equivalently, the series 1 plans listed by Gupta, Das and Kageyama, 1995). We list these designs below under the headings of Methods 1 and 2. Selected subsets of blocks from Methods 1 and 2 will be called *Series A* and *B designs* respectively.

In each of the above constructions, the basic PDC design satisfies the conditions of Theorem 3.1 and is MS-optimal in $D_0(p, bk)$. In addition, since each design is orthogonally blocked, the block design is also MS-optimal in $D(p, b, k)$, by Theorem 3.2. In Section 5, we compare the best series A and B designs with a large number of series C designs. Except for a few cases, the series A and B designs perform better.

Method 1 (p even). For any integer $t \geq 2$ and $p = 2t$ lines, we construct the following set of $p-1$ blocks, each of size $k = p/2$. For $j = 1, 2, \ldots, p-1$, we define block j as

Block j: $\{(j, 2t-3+j), (1+j, 2t-4+j), \ldots, (t-2+j, t-1+j), (j-1, \infty)\}$
In each block, the symbols are reduced modulo $(p-1)$ and ∞ is an invariant symbol. An MS-optimal series A design $d^* \in D(p, b, k)$ with $p = 2t,\ t \geq 2,\ k = p/2,\ b < p-1$ is obtained by selecting any b distinct blocks. Such a design has $n = pb/2$ with $s = b$.

Method 2 (p odd). For any integer $t \geq 1$ and $p = 2t+1$ lines, we construct the following $t = (p-1)/2$ blocks each of size $k = p$. For $j = 1, 2, \ldots, t$, we define block j as

Block j: $\{(j, 2t+1-j), (1+j, 1-j), (2+j, 2-j), \ldots, (2t-1+j, 2t-1-j), (2t+j, 2t-j)\}$
In each block, the symbols are reduced modulo $p = (2t+1)$. An MS-optimal series B design $d^* \in D(p, b, k)$ with $p = k = 2t+1,\ b < t,\ t \geq 1$ is obtained by

selecting any b of the t blocks. Such a design has $n = ps/2$ with $s = 2b$.

For given t and b, Method 1 gives rise to $\binom{2t-1}{b}$ possible MS-optimal block designs with $b < 2t-1$ and Method 2 gives rise to $\binom{t}{b}$ possible MS-optimal block designs with $b < t$. Any of the Series A designs can be enlarged by appending a complete set of $p-1$ blocks from Method 1 and the Series B designs can be enlarged by appending a complete set of $(p-1)/2$ blocks from Method 2.

Example 1. Suppose we have $p = 8$ lines, so that $t = 4$. A Series A design with $b = 4$ blocks of $k = 4$ crosses can be obtained by selecting any four blocks from the Method 1 construction. Suppose we select the blocks with $j = 1, 2, 3, 5$. Leaving the symbol ∞ fixed, and reducing all other symbols modulo $p - 1 = 7$, we obtain the design:

(1, 6)	(2, 5)	(3, 4)	(0, ∞)
(2, 0)	(3, 6)	(4, 5)	(1, ∞)
(3, 1)	(4, 0)	(5, 6)	(2, ∞)
(5, 3)	(6, 2)	(0, 1)	(4, ∞)

This design is MS-optimal in $D(8, 4, 4)$. An MS-optimal design in $D(8, 11, 4)$ with $b = 11$ blocks of size $k = 4$ can be obtained by appending the full set of $p - 1 = 7$ blocks from Method 1 to the above design.

Example 2. The set of $bk = 16$ crosses in Example 1 provides an MS-optimal completely randomized design in $D_0(8, 16)$. Condition (i) of Theorem 3.1 is satisfied since each line occurs $s_{d_0^* i} = 4$ times in the design and condition (ii) is satisfied since every cross (i, i') appears 0 or 1 time in the design.

5. A- and D- Efficiency

In this section, we show that the Series A and B PDC orthogonal block designs constructed in Section 4, are not only MS-optimal, but also have high efficiencies with respect to the A- and D-optimality criteria. We also show that these efficiencies compare extremely well with the best Series C designs obtained from a number of different sources, and with the E-M designs of Singh and Hinkelmann (1995), and the E-optimal designs of Mukerjee (1997).

Let $z_{d1} \le z_{d2} \le \ldots z_{d,p-1}$ be the non-zero eigenvalues of C_d for a connected design $d \in D(p, b, k)$. For any such design, the A-value is defined as $\phi_A(d) = tr(C_d^-) = \Sigma z_{di}^{-1}$ and the D-value as $\phi_D(d) = \prod z_{di}^{-1}$. Let d_A (d_D) be the A-optimal (D-optimal) design in $D(p, b, k)$, then the A- and D- efficiencies of design d are defined as

$$e_A(d) = \phi_A(d_A)/\phi_A(d)$$

and

$$e_D(d) = \{\phi_D(d_D)/\phi_D(d)\}^{1/(p-1)}.$$

We now give the following result on lower-bounds for A- and D- efficiencies in $D(p,b,k)$.

LEMMA 5.1. *The A- and D-efficiency lower-bounds* $e'_A(d)$ *and* $e'_D(d)$ *for design* $d \in D(p,b,k)$ *are given by*

$$e'_A(d) = \frac{(p-1)^2}{s(p-2)\phi_A(d)} \qquad \ldots (5)$$

and

$$e'_D(d) = \frac{(p-1)}{s(p-2)\{\phi_D(d)\}^{1/(p-1)}} \qquad \ldots (6)$$

where $s = 2n/p$.

PROOF. Let d be a block design in $D(p,b,k)$, and let d_0 be the corresponding unblocked design in $D_0(p,n)$ with $n = bk$. Let C_d and C_{d_0} be the information matrices for estimating the general combining ability effects, as defined in Section 2, and let $z_{d1} \leq \ldots \leq z_{d,p-1}$ and $z_{d_01} \leq \ldots \leq z_{d_0,p-1}$ be the sets of their non-zero eigenvalues, respectively. Since the second term of the right hand side of (4) is non-negative definite, it is true that

$$z_{di} \leq z_{d_0i} \quad , \quad i = 1, \ldots, p-1.$$

Using this fact, together with (3) and the fact that the harmonic mean is smaller than the arithmetic mean, we have

$$\phi_A(d) = \sum_{i=1}^{p-1} z_{di}^{-1} \geq \sum_{i=1}^{p-1} z_{d_0i}^{-1} \geq \frac{(p-1)^2}{\sum_{i=1}^{p-1} z_{d_0i}} \geq \frac{(p-1)^2}{s(p-2)}.$$

Consequently, $\phi_A(d_A) \geq (p-1)^2/s(p-2)$ and $e_A(d) \geq e'_A(d)$.

The proof that $e_D(d) \geq e'_D(d)$ follows along similar lines using the fact that the geometric mean is smaller than the arithmetic mean. □

It is clear from the proof of Lemma 5.1, that the lower bounds (5) and (6) also hold for designs in $D_0(p,n)$. We note that (5) is equivalent to the average efficiency factor for a PDC design relative to a complete diallel cross as calculated by Singh and Hinkelmann (1995) with $2r$ replaced by s in their formula (17).

EXAMPLE 3. In Example 1, we presented an MS-optimal design d^* in $D(8,4,4)$. The A-value and D-value for this design are $\phi_A(d^*) = 2.4811$ and $\phi_D(d^*) = 0.00034$. The lower bounds (5) and (6) on the A- and D- efficiencies are $e'_A(d^*) = .8229$. and $e'_D(d^*) = .9112$. The design is listed in Table 1 and is the best Series A design in terms of the A- and D- values.

Table 1. Series A orthogonal block designs with block size $p/2$

p	n	$e'_A(d^*)$	$e'_D(d^*)$	Building block selection
4	8	.9000	.9449	1 1–3
4	10	.9000	.9524	1 3 1–3
6	9	.7143	.8532	1 2 3
6	12	.8929	.9473	1 2 3 4
6	18	.9615	.9801	2 1–5
6	21	.9542	.9769	1 2 1–5
6	24	.9637	.9819	1 2 5 1–5
6	27	.9804	.9903	1 2 4 5 1–5
8	12	.6405	.8242	1 2 3
8	16	.8229	.9112	1 2 3 5
8	20	.9026	.9520	1 2 3 4 5
8	24	.9608	.9806	1 2 3 4 5 6
8	32	.9800	.9898	3 1–7
8	36	.9728	.9863	1 2 1–7
8	40	.9736	.9867	2 3 6 1–7
8	44	.9782	.9890	1 2 3 6 1–7
8	48	.9842	.9921	1 2 3 4 5 1–7
10	15	.5571	.7920	1 2 4
10	20	.7826	.8915	1 2 3 5
10	25	.8782	.9373	1 2 3 4 7
10	30	.9184	.9593	1 2 3 4 5 7
10	35	.9530	.9767	1 2 3 4 5 6 7
10	40	.9798	.9900	1 2 3 4 5 6 7 8
10	50	.9878	.9938	3 1–9
10	55	.9820	.9910	3 4 1–9

In Tables 1 and 2, we list the best Series A designs for given $p = 2t$ and $n = tb$ (with $p \leq 16,\ n < 100$), together with their efficiency lower bounds (5) and (6). For each design listed, we give the blocks $j_1, \ldots, j_b$ from Method 1 used to construct the design. Similarly, in Table 3, we list the best Series B designs constructed from the blocks of Method 2 for given $p = 2t+1$ and $n = (2t+1)b$, (with $p \leq 15,\ n < 100$). Larger Series A designs can be formed by appending to the base design one or more full sets of $u = p-1$ blocks (denoted 1–u in Tables 1 and 2), and larger Series B designs can be formed by appending one or more full sets of $u = (p-1)/2$ blocks (denoted 1–u in Table 3). We have included such designs with an appended complete set of blocks provided that either $n \leq 50$ or the base design is disconnected. For these larger designs, the A- and D-efficiency bounds are considerably higher than the corresponding values for the base designs. Apart from the very small designs, the efficiency lower bounds of Series A and B designs tend to be above .85, and most often above .90, with $e'_D(d^*) > e'_A(d^*)$. Some of the smaller Series A designs are disconnected and these are not listed.

Table 2. Series A orthogonal block designs with block size $p/2$, continued

p	n	$e'_A(d^*)$	$e'_D(d^*)$	Building block selection
12	18	.5698	.7941	1 2 4
12	24	.7590	.8803	1 2 3 6
12	30	.8595	.9271	1 2 3 5 8
12	36	.8975	.9484	1 2 3 4 5 8
12	42	.9304	.9650	1 2 3 4 5 6 9
12	48	.9537	.9769	1 2 3 4 5 6 7 9
12	54	.9725	.9863	1 2 3 4 5 6 7 8 9
12	60	.9878	.9939	1 2 3 4 5 6 7 8 9 10
12	72	.9918	.9959	6 1–11
12	78	.9393	.9773	3 4 1–11
14	21	.6142	.8029	1 2 5
14	28	.7467	.8748	1 2 3 6
14	35	.8393	.9182	1 2 3 5 8
14	42	.8824	.9406	1 2 3 5 6 9
14	49	.9156	.9573	1 2 3 4 5 7 10
14	56	.9382	.9690	1 2 3 4 5 6 8 11
14	63	.9566	.9782	1 2 3 4 5 6 8 10 11
14	70	.9702	.9851	1 2 3 4 5 6 7 8 9 11
14	77	.9819	.9910	1 2 3 4 5 6 7 8 9 10 11
14	84	.9918	.9959	1 2 3 4 5 6 7 8 9 10 11 12
14	98	.9941	.9970	2 1–13
16	24	.5904	.7959	1 2 5
16	32	.7387	.8708	1 2 3 7
16	40	.8282	.9128	1 2 3 5 11
16	48	.8730	.9359	1 2 3 5 8 13
16	56	.9054	.9521	1 2 3 4 7 10 12
16	64	.9282	.9637	1 2 3 4 5 7 10 12
16	72	.9453	.9724	1 2 3 4 5 6 8 11 12
16	80	.9588	.9793	1 2 3 4 5 6 7 10 11 14
16	88	.9701	.9850	1 2 3 4 5 6 7 8 10 12 13
16	96	.9793	.9896	1 2 3 4 5 6 7 8 9 10 12 13

We searched the following sources for auxiliary resolvable and 2–resolvable incomplete block designs with blocks of size 2 and pairs of treatments occurring in λ or $\lambda + 1$ blocks:

Clatworthy (1973): partially balanced incomplete block designs
John, Wolock and David (1972): cyclic designs
Mitchell and John (1976): regular graph designs

From each such auxiliary design, we formed the series C design as explained in Section 4. The series C designs based on resolvable plans can be compared with the Series A designs and those based on the 2-resolvable plans can be compared with the Series B designs. We found only four incomplete block designs d from these sources that have higher values of $e'_A(d)$ and $e'_D(d)$ than our listed Series A and B designs, and these are listed with references c.Xx in Table 4. We note that the improvement over the Series A and B designs is only of the order of .02. The corresponding incomplete block designs are listed by Clatworthy (1973) in resolvable sets of blocks.

Table 3. Series B orthogonal block designs with block size p

p	n	$e'_A(d^*)$	$e'_D(d^*)$	Building block selection
5	5	.4444	.6667	1
5	15	.9383	.9686	1 1–2
7	7	.3000	.6000	1
7	14	.8419	.9217	1 2
7	28	.9653	.9825	1 1–3
7	35	.9769	.9885	1 3 1–3
9	9	.7033	.8081	3
9	18	.8472	.9168	1 2
9	27	.9345	.9676	1 2 3
9	45	.9812	.9901	3 1–4
9	54	.9812	.9903	1 2 1–4
11	11	.1852	.5556	1
11	22	.8289	.9079	1 2
11	33	.9222	.9587	1 2 3
11	44	.9647	.9825	1 2 3 4
13	13	.1558	.5455	1
13	26	.8228	.9039	1 2
13	39	.9156	.9541	1 2 3
13	52	.9511	.9745	1 2 3 5
13	65	.9779	.9890	1 2 3 4 5
15	15	.1346	.5385	1
15	30	.8869	.9292	3 6
15	45	.9080	.9500	1 2 3
15	60	.9490	.9723	1 4 5 6
15	75	.9661	.9827	2 3 5 6 7
15	90	.9849	.9925	1 2 3 4 5 6

Table 4. Resolvable and 2-resolvable incomplete block designs

p	n	$e'_A(d)$	$e'_D(d)$	Reference
6	21	.9653	.9809	c.R20
8	12	.8596	.9099	m.8.4
8	40	.9800	.9890	c.R32
9	9	.7033	.8081	m.9.3
10	30	.9265	.9618	c.T1
12	18	.8345	.8955	m.12.4
12	30	.9308	.9564	m.12.6
15	15	.6853	.8002	m.15.3
16	24	.8242	.8898	m.16.4
16	48	.8929	.9425	c.M2 or c.LS3
16	56	.9547	.9717	m.16.8

The PDC E-optimal designs of Mukerjee (1997) are based on auxiliary disconnected group divisible designs. Ten of these have parameter values that coincide with the listed series A and B designs, and seven of these can be formed into similar sized orthogonal block designs. These seven design are indicated in Table 4 with reference m.p.x, where p is the number of lines and x is the group size of the group divisible design used in the construction. Six of the seven designs have higher efficiencies than the corresponding series A or B designs. The two designs with $x = 3$ were proved by Mukerjee to be A- and D-optimal.

The blocked E-M designs of Singh and Hinkelmann (1995) have $ps/2$ distinct crosses in the PDC replicated $r \geq 2$ times; that is, they have $psr/2$ observations with $r \geq 2$. We have not listed the the M-S optimal Series A and B designs for such large sizes. However, since replication of an entire design does not affect its efficiency calculation, we compared the efficiency of each listed E-M design with $psr/2$ observations with a replications of the corresponding listed series A or B design having $n = psr/2a$ distinct crosses each replicated once, where a is the smallest integer for which a design is listed. For example, Singh and Hinkelmann's design for $p = 8$ lines, $ps/2 = 12$ distinct crosses replicated $r = 11$ times each (132 observations), in $b = 33$ blocks of size $k = 4$, was compared with $a = 3$ replications of our series A design with $n = 44$ distinct crosses each replicated once (132 observations) with the same block structure. Such designs do not satisfy Theorem 3.1 and are not MS-optimal. However, the series A designs have more distinct crosses than the E-M designs and, not surprisingly, over the range of matching parameter values, the listed E-M designs have lower efficiencies than replications of the listed series A and B designs.

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