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Comment on 'Amitava Bose: A Discussion on Continuous Time Growth Models'

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The Problem

A well known feature of the neoclassical growth model of Solow (1956) is that if capital is essential, an economy without any capital cannot reach the non-trivial steady state. With no capital, there is no output, and no investment and savings to kick-start accumulation. However, consider the following special case of the Solow Model,

$$\begin{aligned} \dot{k} &= sf(k(t)) - \delta k(t) \\ k(0) &= 0. \end{aligned} \tag{1}$$

Assume that

$$\begin{aligned} f(k(t)) &= 4\sqrt{k(t)} \\ s &= \frac{1}{2} \end{aligned}$$

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where $f(.)$ is an intensive form production function defined over k, the capital labor ratio, and $s \in [0,1]$, $\delta \in [0,1]$ are the savings rate and depreciation rate, respectively. We abstract from population growth ($n = 0$). Equation (1) with the initial condition, $k(0) = 0$, reduces to the following initial value problem (IVP),

$$\begin{aligned} \dot{k} &= 2\sqrt{k(t)} \\ k(0) &= 0. \end{aligned} \tag{2}$$

Note that $k(t) = 0$ for all $t \geq 0$ is a solution. However, this is not the only solution. It is easy to verify that $k(t) = t^2$ for all $t \geq 0$ is also a solution.[2] In fact, for each $\mu > 0$

$$k_\mu(t) = \left\{ \begin{array}{cc} 0, & 0 \leq t < t_\mu \\ (t - t_\mu)^2, & t \geq t_\mu \end{array} \right\}$$

are also a class of solutions to the IVP, (2).[3] We can interpret t_μ as the take-off time, which occurs spontaneously.

[2]Using separation of variables,

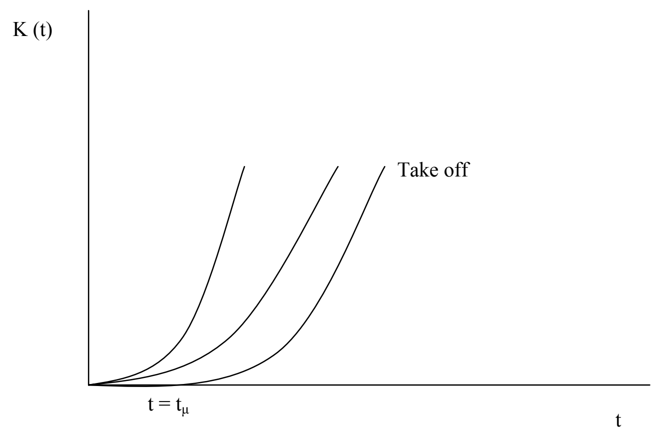
$$\begin{aligned} \int \frac{dk}{2\sqrt{k}} &= \int dt \\ \sqrt{k} &= t + c \\ k(t) &= (t + c)^2. \end{aligned}$$

Since $k(0) = 0$,this implies $c = 0$. Hence, $k(t) = t^2$ for all $t \geq 0$, is also a solution to the initial value problem, (2)

[3]Suppose we consider a slight variant of the initial value problem,

$$\begin{aligned} \frac{dk}{dt} &= 3k^{\frac{2}{3}} \\ k(0) &= 0. \end{aligned}$$

This problem also does not have a unique solution. To see this, $k(t) = 0$ is a solution since



Two things should be noted. First, the puzzling feature is that even though capital is essential ($k(0) = 0$), spontaneous take off is possible. Is there causality, or spontaneous take off ? An economy with zero capital may go on without accumulation forever, or depart on a trajectory of positive growth of capital, although with no apparent cause. Why does this happen ? Second, the non-uniqueness of the solution stems from the fact that a Lipschitz

it satisfies both conditions. Using separation of variables, it is easy to see that $k(t) = t^3$, for all $t \geq 0$ is also a solution. In fact, one can show that there exist an infinite two parameter family of solutions,

$$\begin{aligned} &= (t - c_1)^3 \text{ if } t < c_1 \\ k_{c_1,c_2}(t) &= 0 \qquad\qquad c_1 \leq t \leq c_2 \\ &= (t - c_2)^3 \text{ if } t \geq c_2, \end{aligned}$$

where $c_1 < 0 < c_2$.

condition is violated.[4] In equation (2),

$$\lim_{k\to 0}\frac{\partial \dot{k}}{\partial k} = \lim_{k\to 0}\frac{1}{k^{\frac{1}{2}}} = \infty$$

Because the differential equation is not Lipschitz continuous at $k = 0$, the solution is not unique.[5] Amitava's insightful paper is devoted to understanding why a spontaneous take-off can happen even though capital is essential.

A Discrete Time Interpretation

Amitava tries to explain the possible causality by discretizing the continuous time growth trajectory in (2) and thinking about the problem in the context of a production lag. He assumes that there are two inputs, a fixed capital stock, k_t, at any date, t, and a waiting time, h. The capital stock, k, and h are separable. The period lasts for a length of time h at the end of which the capital stock grows by, $F(k_t, h)$.Unlike the intensive form production function

[4] We use the following definition of Lipschitz continuity. A function, $f : D \subseteq R\times R^n \to R$ is said to be Lipschitz continuous in X on D, if $\exists$ a constant k s.t.

$$\| f(t,x) - f(t,y) \| \leq k \| x - y \|$$

for all x, y, s.t. (t,x) and $(t,y) \in D$. k is called a Lipschitz constant for f. If the function is differentiable and the derivative is bounded ($< \infty$), Lipschitz continuity obtains. See Cronin (1994, p. 12).

[5] This implies that the continuity of of a function is not sufficient to gaurantee uniqueness, but is sufficient to ensure the existence of a solution. The additional condition that a function satisfy a Lipschitz condition is really needed to only to prove the uniqueness of a solution.

before, $F(\cdot)$ is a stock production function, as opposed to the flow production function, $f(\cdot)$.The time interval, $[t, t+h]$, is the production period. Finally, there is no consumption while the capital stock is experiencing pure growth within the period. Consumption takes place at the end of the production period after which a fraction, $1-s$, of additional output is consumed, while a fraction, s, is saved and can be used as an input in the next period. Abstracting from depreciation and population growth, k_t satisfies

$$k_{t+h} = sF(k_t, h) + k_t, \quad t = 0, h, 2h...... \tag{3}$$

Given $k(0)$, the above sequence defines $< k_t >$ for $t = 0, h, 2h...$ Now assume that

$$F(k, h) = 2h(h + 2\sqrt{k}), \tag{4}$$

and $s = \frac{1}{2}$. Then the solution to (3) is given by

$$k(t) = t^2, \qquad t = 0, 1, 2....$$

which is identical to the non-trivial solution of the initial value problem, (2). Since h is arbitrary, Amitava argues that we can explore the link between discrete time and continuous time models by taking h to be smaller. Further, (4) has a unique solution, $k(t) = t^2$ for any $h > 0$. Equation (3) does not generate the null solution ad infinitum because it is not the case that $k_{t-h} = 0$ when $k_t = 0$. However, if $F(k) = 4h\sqrt{k}$, then the author shows that the only solution for the case, $k_0 = 0$ is $k_t = 0$, for $t = 0, 1, 2....$Therefore, Amitava's main point is that for $k_0 = 0$, different reduced form models (such as equation (4)) in discrete time converge to a single reduced form model in continuous time which leads to multiple solutions.

Another Explanation

Another approach would be to retain the continuous time framework and think about the tension between the Inada condition and the essentiality of capital. Whether there is take-off or no accumulation (forever) depends on what "force" gets the upper hand at $k = 0$.[6] On the one hand, no capital can be accumulated since capital is essential. On the other hand, at $k = 0$, the marginal product of capital is infinity. Therefore, even a zero amount of capital can lead to positive output and to accumulation. Which of these forces dominates at $t = t_\mu$ is unpredictable. Either the essentiality of capital dominates. This produces the trivial solution ($k = 0$). Or the Inada condition dominates and this triggers an instantaneous take off.

What happens when the production function violates the Inada Condition ? Consider the aggregate law of motion of capital, K,

$$\dot{K}(t) = sF(K, L) - \delta K(t). \tag{5}$$

Then,

$$\lim_{K\to 0} \frac{\partial \dot{K}}{\partial K} = s \lim_{K\to 0} \frac{\partial F}{\partial K} - \delta < \infty$$

Because the derivative is bounded ($< \infty$), equation (5) is Lipschitz continuous. However, since capital is essential, $F(0, L) = 0$. Hence, $\dot{K}(t) = 0$, and

[6] Hakenes and Irmen (2006) formalize this line of reasoning. They posit a broad class of production functions (which encompasses the neo-classical production function) and show that take-off is possible even though the initial capital stock is zero and capital is essential. Since the marginal product of capital is infinite, the trivial steady state becomes so unstable that take off becomes possible.

take-off is excluded. Essentiality alone is not sufficient for a take off. However, the Inada condition (or violation of Lipschitz continuity) alone implies an immediate take off. The important point to note that is when both essentiality of capital and the Inada condition hold, a take off is possible, but need not happen.[7]

References

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[7] Barro and Sala-i-Martin (2004, p. 77) show that the condition of constant returns to scale and the Inada condition implies that capital is essential for production.