A USEFUL CONVERGENCE THEOREM

By V. S. VARADARAJAN

Indian Statistical Institute, Calcutta

SUMMARY. Let $X_1, X_2, \ldots$ be a sequence of k-dimensional random vectors. It is proved that in order that the distributions of the X_n should be convergent to a limit, it is necessary and sufficient that the distributions of $l(X_n)$ should converge to some limit for every linear function l.

The present paper is concerned with convergence of probability distributions on finite-dimensional Euclidean spaces. It is proved that for a given sequence $\{P_n\}$ of probability distributions to have a limit, it is necessary and sufficient that the one-dimensional distributions $\{P_n\lambda^{-1}\}$ have a limit for each linear function λ. This theorem seems useful in situations where the multidimensional theorem involves heavier computation. For example, this result ensures that in order that the sequence $\{P_n\}$ be asymptotically normal, it is necessary and sufficient that the sequence $\{P_n\lambda^{-1}\}$ be asymptotically normal for each linear function λ.

In what follows, X is a fixed k-dimensional space and S the σ-field of its Borel sets. Given a measure P on S and a measurable function λ on X, there exists a measure on the Borel sets of the real line, denoted by $P\lambda^{-1}$, such that $P\lambda^{-1}(E) = P\{\lambda^{-1}(E)\}$ for every Borel set E on the reals. If P_1 and P_2 are two measures on S, $P_1 = P_2$ when and only when $P_1\lambda^{-1} = P_2\lambda^{-1}$ for each linear function λ on X. A sequence $\{P_n\}$ of measure on S will be said to converge weakly to P if

$$\int_X g dP_n \to \int_X g dP$$

for every bounded continuous function g on X. This convergence is the classical convergence of P. Levy. It is possible to prove as in the one-dimensional case, that on the space of all measures on X this convergence arises through a metric (Varadarajan 1958, where a result of much greater generality is proved). The importance of this result lies in the fact that we can use simple topological arguments in many situations. We now state a lemma, which is true under more general situations (Kolmogorov and Prohorov 1954) but whose proof in the present case is very simple and well-known.

Lemma : *Let Γ be a set of probability measures on X. In order that Γ is conditionally compact* (i.e. $\bar{\Gamma}$ is compact) *it is necessary and sufficient that for each $\epsilon > 0$, there should exist a compact set K_ϵ such that*

$$P(K_\epsilon) > 1-\epsilon$$

for all $P \epsilon \Gamma$.

We can now state and prove our theorem.

Theorem : *Let* $\{P_n\}$ *be a sequence of probability distributions on* S. *In order that there should exist a* P *such that* $P_n \Longrightarrow P$, *it is necessary and sufficient that for each linear function* λ *on* X *the sequence of one-dimensional distributions* $\{P_n\lambda^{-1}\}$ *converges to a limit.*

Proof : The necessity of the condition is obvious. We will prove the sufficiency now. Let $\Gamma = \{P_1, P_2, \ldots\}$. Let λ_i ($i = 1, 2, \ldots k$) be the projection of X on its i-th component. Since $\{P_n\lambda_i^{-1}\}$ has a limit we can find numbers $a_{i\epsilon}$ and $b_{i\epsilon}$ such that

$$P_n\lambda_i^{-1}[a_{i\epsilon}, b_{i\epsilon}] > 1-\epsilon/k$$

for all n. If we now define $K_\epsilon = \{(x_1, \ldots x_k) : a_{i\epsilon} \leqslant x_i \leqslant b_{i\epsilon} \text{ for } i = 1, 2, \ldots k\}$, we have

$$P_n(K_\epsilon) > 1-\epsilon$$

for all n. This shows, in virtue of the lemma, that Γ is conditionally compact. If Γ has two limit points Q_1 and Q_2, then for any linear function λ, $Q_1\lambda^{-1}$ and $Q_2\lambda^{-1}$ are limit points of the sequence $\{P_n\lambda^{-1}\}$. This sequence is given to converge and hence it follows that $Q_1\lambda^{-1} = Q_2\lambda^{-1}$. Since λ is arbitrary, $Q_1 = Q_2$. Thus Γ can have only one limit point, say P. Since we have already shown that Γ is conditionally compact, it follows that $P_n \Longrightarrow P$. This completes the proof.

As an interesting application we have the following.

Corollary : *In order that* $\{P_n\}$ *may converge to a normal distribution on* X, *it is sufficient (and also necessary) that the sequence* $\{P_n\lambda^{-1}\}$ *converges to some normal distribution for each* λ.

REFERENCES

KOLMOGOROV, A. N. AND PROHOROV, YU. V. (1954): *Bericht Über Die Tagung Wahrscheinlichkeitsrechnung und Mathematische Statistik*, Berlin, 119.

VARADARAJAN, V. S. (1958): Weak convergence of measures in separable metric spaces. *Sankhyā*, **19**, 15.

Paper received : May, 1958.