A Generalized Design of the Mexican Hat and other Even-order Hermitian Wavelets in a Gaussian Scale Space

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Abstract- **A generalized methodology of constructing a Mexican hat wavelet family involving even order Gaussian derivatives has been devised in a Gaussian scale space only. The optimization has been carried out in Fourier domain and the kernels in Gaussian scale space domain are found to be exact replica of their derivative wavelet counterpart for low as well as high order. Wavelet properties of the lowest order (2), has been discussed and the results are shown to be better and different from the well-known LOG-DOG equivalence of Marr-Hildreth. Such filters, simple to implement in Gaussian scale space, are likely to be important in vision, analysis of seismic signals, cosmic microwave background (CMB) maps and possibly in the general cases of signals from Gaussian point sources.**

I. INTRODUCTION

In the very first chapter of her well-known textbook [1], Daubechies introduces the reader to the world of wavelets mentioning that a typical choice for wavelet function is "the second derivative of Gaussian, sometimes called the Mexican hat function because it resembles a cross section of a Mexican hat." Thus if,

$$\psi^{a,b}(s) = |a|^{-1/2}\psi\left(\frac{s-b}{a}\right)$$

represents a typical wavelet function, then

$$\psi(t) = (1 - t^2)\exp(-t^2/2)$$

is the second derivative of Gaussian, mentioned above, where the scale parameter of the Gaussian is unity. Daubechies [1] goes on to show that this Mexican hat function is well localized in both time and frequency and satisfies the condition

$$\int dt\,\psi(t) = 0.$$

In Geophysics the Mexican hat function is popularly known as the Ricker wavelet. It is frequently used by the geophysicists to analyze seismic data [2]. The same function in two dimension, happens to be an oft-used tool for the computer vision and image processing community. This is the well-known Laplacian of Gaussian function which produces a circularly symmetric Mexican hat:

$$\nabla^2 g(r) = -\frac{1}{\pi\sigma^2}\left[1 - \frac{r^2}{2\sigma^2}\right]e^{-\frac{r^2}{2\sigma^2}}$$

Such a function was introduced by Marr-Hildreth [3] as an edge detector in low-level vision. The Gaussian part of this operator effectively blurs the image by wiping out structures that at scales smaller than the space constant σ of the Gaussian. The Gaussian function has the desirable characteristic of being smooth and localized in both spatial and frequency domain and is a unique distribution that is simultaneously optimally localized in both domains. On the other hand, the Laplacian is the lowest order isotropic differential operator having the great advantage of economy in computation, over the directional derivatives.

Marr [4] went on to show that that the same Mexican hat can also be generated by the difference of two Gaussian functions, the physiologist's model for low-level vision, for a certain ratio of the scale parameters (σ_1, σ_2) of the two Gaussians (i.e., for a certain value of σ_1:σ_2). We shall talk about this particular conclusion of Marr, regarding the accuracy of this equivalence later on in this paper. But one thing about this equivalence is very interesting, an even order Gaussian derivative filter has been expressed here as a linear combination of two Gaussian functions. Can this idea have an element of universality for any even order Gaussian derivative and at any possible scale, for that would enable the filter designer to easily implement any high order Gaussian derivative as a linear combination of Gaussians only or in other words in a Gaussian scale space? This paper answers this query in the affirmative as we are going to see with the help of related work by Ma and Li [5]

Such Gaussian derivatives of order 2 or greater, may be termed either as even-order Hermitian wavelets (because of the close connectivity with Hermite polynomials) or as the Mexican hat family. These are generated by applying second order derivative operator to the Mexican hat wavelet, obtain the next higher order isotropic wavelet and iterate the process to get a whole family.

A Gaussian-Mexican hat wavelet pair has been extensively used to detect structure from astrophysical images, particularly for detecting point sources in cosmic microwave background (CMB) maps [6]. More recently, it has been shown, a generalized Gaussian-Mexican hat family wavelet pair improves such point source detection in CMB maps [7]. Thus a simple scheme for implementing this wavelet family is a problem worth dealing for the signal processing community. It has already been shown that the Gaussian derivative family can provide a very effective tool for designing good quality low-pass filters [8, 9].

II. THE PROPOSED METHODOLOGY OF DESIGN

We are now going to see how by the method of vanishing moments any (2k)th order Gaussian derivative filter of arbitrary scale can be expressed as the linear combination of (k+1) smooth filters, i.e.:

$$h_{2k}(x) = \sum_{j=0}^{k} \frac{\alpha_j}{\sigma_j} g\left(\frac{x}{\sigma_j}\right) \quad (1)$$

Where α_js are the weighting coefficients and h_{2k} is the weighted sum of k−1 multi-scales (σ_j) function of the same Gaussian kernel.

This will be an improvement and generalization of the method of Ma and Li [5], since they did not handle the scale of the left hand side of Equation (1).

As $h_{2k}(x)$ is a (2k)-th order derivative filter so the weighting coefficients will satisfy the following linear moment equations:

$$\begin{aligned}
&\alpha_0 + \alpha_1 + \alpha_2 + \dots\dots\dots + \alpha_k = 0\\
&\alpha_0\sigma_0^2 + \alpha_1\sigma_1^2 + \alpha_2\sigma_2^2 + \dots\dots\dots + \alpha_k\sigma_k^2 = 0\\
&\qquad \dots \qquad \dots \qquad \dots\\
&\qquad \dots \qquad \dots \qquad \dots\\
&\alpha_0\sigma_0^{2k} + \alpha_1\sigma_1^{2k} + \alpha_2\sigma_2^{2k} + \dots\dots\dots + \alpha_k\sigma_k^{2k} = \frac{(2k)!}{m_{g,2k}}
\end{aligned} \quad (2)$$

$m_{g,2k}$ is the (2k)th order moment of the primitive function g(x):

$$m_{g,2k} = \int_{-\infty}^{\infty} x^{2k} g(x)\,dx$$

and

$$M_\sigma = \begin{pmatrix} 1 & 1 & \dots & \dots & \dots & 1 \\ \sigma_0^2 & \sigma_1^2 & \dots & \dots & \dots & \sigma_k^2 \\ \vdots & & \vdots & & & \vdots \\ \sigma_0^{2k} & \sigma^{2k} & \dots & \dots & & \sigma_k^{2k} \end{pmatrix} \quad (3)$$

If the matrix M_σ that encompasses the given scales σ_j, is not singular then the coefficients α_j can be calculated as.

$$\alpha_j = \frac{K}{\prod_{l\neq j}(\sigma_j^2 - \sigma_l^2)} \quad (4)$$

Where, $K = \frac{(2k)!}{m_{g,2k}}$.

In this paper, we have computed the (2k)th order derivative of a Gaussian smooth function of a particular scale (σ) by a linear combination of multi-scale Gaussian function. The important issue of this design is to find the optimum scales of the (k+1) smooth functions so that the linear combination can yield the best approximation of the (2k)th order derivative ((2k)th order derivative of a smooth function [5]) of Gaussian of scale σ. In order to achieve the optimum scales, we have used the least-square curve-fitting algorithm. This optimization is performed in the frequency domain rather than in space/time domain, though we shall show a comparison of the two second order kernels in spatial domain with respect to the best engineering approximation presented by Marr [4].

So, in frequency domain Equation (1) is transformed as:

$$H_{2k}(\omega) = \sum_{j=0}^{k} \alpha_j(\sqrt{2\pi})G(\omega, \sigma_j) \quad (5)$$

Where, $G(\omega, \sigma_j) = \exp\left(-\frac{\sigma_j^2\omega^2}{2}\right)$

LHS of Equation (5) can be expressed in normalized form

$$H_{2k}(\omega) = (\omega\sigma)^{2k} n^{-k} \exp\left(-\frac{\sigma^2\omega^2}{2} + k\right) \quad (6)$$

During the optimization the weight factors (α) are calculated from Equation (4) to minimize the square norm of the following error function.

$$\min E_j = \frac{1}{2}\sum_i \left\| G_{2k}(\omega_i, \sigma) - C_{2k}\sum_{j=0}^{k}\alpha_j(\sqrt{2\pi})G(\omega_i, \sigma_j)\right\|^2 \quad (7)$$

C_{2k} is the normalization constant and G_{2k} is the normalized $2k^{th}$ order Gaussian derivative.

III. RESULTS AND DISCUSSION

We are now going to present a comparative study of the Mexican hat and various other even order Hermitian wavelets in the Fourier domain with corresponding designs in Gaussian scale space. In all the cases we have compared the designs for three different scales, including one very wide scale viz. $\sigma = 28$. Let us begin with the Mexican hat wavelet. The results in Table 1 and the corresponding diagram in Fig.1 proves that the proposed filter in Gaussian scale space practically merges with the second order derivative in Fourier domain. The kernels from the proposed design have also been shown in Fig. 1. We find that the values of σ in the Gaussian scale space are close to the scale of the derivative filter.

In table 2 and Fig. 2, we make a similar comparison for fourth order Gaussian derivative, with similar results. This function in two dimension is the Bi-Laplacian of Gaussian and a corresponding linear combination of three Gaussian functions can be a physiological equivalent to the non-classical receptive field in low-level vision, the widest Gaussian representing an extended surround beyond the

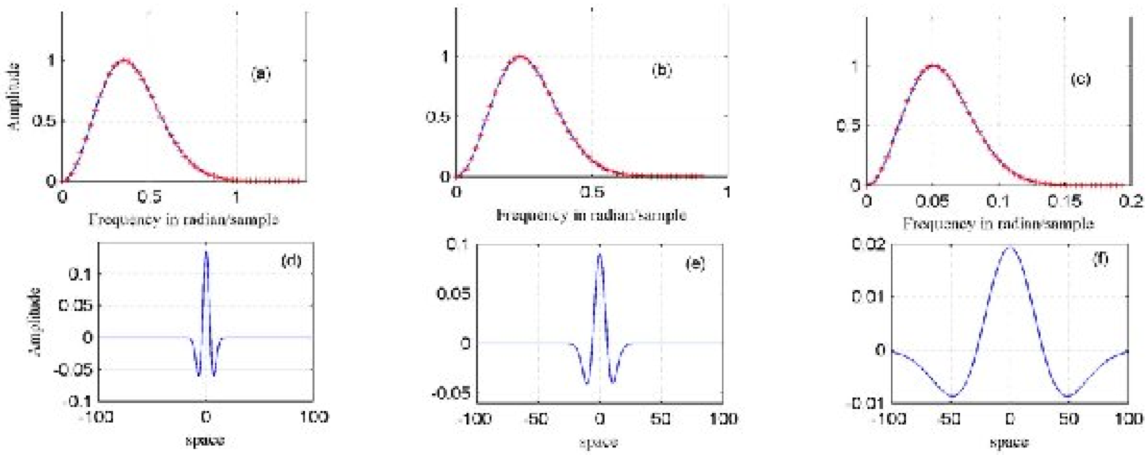

Fig. 1. *(+ denotes Linear combination of multi-scale Gaussians and solid line denotes Gaussian derivative);* (a)-(c) Comparison of amplitude spectrum of 2nd order Gaussian derivative and the linear combination of 2 multi-scale Gaussians for σ=4, 6 and 28. (d)-(f) corresponding kernels of the linear combinations.

classical Difference of Gaussian modeled receptive field [10]. Young [11, 12] had also demonstrated that in human visual system though in rare cases one may have to include up to tenth order of differentiation, generally inclusion up to fourth order suffices. We have performed similar analysis with Gaussian derivatives of higher order and obtained similar results for order 6, 10 and 18. However due to lack of space, we are only showing the result for derivative order 18 only in table 3 and Fig. 3.

Having thus implemented any even order Gaussian derivative in Gaussian scale space, we come back to the findings of Marr-Hildreth and compare our findings with theirs [3]. Marr-Hildreth has reported about a second order derivative of Gaussain of half-power bandwidth 1.3 octave. We have found that the corresponding scale is given by $\sigma = 2$. We now implement our method for this scale. The results have been shown in Fig. 4. The multi-scale filter and the derivative kernels practically overlap and in Fourier domain also we find the same coincidence. This occurs at a scale ratio of 1:1.01 in the Gaussian scale space and appears to be a better engineering approximation than what was reported by Marr-Hildreth at a scale ratio of 1:1.6 and as exhibited in Fig. 11 of Marr-Hildreth's paper [3].

TABLE 1 COMPARATIVE STUDY OF THE SCALES OF THE MULTI-SCALE GAUSSIANS AND THAT OF THE GAUSSIAN DERIVATIVE OF 2ND ORDER

Order of Derivative N	σ	σ_1	σ_2	L^2 norm $x 10^{-5}$
2	4	3.9446	4.0565	6.245
2	6	5.9292	6.0720	2.198
2	28	27.8232	28.1802	1.631

Finally we are going to show, that the family of Gaussian derivatives thus designed in a Gaussian scale space can indeed be claimed to form a wavelet family [13] if we study the coefficients of the linear combination as filter for perfect reconstruction. This we show for multi-scale equivalent of second order derivative and can be extended for the higher order derivatives as well.

From 1st condition of equation (2), considering the case of second order Gaussian derivative, we get:

$$\alpha_1 = -\alpha_2$$

TABLE 2 COMPARATIVE STUDY OF THE SCALES OF THE MULTI-SCALE GAUSSIANS AND THAT OF THE GAUSSIAN DERIVATIVE OF 4TH ORDER

Order of Derivative N	σ	σ_1	σ_2	σ_3	L^2 norm $x 10^{-5}$
4	4	3.9193	3.9569	4.1296	7.5847
4	6	5.8918	5.9446	6.1703	3.026
4	28	27.8273	27.9142	28.2633	0.11568

It is thus observed that the coefficients of two multi-scale Gaussians form a high-pass filter which thus resembles one of the pair of Haar filters for two-channel Perfect Reconstruction (PR), except the normalization constant. Then, one of the analysis filters is

$$H_1(Z) = \alpha_0(1 - Z^{-1})$$

and the simplest choice of the other analysis filter is

$$H_0(Z) = \alpha_0(1 + Z^{-1})$$

As the system will be alias free, so the two synthesis filters will be of the following form:

$$G_1(Z) = \frac{-H_0(Z)}{\alpha_0} \quad G_0(Z) = \frac{H_1(Z)}{\alpha_0}$$

This results the distortion transfer function

$$T(Z) = \frac{1}{2}\left[H_0(Z)H_1(-Z) - H_1(Z)H_0(-Z)\right]$$

to be a delay filter.

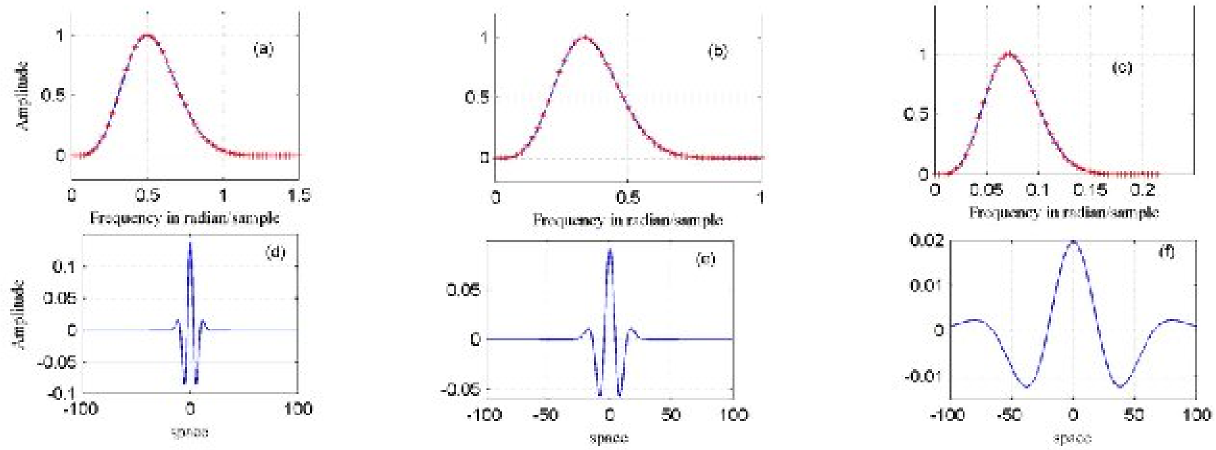

Fig. 2. *(+ denotes Linear combination of multi-scale Gaussians and solid line denotes Gaussian derivative)*: (a)-(c) Comparison of amplitude spectrum of 4th order Gaussian derivative and the linear combination of 3 multi-scale Gaussians for $\sigma = 4$, 6 and 28. (d)-(f) corresponding kernels of the linear combinations.

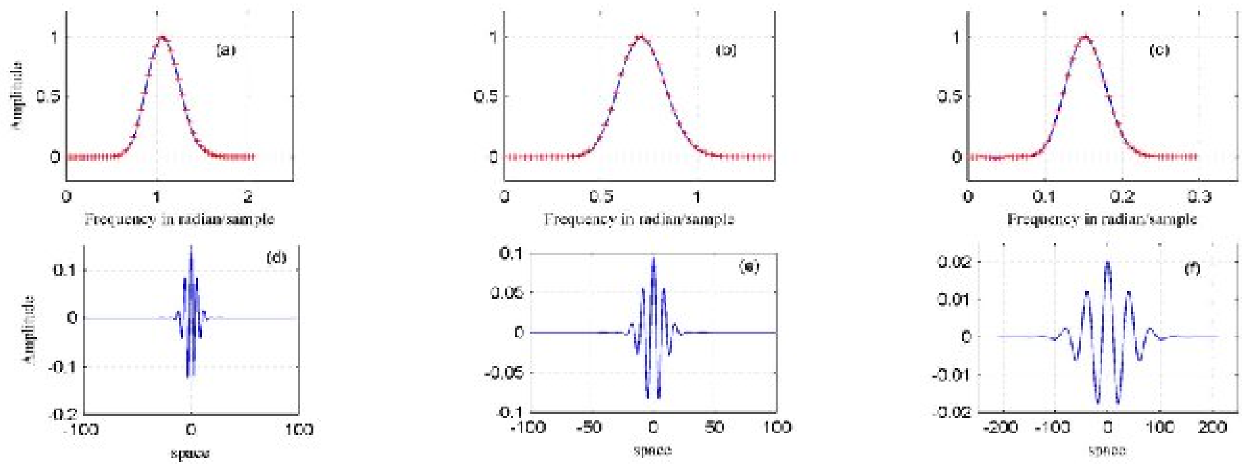

Fig. 3. *(+ denotes Linear combination of multi-scale Gaussians and solid line denotes Gaussian derivative)*; (a)-(c) Comparison of amplitude spectrum of 18th order Gaussian derivative and the linear combination of 10 multi-scale Gaussians for $\sigma=4$, 6 and 28. (d)-(f) corresponding kernels of the linear combinations.

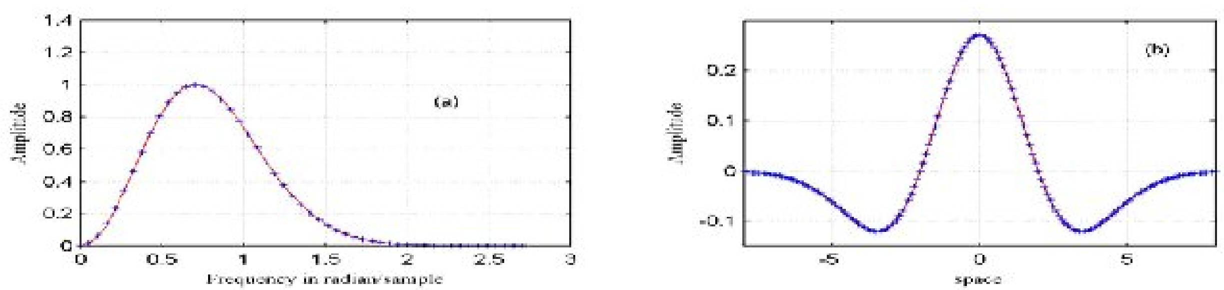

Fig. 4. *(+ denotes Linear combination of multi-scale Gaussians and solid line denotes Gaussian derivative)*; (a)-(c) Comparison of amplitude spectrum of 18th order Gaussian derivative and the linear combination of 10 multi-scale Gaussians for $\sigma=4$, 6 and 28. (d)-(f) corresponding kernels of the linear combinations.

With the above choices of four filters we get.

$T(Z) = 2Z^{-1}$

The results are apparent from Fig. 5

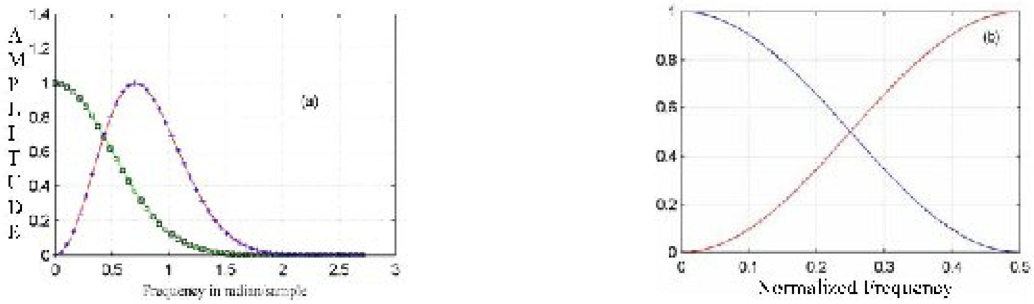

Fig. 5. Comparison of power spectrum (a) *(blue '—' denotes 2nd order Gaussian derivative, red solid line denotes linear combination of 2 multi-scale Gaussians, black 'box' denotes linear combination of 2 multi-scale Gaussians when coefficients form $H_0(Z)$, green solid line represents the Gaussian of the scale(σ1.7).* (b) Power spectrum *(blue line denotes $H_0(Z)$, red solid line represents $H_1(Z)$.*

TABLE 3. COMPARATIVE STUDY OF THE SCALES OF THE MULTI-SCALE GAUSSIANS AND THAT OF THE GAUSSIAN DERIVATIVE OF 18TH ORDER

N	σ	σ_1	σ_2	σ_3	σ_4	σ_5	σ_6	σ_7	σ_8	σ_9	σ_{10}	L^2 norm
18	4	3.9621	3.734	3.7287	3.6236	3.5066	4.7907	4.3726	4.5458	4.1138	4.0303	0.0751
18	6	5.9025	5.6246	5.5709	5.4584	5.3808	6.8977	6.6628	6.4844	6.3485	6.2129	0.0214
18	28	27.8	27.6	27.50	27.3	27.1	28.9	28.7	28.5	28.4	28.2	0.0057

IV. CONCLUSION

Isotropic Gaussian derivatives were well-studied by Young in the context of retinal and cortical receptive fields. Young demonstrated that the receptive fields of many neurons, both at the cortical and retinal level, in the mammalian visual system could be approximated by a combination of higher derivatives of Gaussian. For example, spatial frequency spectra of simple cells in monkey visual cortex showed a large variation in the bandwidth and that variation could be accounted for by a Gaussian Derivative model [14]. The present work shows how even high order Gaussian derivatives may be computed by the visual system in a Gaussian scale space only. The proposed wavelet family in Gaussian scale space is also likely to be an important tool in seismic signal analysis, in cosmic microwave background (CMB) maps and may be in the general cases involving Gaussian point sources.

REFERENCES

[1] I. Daubechies, *Ten Lectures on Wavelets*, Fourth Printing, Society for Industrial and Applied Mathematics, Philadelphia, Pennsylvania (1992), 3.

[2] Z. Zhou, H. Adeli, "Time-frequency signal analysis of earthquake records using Mexican hat wavelets," *Computer-Aided Civil and Infrastructure Engineering* 18 (2003), 379–389.

[3] D. Marr, E. Hildreth, "Theory of edge detection," *Proceedings of the Royal Society of London* B 207, pp. 187-217, 1980.

[4] D. Marr, *Vision: A Computational Investigation into the Human Representation and Processing of Visual Information*, W. H. Freeman and Company, New York 1982.

[5] S. D. Ma, B. Li, "Derivative computation by multiscale filters," *Image and Vision Computing* 16, 43-53, 1998.

[6] P. Vielva, et al., "Point source detection using the spherical mexican hat wavelet on simulated all-sky Planck maps," *Monthly Notices of Royal Astronomical Society* 344, 89, 2003.

[7] F. Arg'ueso, J. Gonz alez-Nuevo, J. L. Sanz, L. Toffolatti, P. Vielva, D. Herranz, M. L'opez-Caniego, " The Mexican hat wavelet family — application to point source detection in cosmic microwave background maps," *Proceedings of Thirteenth European Signal Processing Conference (EUSIPCO-2005)*, September 4-8, 2005, Antalya, ISBN: 975-00188-0-X, 2005.

[8] S. Karmakar, K. Ghosh, S. Sarkar, S. Sen, "Design of a low-pass filter by multi-scale even order Gaussian derivatives," *Signal Processing* 86, pp. 3923-3933, 2006.

[9] S. Karmakar, K. Ghosh, S. Sarkar, S. Sen, " A new design of low pass filter by Gaussian derivative family," *IEEE Proceedings, Third International Conference on Intelligent Sensing and Information Processing*, 14-17 December, 2005, Bangalore, India (2005), pp. 177-182.

[10] K. Ghosh, S. Sarkar, K. Bhaumik, "A possible mechanism of zero-crossing detection using the concept of extended classical receptive field model of retinal ganglion cells," *Biological Cybernetics* 93, pp. 1-5 2005.

[11] R.A. Young, "The Gaussian derivative theory of spatial vision: analysis of cortical cell receptive field line weighting profiles," *General Motors Research Publication* GMR-4920, 1985.

[12] R.A. Young, "The Gaussian derivative model for spatial vision: I. Retinal mechanisms," *Spatial Vision* 2, pp. 273–293, 1987.

[13] S. G. Mallat, "A theory of multiresolution signal decomposition: the wavelet representation," *IEEE Trans. PAMI*, 11 pp. 674-693, 1989.

[14] R.L. De Valois, D.G. Albrecht, L.G., "Thorell, Spatial frequency selectivity of cells in macque visual cortex," *Vision Research* 22 pp. 545–560, 1982.