On some efficient partial diallel cross designs

Himadri Ghosh[a], Ashish Das[b,*], C.K. Midha[c]

[a] I.A.S.R.I., Library Avenue, New Delhi 110 012, India
[b] Indian Statistical Institute, Stat-Math Division, New Delhi 110016, India
[c] The University of Akron, Department of Statistics, Akron, OH 44325, USA

Abstract

A general A- and D-efficiency lower bound has been obtained for partial diallel cross designs. These bounds have been used to show that a class of E-optimal designs, obtained by Mukerjee [1997. Optimal partial diallel crosses. Biometrika 84, 939–948], have high A- and D-efficiencies. Also, a class of block designs, introduced by Mukerjee [1997. Optimal partial diallel crosses. Biometrika 84, 939–948], is shown to be nearly E-optimal. General eigenvalues of the information matrix of these designs is obtained, which enable us to show that the block designs have high A- and D-efficiencies.

Keywords: A-efficiency; D-efficiency; E-optimality; Partial diallel crosses

1. Introduction

Genetic properties of inbred lines in plant breeding experiments are investigated by carrying out diallel crosses. Design of experiments for diallel crosses has received considerable attention in the literature; see Curnow (1963), Hinkelmann (1975) and Gupta et al. (1995) for references. Let p denote the number of lines and let a cross between lines i and i' be denoted by (i, i'), $i < i' = 1, 2, \ldots, p$. Let n denote the total number of crosses observed in the experiment. Our interest lies in comparing the lines with respect to their general combining ability effects.

Complete diallel cross designs involve equal numbers of occurrences of each of the $\binom{p}{2}$ distinct crosses among p inbred lines. Gupta and Kageyama (1994), Dey and Midha (1996) and Das et al. (1998a) investigated the issue of optimality of complete diallel crosses. When p is large, it becomes impractical to carry out an experiment using a complete diallel cross design. In such situations, we use partial diallel cross designs where a subset of $\binom{p}{2}$ crosses are used. In the literature designs for partial diallel crosses have been discussed for $n = ps/2$ $(s < p - 1)$, distinct crosses. Although efficient designing of partial diallel crosses has been studied by several authors (Hinkelmann and Kempthorne, 1963; Arya, 1983; Singh and Hinkelmann, 1990, 1995), no formal optimality results within adequately general classes has been reported except for the recent works of Mukerjee (1997) and Das et al. (1998b). Sometimes partial diallel crosses can, themselves, be quite large and thus it is desirable to use a block design for the experiment. Gupta et al. (1995) and Mukerjee (1997) provide orthogonal blocking schemes for partial diallel cross designs.

The objective of the present paper is to investigate the A- and D-optimality properties of certain partial diallel cross designs. For this, we first derive a general A- and D-efficiency lower bound for partial diallel cross designs. These bounds are used to show that a class of E-optimal designs, obtained by Mukerjee (1997), have high A- and D-efficiencies. Furthermore, a class of block designs, introduced by Mukerjee (1997), is shown to be nearly E-optimal. General eigenvalues of the information matrix of these designs is obtained, which enable us to show that the block designs have high A- and D-efficiencies. As such, presently there hardly exists any general A- or D-optimality results for such partial diallel cross designs and therefore our results contribute towards identifying designs having high A- and D-efficiencies.

In a diallel cross experiment, we consider a design d involving p inbred lines, giving rise to a total of $\binom{p}{2}$ possible distinct crosses. Let $s_d = (s_{d1}, \ldots, s_{dp})'$ where s_{di} denotes the replication number of the ith line that occurs among the crosses in the design d, $i = 1, 2, \ldots, p$. Then $1'_p s_d = 2n$, where $'$ denotes transpose of a matrix and 1_t denotes a t-component column vector of all ones. We use the model M1: $Y = \mu 1_n + \Delta_1 g + \varepsilon$, for an unblocked diallel cross experiment. For a blocked diallel cross experiment, we consider the model M2: $Y = \mu 1_n + \Delta_1 g + \Delta_2 \beta + \varepsilon$. Here, Y is the $n \times 1$ vector of observed responses, μ is a general mean effect, g and β are vectors of p general combining ability effects and b block effects, respectively, Δ_1, Δ_2 are the corresponding design matrices, that is, the (h,l)th element of Δ_1 (respectively, of Δ_2) is 1 if the hth observation pertains to the lth line (respectively, to the lth block), and is zero otherwise; ε is the vector of random error components.

Let $\mathscr{D}(p,n)$ denote the class of all unblocked designs with p lines and n crosses. For a design, $d_0 \in \mathscr{D}(p,n)$, under model M1, Mukerjee (1997) has shown that the information matrix of the reduced normal equations for estimating linear functions of general combining ability effects g is $C_{d_0} = G_{d_0} - (1/n) s_{d_0} s'_{d_0}$, where $s_{d_0} = (s_{d_0 1}, s_{d_0 2}, \ldots, s_{d_0 p})$, $G_{d_0} = (g_{d_0 ii'})$, $g_{d_0 ii} = s_{d_0 i}$, and for $i \neq i'$, $g_{d_0 ii'}$ denotes the number of times the cross (i,i') appears in d_0. Similarly, let $\mathscr{D}(p,b,k)$ denote the class of all block designs with p lines, and b blocks each with k crosses. Following Gupta and Kageyama (1994), for a block design $d \in \mathscr{D}(p,b,k)$ under model M2, the information matrix for g is given by $C_d = G_d - (1/k) N_d N'_d$, where $N_d = (n_{dij})$; n_{dij} is the number of times that line i occurs in block j of d; and $G_d = (g_{dii'})$ is as defined earlier. For such a block design, $n = bk$ and $N_d 1_b = s_d$.

A design $d_0(d)$ will be called connected if and only if the rank of its information matrix is $p-1$. Equivalently, $d_0(d)$ is connected if and only if all elementary comparisons among general combining ability effects are estimable. In this paper, we consider only connected designs.

For a design $d_0 \in \mathscr{D}(p,n)$, let $\lambda_{d_0 1} \leqslant \lambda_{d_0 2} \leqslant \cdots \leqslant \lambda_{d_0 (p-1)}$ denote the non-zero eigenvalues of the information matrix C_{d_0}. Then, a design $d_0^* \in \mathscr{D}(p,n)$ is A-optimal if $\sum_{i=1}^{p-1} \lambda_{d_0^* i}^{-1} = \min_{d_0 \in \mathscr{D}(p,n)} \sum_{i=1}^{p-1} \lambda_{d_0 i}^{-1}$, is D-optimal if $\prod_{i=1}^{p-1} \lambda_{d_0^* i} = \max_{d_0 \in \mathscr{D}(p,n)} \prod_{i=1}^{p-1} \lambda_{d_0 i}$, and is E-optimal if $\lambda_{d_0^* 1} = \max_{d_0 \in \mathscr{D}(p,n)} \lambda_{d_0 1}$. Similarly, A-, D- and E-optimality are defined for connected block designs $d \in \mathscr{D}(p,b,k)$. One may refer to Shah and Sinha (1989) for the above definitions and the statistical interpretation of these optimality criteria.

2. Improved lower bounds for A- and D-efficiencies

In this section, while dealing with unblocked partial diallel cross designs (based on model M1), we first derive general A- and D-efficiency lower bounds. Mukerjee (1997) has investigated the optimality of certain partial diallel crosses, under fixed effects model, which are linked with a certain class of group divisible designs. Though his results are on E-optimality, he also presented results on A- and D-optimality in the saturated case. In general, the E-optimal designs turn out to be highly efficient under the A- and D-optimality criteria.

Let $p = n_1 n_2$ and $n = \frac{1}{2} p(n_2 - 1)$, where $n_1 \geqslant 2$, $n_2 \geqslant 3$. Partition the set $\{1, 2, \ldots, p\}$ into n_1 mutually exclusive and exhaustive subsets $\{S_1, S_2, \ldots, S_{n_1}\}$ each of cardinality n_2. Let

$$d_0^* = \{(i,j) : 1 \leqslant i < j \leqslant p \text{ and } i, j \in S_u \text{ for } u = 1, 2, \ldots, n_1\}. \tag{2.1}$$

Then, for $i = 1, \ldots, p - n_1$, $\lambda_{d_0^* i} = n_2 - 2$ and for $i = p - n_1 + 1, \ldots, p-1$, $\lambda_{d_0^* i} = 2(n_2 - 1)$ with $d_0^* \in \mathscr{D}(p,n)$. Also, Mukerjee (1997) has shown that for $d_0 \in \mathscr{D}(p,n)$, $\lambda_{d_0 1} \leqslant n_2 - 2$. Thus, d_0^* is E-optimal in $\mathscr{D}(p,n)$ and we can write

$$\lambda_{d_0 1} \leqslant \lambda_{d_0^* 1} = n_2 - 2 \quad \text{for } d_0 \in \mathscr{D}(p,n). \tag{2.2}$$

Mukerjee (1997) has shown that d_0^* is *A*- and *D*-optimal in $\mathscr{D}(p,n)$ when $n_2 = 3$, i.e., when the design is saturated ($p = 3n_1$, $n = 3n_1$).

Retaining clarity, where ever convenient, we write λ_i in place of $\lambda_{d_0 i}$. Now, since $trace(C_{d_0}) \leqslant 2(n - n_2 + 1)$, a conservative lower bound to $\sum_{i=1}^{p-1} \lambda_i^{-1}$ for $d_0 \in \mathscr{D}(p,n)$ is given by

$$\sum_{i=1}^{p-1} \lambda_i^{-1} \geqslant \frac{(p-1)^2}{2(n - n_2 + 1)}. \tag{2.3}$$

The *A*-efficiency findings of Mukerjee (1997) are based on the lower bound as given in (2.3). It can be verified that the lower bound in (2.3) will be attained by d_0 only when $\lambda_1 = \lambda_2 = \cdots = \lambda_{p-1} = 2(n - n_2 + 1)/(p-1)$, i.e., d_0 is a complete diallel cross design. But, since $n_2 < p$, we have $\lambda_1 = 2(n - n_2 + 1)/(p-1) = (n_2 - 1)(p-2)/(p-1) = (n_2 - 1) - (n_2 - 1)/(p-1) > n_2 - 2$. This leads to a contradiction since from (2.2), for $d_0 \in \mathscr{D}(p,n)$, $\lambda_1 \leqslant n_2 - 2$. Based on this fact, for $n_1 > 1$, the following theorem gives an improved lower bound for $\sum_{i=1}^{p-1} \lambda_i^{-1}$.

Theorem 2.1. *Given the class $\mathscr{D}(p,n)$ of partial diallel cross designs, where $p = n_1 n_2$, $n = \frac{1}{2}p(n_2 - 1)$,*

$$\sum_{i=1}^{p-1} \lambda_{d_0 i}^{-1} \geqslant \frac{1}{n_2 - 2} + \frac{(p-2)^2}{(n_2 - 1)(p-3) + 1} \quad (= \alpha(n_1, n_2),\ \textit{say}) \tag{2.4}$$

for all $d_0 \in \mathscr{D}(p,n)$.

Proof. Using the arithmetic mean and geometric mean inequality, we have

$$\begin{aligned}
\sum_{i=1}^{p-1} \lambda_i^{-1} &= \lambda_1^{-1} + \sum_{i=2}^{p-1} \lambda_i^{-1} \\
&\geqslant \lambda_1^{-1} + \frac{(p-2)^2}{\sum_{i=1}^{p-1} \lambda_i - \lambda_1} \\
&\geqslant \lambda_1^{-1} + \frac{(p-2)^2}{(n_2 - 1)(p-2) - \lambda_1}, \quad \text{since } \sum_{i=1}^{p-1} \lambda_i \leqslant (n_2 - 1)(p-2).
\end{aligned}$$

Let $f(\lambda_1) = \lambda_1^{-1} + (p-2)^2/((n_2 - 1)(p-2) - \lambda_1)$. Then

$$\begin{aligned}
f'(\lambda_1) = &-\lambda_1^{-2} + \frac{(p-2)^2}{\{(n_2 - 1)(p-2) - \lambda_1\}^2} = 0 \\
&\Leftrightarrow \frac{(p-2)^2 \lambda_1^2 - \{(n_2 - 1)(p-2) - \lambda_1\}^2}{\{(n_2 - 1)(p-2) - \lambda_1\}^2 \lambda_1^2} = 0 \\
&\Leftrightarrow \{(n_2 - 1)(p-2) + (p-3)\lambda_1\}\{(p-1)\lambda_1 - (n_2 - 1)(p-2)\} = 0.
\end{aligned}$$

Thus, the possible stationary values of the function $f(\lambda_1)$ are $-(n_2 - 1)(p-2)/(p-3)$ and $(n_2 - 1)(p-2)/(p-1)$ out of which only the second one is admissible.

Now,

$$f'(\lambda_1) = \frac{(p-3)(p-1)[\lambda_1 + (n_2 - 1)(p-2)/(p-3)][\lambda_1 - (n_2 - 1)(p-2)/(p-1)]}{\{(n_2 - 1)(p-2) - \lambda_1\}^2 \lambda_1^2}$$

is negative when $0 < \lambda_1 < (n_2 - 1)(p-2)/(p-1)$ and positive when $\lambda_1 > (n_2 - 1)(p-2)/(p-1)$.

Thus $f(\lambda_1)$ is a decreasing function for $0 < \lambda_1 < (n_2 - 1)(p-2)/(p-1)$ and since $\lambda_1 = (n_2 - 1)(p-2)/(p-1) > n_2 - 2$, from (2.2), the minimum is attained at $\lambda_1 = n_2 - 2$. Hence a sharper lower bound for $f(\lambda_1)$ is

$$f(n_2-2)=\frac{1}{n_2-2}+\frac{(p-2)^2}{(n_2-1)(p-3)+1}.$$

From (2.4), the A-efficiency of the design $d_0^* \in \mathscr{D}(p,n)$ is at least as large as $e_A(n_1,n_2)$ where

$$e_A(n_1,n_2)=\frac{\alpha(n_1,n_2)}{\sum_{i=1}^{p-1}\lambda_{d_0^*i}^{-1}},$$

i.e. substituting the values of $\lambda_{d_0^*i}$, we have

$$e_A(n_1,n_2)=\frac{\alpha(n_1,n_2)}{(p-n_1)\{n_2-2\}^{-1}+(n_1-1)\{2(n_2-1)\}^{-1}}. \qquad \square \tag{2.5}$$

Next, we consider improved lower bounds to D-efficiency. For $n_1>1$, the following theorem establishes a sharper lower bound for $(\prod_{i=1}^{p-1}\lambda_i)^{-1}$.

Theorem 2.2. *Given the class $\mathscr{D}(p,n)$ of partial diallel cross designs, where $p=n_1n_2$, $n=\frac{1}{2}p(n_2-1)$,*

$$\left(\prod_{i=1}^{p-1}\lambda_{d_0i}\right)^{-1}\geqslant\frac{(p-2)^{p-2}}{(n_2-2)\{(n_2-1)(p-3)+1\}^{p-2}} \quad (=\delta(n_1,n_2),\ say) \tag{2.6}$$

for all $d_0\in\mathscr{D}(p,n)$.

Proof. As before, using the arithmetic mean and geometric mean inequality, we get

$$\frac{1}{(\prod_{i=1}^{p-1}\lambda_i)^{1/(p-1)}}\geqslant\frac{p-1}{\sum_{i=1}^{p-1}\lambda_i},$$

i.e.,

$$\left(\prod_{i=1}^{p-1}\lambda_i\right)^{-1}\geqslant\frac{(p-1)^{p-1}}{(\sum_{i=1}^{p-1}\lambda_i)^{p-1}}.$$

Now again using the arithmetic mean and geometric mean inequality on $\lambda_2,\lambda_3,\ldots,\lambda_{p-1}$, we have

$$\left(\lambda_1\prod_{i=2}^{p-1}\lambda_i\right)^{-1}\geqslant\left(\frac{1}{\lambda_1}\right)\left(\frac{p-2}{\sum_{i=2}^{p-1}\lambda_i}\right)^{p-2}. \tag{2.7}$$

From (2.7) and using the inequality $\sum_{i=1}^{p-1}\lambda_i\leqslant(n_2-1)(p-2)$, it follows that

$$\left(\prod_{i=1}^{p-1}\lambda_i\right)^{-1}\geqslant\frac{(p-2)^{p-2}}{\lambda_1\{(n_2-1)(p-2)-\lambda_1\}^{p-2}}=g(\lambda_1),\quad \text{say}. \tag{2.8}$$

Then,

$$g'(\lambda_1)=\lambda_1^{-2}(p-2)^{p-2}\{(n_2-1)(p-2)-\lambda_1\}^{-(p-1)}\{\lambda_1(p-1)-(n_2-1)(p-2)\}=0,$$

yields that the only admissible stationary value of the function $g(\lambda_1)$ is $(n_2-1)(p-2)(p-1)^{-1}$.

Clearly, the function $g'(\lambda_1)$ is negative when $0<\lambda_1<(n_2-1)(p-2)(p-1)^{-1}$ and positive when $\lambda_1>(n_2-1)(p-2)(p-1)^{-1}$. Thus, $g(\lambda_1)$ is a decreasing function for $0<\lambda_1<(n_2-1)(p-2)(p-1)^{-1}$ and since $\lambda_1=(n_2-1)(p-2)(p-1)^{-1}>n_2-2$, the minimum is attained at $\lambda_1=n_2-2$. Hence a sharper lower bound for $g(\lambda_1)$ is

$$g(n_2-2)=\frac{(p-2)^{p-2}}{(n_2-2)\{(n_2-1)(p-3)+1\}^{p-2}}.$$

From (2.6), the D-efficiency of the design $d_0^* \in \mathscr{D}(p,n)$ is at least as large as $e_D(n_1,n_2)$ where

$$e_D(n_1,n_2) = \frac{\{\delta(n_1,n_2)\}^{1/(p-1)}}{\{\prod_{i=1}^{p-1}\lambda_{d_0^*i}\}^{-1/(p-1)}},$$

i.e. substituting the values of $\lambda_{d_0^*i}$, we have

$$e_D(n_1,n_2) = \{\delta(n_1,n_2)(n_2-2)^{p-n_1}(2(n_2-1))^{n_1-1}\}^{1/(p-1)}. \qquad \square \tag{2.9}$$

Based on Theorems 2.1 and 2.2, the efficiency lower bounds $e_A(n_1,n_2)$ and $e_D(n_1,n_2)$ have been calculated for the designs d_0^* as given in (2.1). We consider designs d_0^* having parameters in the practical range $n_1 \geqslant 2, n_2 \geqslant 4, p \leqslant 200$. There are a total of 535 possible designs within this range. It is observed that, 100%, 91.0%, 78.7% and 61.5% of the designs have e_A greater than 0.8, 0.85, 0.9 and 0.95, respectively. Also, if we restrict to designs having $n_1 \leqslant n_2$ then of the 353 possible designs, 100%, 99.4%, 98.3% and 90.1% of the designs have e_A greater than 0.8, 0.85, 0.9 and 0.95, respectively. Similarly, 100%, 91.0% and 73.6% of the designs have e_D greater than 0.85, 0.9 and 0.95, respectively. Also, restricting to designs having $n_1 \leqslant n_2$, 100%, 99.4% and 96.6% of the designs have e_D greater than 0.85, 0.9 and 0.95, respectively.

3. Nearly *E*-optimal block designs and their *A*- and *D*-efficiencies

Mukerjee (1997) has discussed blocking for partial diallel crosses and constructed E-optimal block designs for the case when n_2 is odd (case 1) and even (case 2). We now study an alternative method for constructing block designs when n_2 is odd. The resulting designs have the advantage of block sizes being considerably smaller than Mukerjee's case 1 designs. Also, from the optimality considerations, we observe that these alternative designs perform well with respect to A-, D- and E-criteria.

In the construction of the block design, that follows, there are $p\ (= n_1n_2)$ lines, and the lines are denoted by $a_j^u,\ 1 \leqslant u \leqslant n_1,\ 0 \leqslant j \leqslant n_2-1$. In (2.1) take $S_u = \{a_0^u, a_1^u, \ldots, a_{n_2-1}^u\}$. Then from (2.1) d_0^* consists of $n_1n_2(n_2-1)/2$ crosses where in a cross the two lines are from the same S_u. For $n_2\ (\geqslant 5)$ odd, a general approach for grouping the crosses in d_0^* into blocks is now given.

Let M_1 be the incidence matrix of a general block design d_1 involving n_1 treatments and n_2 blocks such that each block has size n_1 and each treatment is replicated n_2 times. For $1 \leqslant u \leqslant n_1,\ 0 \leqslant l \leqslant n_2-1$, in the lth occurrence of treatment u in d_1, replace treatment u by the $(n_2-1)/2$ crosses $\{(a_{j+l}^u, a_{n_2-j+l}^u) : 1 \leqslant j \leqslant (n_2-1)/2\}$, where $j+l$ and n_2-j+l are reduced $(\mathrm{mod}\, n_2)$. This gives the diallel cross block design, d^*, constructed through an alternative method. Clearly, $d^* \in \mathscr{D}(n_1n_2, n_2, n_1(n_2-1)/2)$ and represents a partitioning of the crosses in d_0^* into blocks.

Example 1. Let $n_1 = 3, n_2 = 5$. Then $p = 15, b = 5, k = 6$ and the design d^* based on the above construction is

$$[(a_1^1,a_4^1),(a_2^1,a_3^1),(a_1^2,a_4^2),(a_2^2,a_3^2),(a_1^3,a_4^3),(a_2^3,a_3^3)],$$

$$[(a_2^1,a_0^1),(a_3^1,a_4^1),(a_2^2,a_0^2),(a_3^2,a_4^2),(a_2^3,a_0^3),(a_3^3,a_4^3)],$$

$$[(a_3^1,a_1^1),(a_4^1,a_0^1),(a_3^2,a_1^2),(a_4^2,a_0^2),(a_3^3,a_1^3),(a_4^3,a_0^3)],$$

$$[(a_4^1,a_2^1),(a_0^1,a_1^1),(a_4^2,a_2^2),(a_0^2,a_1^2),(a_4^3,a_2^3),(a_0^3,a_1^3)],$$

$$[(a_0^1,a_3^1),(a_1^1,a_2^1),(a_0^2,a_3^2),(a_1^2,a_2^2),(a_0^3,a_3^3),(a_1^3,a_2^3)].$$

Let, for a design d, $\lambda_{d1} \leqslant \lambda_{d2} \leqslant \cdots \leqslant \lambda_{d(p-1)}$ be the non-zero eigenvalues of the information matrix C_d. Then, after some algebra, we see that for $i = 1, \ldots, n_2-1$, $\lambda_{d^*i} = n_2 - 2 - 2(n_2-1)^{-1}$, for $i = n_2, \ldots, p-n_1$, $\lambda_{d^*i} = n_2 - 2$ and for $i = p-n_1+1, \ldots, p-1$, $\lambda_{d^*i} = 2(n_2-1)$ with $d^* \in \mathscr{D}(p, n_2, n_1(n_2-1)/2)$.

We now give two results on lower bounds to efficiency in case of block designs for diallel crosses. For block designs with $n_1 > 1$, the conservative bounds analogous to (2.3) can be improved. Following Das et al. (1998a), for a block design $d \in \mathscr{D}(p,b,k)$, $trace(C_d) \leqslant k^{-1}b\{2k(k-1-2x)+px(x+1)\}$, where $x = [2k/p]$; $[z]$ being the largest integer not exceeding z. We consider p, b, k such that $p = n_1 n_2$ and $bk = p(n_2 - 1)/2$. Since $C_{d_0} - C_d$ is non-negative definite and $\lambda_{d_0 1} \leqslant n_2 - 2$, it follows that $\lambda_{d1} \leqslant \lambda_{d_0 1} \leqslant n_2 - 2$. On lines similar to Theorem 2.1, the above fact leads to a sharper lower bound for $\sum_{i=1}^{p-1} \lambda_{di}^{-1}$.

Theorem 3.1. *Given the class of designs $\mathscr{D}(p, b, k)$ of partial diallel crosses, with $p = n_1 n_2$, $bk = p(n_2 - 1)/2$ and $x = [2k/p]$,*

$$\sum_{i=1}^{p-1} \lambda_{di}^{-1} \geqslant \frac{1}{n_2 - 2} + \frac{(p-2)^2}{(n_2 - 1)(p-1) + k^{-1} b(px^2 + (p - 4k)x - 2k) + 1} \quad (= \beta(n_1, n_2),\ say) \tag{3.1}$$

for all $d \in \mathscr{D}(p, b, k)$.

From (3.1), the A-efficiency of the design $d^* \in \mathscr{D}(p, b, k)$ is at least as large as $e_{1A}(n_1, n_2)$ where

$$e_{1A}(n_1, n_2) = \frac{\beta(n_1, n_2)}{\sum_{i=1}^{p-1} \lambda_{d^* i}^{-1}},$$

i.e. substituting the values of $\lambda_{d^* i}$, we have

$$e_{1A}(n_1, n_2) = \frac{\beta(n_1, n_2)}{(n_2 - 1)^2 \{(n_2 - 1)(n_2 - 2) - 2\}^{-1} + (n_1 - 1)(n_2 - 1)\{n_2 - 2\}^{-1} + (n_1 - 1)\{2(n_2 - 1)\}^{-1}}. \tag{3.2}$$

Next, we consider lower bounds to D-efficiency. On lines similar to Theorem 2.2, for $n_1 > 1$ the following theorem establishes a sharper lower bound (compared to conservative bounds) for $(\prod_{i=1}^{p-1} \lambda_{di})^{-1}$.

Theorem 3.2. *Given the class of designs $\mathscr{D}(p, b, k)$ of partial diallel crosses, with $p = n_1 n_2$, $bk = p(n_2 - 1)/2$ and $x = [2k/p]$,*

$$\left(\prod_{i=1}^{p-1} \lambda_{di}\right)^{-1} \geqslant \frac{(p-2)^{p-2}}{(n_2 - 2)\{(n_2 - 1)(p - 1) + k^{-1} b(px^2 + (p - 4k)x - 2k) + 1\}^{p-2}} \quad (= \gamma(n_1, n_2),\ say) \tag{3.3}$$

for all $d \in \mathscr{D}(p, b, k)$.

From (3.3), the D-efficiency of the design $d^* \in \mathscr{D}(p, b, k)$ is at least as large as $e_{1D}(n_1, n_2)$ where

$$e_{1D}(n_1, n_2) = \frac{\{\gamma(n_1, n_2)\}^{1/(p-1)}}{\left\{\prod_{i=1}^{p-1} \lambda_{d^* i}\right\}^{-1/(p-1)}},$$

i.e. substituting the values of $\lambda_{d^* i}$, we have

$$e_{1D}(n_1, n_2) = \{\gamma(n_1, n_2)(n_2 - 2 - 2(n_2 - 1)^{-1})^{n_2 - 1}(n_2 - 2)^{(n_1 - 1)(n_2 - 1)}(2(n_2 - 1))^{n_1 - 1}\}^{1/(p-1)}. \tag{3.4}$$

Using Mukerjee's case 1 method of construction, in Example 1 of Mukerjee (1997) an E-optimal design d_1^* in the class $\mathscr{D}(15, 3, 10)$ has been presented. The A-efficiency lower bound of this design d_1^*, as based on (3.2), is 0.889. For the same set of 30 crosses with $p = 15$, the alternative method of construction leads us to the design, as given in our Example 1. This design belongs to the class $\mathscr{D}(15, 5, 6)$ and its A-efficiency lower bound, based on (3.2), is 0.870. Similarly, while considering the D-efficiency in their respective classes, it is seen that our design has higher value of e_{1D} than that for d_1^*, being 0.918 and 0.910, respectively. Thus, the design in Example 1, has a dual advantage of reduced block size and higher value for lower bound to D-efficiency.

Furthermore, since the minimum eigenvalue of the alternative designs d^* is $n_2 - 2 - 2(n_2 - 1)^{-1}$, for large values of n_2 the design is nearly E-optimal, e.g., for $n_2 > 7$, the E-efficiency is more than 0.96 and for $n_2 > 15$, the E-efficiency is more than 0.99.

Theorem 3.3. *For $n_2 > 7$, within the class of designs $\mathscr{D}(p, b, k)$ of partial diallel crosses, with $p = n_1 n_2$, $b = n_2$, $k = n_1(n_2 - 1)/2$, the design d^*, constructed through the alternative method, is nearly E-optimal.*

Finally, we have studied the A- and D-efficiencies of the alternative designs d^*. Based on Theorems 3.1 and 3.2, the efficiency lower bounds $e_{1A}(n_1, n_2)$ and $e_{1D}(n_1, n_2)$ have been calculated for the designs in the practical range $n_1 \geqslant 2, n_2 \geqslant 5, p \leqslant 200$ with n_2 odd. Of the 252 possible designs in this range, 100%, 84.5% and 64.3% of the designs have e_{1A} greater than 0.85, 0.9 and 0.95, respectively. Also, if we restrict to designs having $n_1 \leqslant n_2$ then of the 174 possible designs, 100%, 97.7% and 90.2% of the designs have e_{1A} greater than 0.85, 0.9 and 0.95, respectively. Similarly, 100% and 75.4% of the designs have e_{1D} greater than 0.9 and 0.95, respectively. Also, restricting to designs having $n_1 \leqslant n_2$, 100% and 96.6% of the designs have e_{1D} greater than 0.9 and 0.95, respectively.

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