Some remarks on proximinality in higher dual spaces

T.S.S.R.K. Rao

Stat–Math Unit, Indian Statistical Institute, R. V. College P.O., Bangalore 560059, India

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Abstract

In this paper we consider proximinality questions for higher ordered dual spaces. We show that for a finite dimensional uniformly convex space X, the space $C(K, X)$ is proximinal in all the duals of even order. For any family of uniformly convex Banach spaces $\{X_\alpha\}_{\{\alpha\in\Gamma\}}$ we show that any finite co-dimensional proximinal subspace of $X = \bigoplus_{c_0} X_\alpha$ is strongly proximinal in all the duals of even order of X.

Keywords: Proximinality; Duals of higher order

1. Introduction

We always consider a Banach space as canonically embedded in its bidual. Thus if $Y \subset X$ then $Y \subset Y^{\perp\perp} \subset X^{**}$ is the canonical embedding. We say that Y is proximinal in X if for any $x \in X$ there exists a best approximant $y_0 \in Y$ such that $d(x, Y) = \|x - y_0\|$. The set valued map $x \to P(x)$ where $P(x)$ is the set of best approximants for x in Y, is called the metric projection. We recall that X is said to be proxbid if under the canonical embedding it is a proximinal subspace of X^{**}. This concept received some attention during the eighties. See, for example, [5] and [9] and the references therein. Recently Indumathi [7] proved that any finite co-dimensional proximinal subspace of c_0 continues to be proximinal in its bidual ℓ^∞. Motivated by this in this paper we undertake an investigation of spaces $Y \subset X$ where Y is proximinal (under the canonical embedding) in all the duals of even order of X. Note that if $Y \subset X^{**}$ is proximinal then in particular it is proximinal in its bidual. Thus such a space Y is proximinal in all its duals of even order. It may be noted that proximinality is in general not hereditary or transitive.

We denote by $X^{(2n)}$ the higher duals for $n > 1$. We note that when X is non-reflexive, X^{**} and $X^{\perp\perp}$ are distinct (isometric) subspaces of $X^{(4)}$.

Let K be a compact Hausdorff space and let $C(K, X)$ denote the space of X-valued continuous functions equipped with the supremum norm. Using a result of Lau [9] we show that when X is uniformly convex, any $C(K)$ module M (i.e., $f \in M$, $g \in C(K)$ implies $fg \in M$) is proximinal in $C(K, X)^{**}$.

To see the relation with the proximinality questions in spaces of operators we note that the space of compact operators $\mathcal{K}(X, C(K))$ can be identified with $C(K, X^*)$ and $\mathcal{L}(X, C(K))$ can be identified with the space $W^*C(K, X^*)$ of functions that are continuous when X^* is equipped with the weak*-topology. These embeddings are carried out via the map $T \to T^*|K$ where K is canonically embedded in $C(K)^*$. Moreover when $K = \beta(\Gamma)$, $W^*C(K, X^*)$ can be identified with $\bigoplus_\infty X^*$ (Γ-many copies of X^*). Also since $C(K)$ has the metric approximation property, it follows from [10, Example 1] that in the canonical embedding $C(K, X^*) = \mathcal{K}(X, C(K)) \subset \mathcal{L}(X, C(K))$ is an ideal (in the sense considered in [10]). Consequently one has that $\mathcal{L}(X, C(K))$ is isometric to a subspace of $C(K, X^*)^{**}$ in such a way that the isometric copy is in the canonical embedding, between $C(K, X^*) \subset C(K, X^*)^{**}$.

Therefore if $C(K, X^*)$ is proxbid we in particular have that $\mathcal{K}(X, C(K))$ is proximinal in $\mathcal{L}(X, C(K))$.

We recall [6] that a proximinal subspace $Y \subset X$ is said to be strongly proximinal if for each $x \in X$ and $\epsilon > 0$, there exists $\delta > 0$ such that $\sup\{d(z, P(x)): z \in Y \text{ and } \|x - z\| < d(x, Y) + \delta\} < \epsilon$. See [3] for some characterizations of finite co-dimensional strongly proximinal subspaces.

Extending Theorem 4.1 of [7] we show that for any family $\{X_\alpha\}_{\{\alpha \in \Gamma\}}$ of uniformly convex Banach spaces, any proximinal finite co-dimensional subspace Y of $X = \bigoplus_{c_0} X_\alpha$ is strongly proximinal in all the duals of even order of X.

We refer to Chapter VIII of [2] for standard results from tensor product theory that we will be using. We use the subscript π to denote the projective tensor product.

2. Main results

We recall that a closed subspace $M \subset X$ is said to be an M-ideal if there exists a linear projection $P : X^* \to X^*$ such that $\ker(P) = M^\perp$ and $\|x^*\| = \|P(x^*)\| + \|x^* - P(x^*)\|$ for all $x^* \in X^*$. If $X = M \oplus_\infty N$ for a closed subspace N, then M is said to be an M-summand.

It is easy to see that any M-summand is an M-ideal. Any M-ideal is a strongly proximinal subspace (see [7]).

Lemma 2.1. *If $M \subset X$ is an M-ideal then under the canonical embedding M is proxbid if and only if M is proximinal in X^{**}.*

Proof. One implication is always true. Suppose M is an M-ideal and proxbid. Since $X^{**} = M^{\perp\perp} \oplus_\infty N$, as M is proximinal in its bidual $M^{\perp\perp}$ we get that M is proximinal in X^{**}. □

Lemma 2.2. *Let $Y \subset X$ be the range of a projection P of norm one. If X is proxbid then so is Y.*

Proof. We have that $P^{**} : X^{**} \to Y^{\perp\perp}$ is a projection of norm one. Let $\Lambda \in Y^{\perp\perp} \setminus Y$. Since X is proxbid there exists $x_0 \in X$ such that $d(\Lambda, X) = \|\Lambda - x_0\|$. Now for any $y \in Y$, $\|\Lambda - y\| \geqslant \|\Lambda - x_0\| \geqslant \|P^{**}(\Lambda - x_0)\| = \|\Lambda - P(x_0)\|$. As $P(x_0) \in Y$ we have $d(\Lambda, Y) = \|\Lambda - P(x_0)\|$. □

Question 2.3. *We do not know if X^{**} is proxbid implies X is proxbid. In view of the natural projection $\Lambda \to \Lambda|X$ from X^{***} to X^*, by the above lemma we have that if X^{***} is proxbid then so is X^*.*

Remark 2.4. Let T be a locally compact noncompact Hausdorff space and let X be a Banach space whose dual is isomorphic to $L^1(\mu)$. For a closed subspace $V \subset C_0(T, X) \subset C(\beta(T), X)$, Theorem 3.9 of [1] gives conditions under which V is proximinal in $C(\beta(T), X)$. It follows from Remark 3.25 of [1] that $V = \{f \in c_0\colon\ f(2n) = nf(2n-1),\ n \geqslant 1\}$ is proximinal in c_0 but not in ℓ^∞. Since c_0 is an M-ideal in ℓ^∞ but not an M-summand, this example shows that proximinality is not transitive even when one of the subspaces has the stronger property of being an M-ideal. As already noted during the proof of Lemma 1, if $Z \subset Y \subset X$ and Z is proximinal in Y and Y is an M-summand in X, then Z is proximinal in X.

It follows from [4, Theorem III.1.6] that V is proxbid. It follows from [10, Proposition 1] that there is a projection of norm one $P^*: (\ell^\infty)^{**} \to (\ell^\infty)^{**}$ such that range$(P^*) = V^{\perp\perp}$, i.e., V is an ideal in the sense considered in [10].

It is well known that for a compact set K, $C(K)$ is proximinal in its bidual [5]. Since the bidual of $C(K)$ can be identified with $C(K')$ where K' is the Stone space of $C(K)^{**}$ a simple induction argument shows that all the duals of even order are proxbid. We are interested in conditions under which the space of vector-valued continuous functions $C(K, X)$ and for a family of Banach spaces $\{X_\alpha\}$ the ℓ^∞-direct sum $\bigoplus_\infty X_\alpha$ are proxbid. It is easy to see that this is the case for finite families.

It follows from the above lemma that it is necessary that X or each X_α is proxbid.

When all the X_α's are taken as a finite dimensional space X, it is well known that $\bigoplus_\infty X_\alpha$ can be identified as $C(\beta(\Gamma), X)$ where Γ is the index set for α.

Proposition 2.5. *For any uniformly convex space X, any $C(K)$ module $M \subset C(K, X)$ is proximinal in $C(K, X)^{**}$. In particular, M is proxbid.*

Proof. Since X is uniformly convex it is reflexive. Thus $C(K, X)^* = C(K)^* \oplus_\pi X^*$ (see [2, Chapter VIII]). Hence $C(K, X)^{**} = \mathcal{L}(X^*, C(K)^{**})$. The latter space can now be identified with $W^*C(K', X)$ where K' is the stone space of $C(K)^{**}$. Now under the canonical embedding $C(K)$ is a closed subalgebra of $C(K)^{**}$ containing the identity 1. Thus there exists a continuous onto map $\phi: K' \to K$ such that the canonical embedding of $C(K, X)$ in $W^*C(K', X)$ is implemented by $f \to f \circ \phi$. It now follows from [9, Theorem 4.3] that M is proximinal in $C(K, X)^{**}$. □

In the following corollary we collect several simple consequences. Theorem 2 of [5] is our statement (a) when E is a singleton and X is the scalar field.

Corollary 2.6.

(a) *For a uniformly convex space X, for any $E \subset K$, $M = \{f \in C(K, X)\colon\ f(E) = 0\}$ is proxbid as well as proximinal in $C(K, X)^{**}$.*
(b) *When X is finite dimensional and uniformly convex, $C(K, X)$ and all its duals of even order are proxbid. More over $C(K, X)$ is proximinal in all the duals of even order.*
(c) *For a finite dimensional uniformly convex spaces X and for any index set Γ the same conclusion as in* (b) *holds for the space $\bigoplus_\infty X$.*

Proof. (a) Since M is a $C(K)$ module this follows from the above proposition. It can also be deduced from Lemma 1 since M is an M-ideal in $C(K, X)$.

(b) Following the notation of the proof of the above proposition, as X is finite dimensional and uniformly convex $C(K, X)^{**} = C(K', X)$ and the canonical embedding is implemented by composition with the continuous onto map $\phi: K' \to K$. Now repeating the argument with K', we get a K'' and a continuous onto map $\psi: K'' \to K'$. Now applying the proposition once again using $\phi \circ \psi: K'' \to K$ we see that $C(K, X)$ is proximinal in $C(K, X)^{(4)}$. The conclusion now follows by induction.

(c) We identify the space $\bigoplus_\infty X$ with $C(\beta(\Gamma), X)$ then the conclusion follows from (b). □

Remark 2.7. Let $\{X_\alpha\}$ be a family of finite dimensional uniformly convex spaces such that $\{\dim(X_\alpha)\}$ is bounded. Then using (c) and the fact that finite ℓ^∞-sums are proxbid, we see that $\bigoplus_\infty X_\alpha$ is proxbid. We do not know if this is also the case for a general family of uniformly convex finite dimensional spaces.

Let X be a reflexive Banach space. Note that $C(K, X)^* = C(K)^* \otimes_\pi X^* = L^1(\mu) \otimes_\pi X^* = L^1(\mu, X^*)$ for some positive measure μ. It follows from the remarks on page 200 of [4] that, under the canonical embedding, $C(K, X)^{***} = C(K, X)^* \oplus_1 N$ for some closed subspace N (ℓ^1-direct sum). Thus $C(K, X)^*$ is a Chebyshev subspace (unique best approximation) of its bidual. It can now be deduced from [11, Theorem 6] that $C(K, X)^*$ is a Chebyshev subspace of $C(K, X)^{(2n+1)}$ for $n > 1$. If X is a finite dimensional uniformly convex space, it follows from (b) above that $C(K, X)$ and all of its duals are proximinal in the appropriate biduals.

In the proof of the following theorem we use the fact that strong proximinality is hereditary for uniformly convex spaces. This for example can be seen from [8] that any closed subspace of a uniformly convex space has the stronger property U-proximinality.

Theorem 2.8. *Let $\{X_\alpha\}$ be a family of uniformly convex spaces. Let $X = \bigoplus_{c_0} X_\alpha$. Then any finite co-dimensional proximinal subspace is strongly proximinal in all even ordered duals of X.*

Proof. We will first prove the theorem for $X^{**} = \bigoplus_\infty X_\alpha$. Let $Y \subset X$ be finite co-dimensional and proximinal. Then there exists a finite set A such that $\operatorname{supp} f \subset A$ for all $f \in Y^\perp$. Thus $Y = \bigoplus_{c_0}\{X_\alpha\}_{\alpha\notin A} \bigoplus_\infty (Y \cap \bigoplus_\infty\{X_\alpha\}_{\alpha\in A})$. Since X_α's are reflexive it follows from [4, Theorem III.1.6] that the first summand here is an M-ideal in its bidual $\bigoplus_\infty\{X_\alpha\}_{\alpha\notin A}$ and hence is a strongly proximinal subspace. Also $X^{**} = \bigoplus_\infty\{X_\alpha\}_{\alpha\notin A} \bigoplus_\infty \bigoplus_\infty\{X_\alpha\}_{\alpha\in A}$. Thus we only need to show that $Z = Y \cap \bigoplus_\infty\{X_\alpha\}_{\alpha\in A}$ is strongly proximinal in $\bigoplus_\infty\{X_\alpha\}_{\alpha\in A}$. Now since Z is a finite co-dimensional subspace of $\bigoplus_\infty\{X_\alpha\}_{\alpha\in A}$ and since any closed subspace of a uniformly convex space is strongly proximinal, we see that condition (2) of Theorem 2.2 of [6] is satisfied. Therefore by [6, Theorem 2.2] we have that Z is strongly proximinal. Hence Y is strongly proximinal in X^{**}.

Next note that since A is finite and X_α's are reflexive, $X^{(4)} = (\bigoplus_\infty\{X_\alpha\}_{\alpha\notin A})^{\perp\perp} \bigoplus_\infty\{X_\alpha\}_{\alpha\in A}$.

Again since $\bigoplus_{c_0}\{X_\alpha\}_{\alpha\notin A}$ is a M-ideal in its bidual, it follows from the proof of Theorem 2 in [11] that $\bigoplus_{c_0}\{X_\alpha\}_{\alpha\notin A}$ is an M-ideal in $(\bigoplus_\infty\{X_\alpha\}_{\alpha\notin A})^{\perp\perp}$. Thus as before we conclude that Y is strongly proximinal in $X^{(4)}$. Now the proof can be completed by an induction argument similar to the one given during the proof of Theorem 2 in [11]. □

We recall that for a discrete set Γ, $C(\beta(\Gamma), X)^* = \bigoplus_1 X^* \bigoplus_1 N$ for some closed set N (the set of non-atomic measures). Here and in what follows below the direct sums are taken over the index set Γ. The following result is similar to our proposition, for subspaces which are not necessarily $C(K)$ modules.

Theorem 2.9. *Let X be a finite dimensional uniformly convex space. Let $Y \subset \bigoplus_\infty X = C(\beta(\Gamma), X)$ be a finite co-dimensional subspace determined by elements of $\bigoplus_1 X^*$ that are finitely supported. Then Y is proximinal in $(\bigoplus_\infty X)^{**}$.*

Proof. Since Y is finite co-dimensional we can conclude that the hypothesis implies that

$$Y = \bigoplus_\infty \{X_\alpha\}_{\alpha \notin A} \bigoplus_\infty \Bigl(Y \cap \bigoplus_\infty \{X_\alpha\}_{\alpha \in A} \Bigr)$$

where each $X_\alpha = X$. We have

$$\Bigl(\bigoplus_\infty X \Bigr)^{**} = \Bigl(\bigoplus_\infty \{X_\alpha\}_{\alpha \notin A} \Bigr)^{\perp\perp} \bigoplus_\infty \Bigl(Y \cap \bigoplus_\infty \{X_\alpha\}_{\alpha \in A} \Bigr).$$

Now from the last part of the corollary applied to the index set $\Gamma \setminus A$ we have that $\bigoplus_\infty \{X_\alpha\}_{\alpha \notin A}$ is proximinal in its bidual $(\bigoplus_\infty \{X_\alpha\}_{\alpha \notin A})^{\perp\perp}$. Since the other summand is reflexive, we conclude that Y is proximinal in $(\bigoplus_\infty X)^{**}$. □

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