# Chaplygin Gravitodynamics

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**Abstract:**

We consider a new approach for gravity theory coupled to Chaplygin matter in which the *relativistic* formulation of the latter is of crucial importance. We obtain a novel form of matter with dust like density ($\sim (volume)^{-1}$) and negative pressure. We explicitly show that our results are compatible with a relativistic generalization of the energy conservation principle, derived here.

The equation of state,

$$p = -\frac{A}{\rho} \tag{1}$$

where $p$ and $\rho$ denote pressure and density respectively and $A$ being a parameter, was introduced way back by Chaplygin [1] as an effective model in computing the lifting force on a wing of an airplane. However, in modern cosmological context, the above model has turned out to be an interesting option in the Dark Energy models that initiate the acceleration of the universe. Being a perfect fluid system with negative pressure, this new role of Chaplygin gas was anticipated in the work of Jackiw et.al. [2] and others [3, 4, 5] and was first explicitly introduced in [6] as a model for Dark Energy.

However we observe that there exists a relativistic formulation of Chaplygin gas that yields a generalized expression for pressure [2],

$$p = -\frac{ma^2}{\sqrt{a^2 + \rho^2}}, \tag{2}$$

with $m$ and $a$ being constant parameters. In the non-relativistic limit,

$$p \approx -\frac{ma^2}{\rho} + O(\frac{1}{\rho^3})$$

so that the relativistic equation of state (2) reduces to the conventional non-relativistic one (1).

Our analysis using FRW gravity coupled with relativistic Chaplygin matter shows that *for all times* the energy density will be of the dust-like form,

$$\rho = \frac{b}{R^3}, \tag{3}$$

where $b$ is an integration constant and $R$ is the scale factor. Although the form of density (3) is similar to a conventional dust it has a negative pressure, characteristic of the Chaplygin gas. This feature of the Chaplygin gas is shown to be compatible with a generalized energy conservation principle, pertaining to the relativistic Chaplygin dynamics. This is our main result.

The paper is divided into two major sections. In *Section I*, we present our analysis of (relativistic) Chaplygin cosmology, leading to (3). *Section II* discusses the points of distinction between our approach and the conventional one of [6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19]. Also we establish the compatibility between our result (3) with (2) by exploiting a relativistic generalization of energy conservation principle.

*Section I: Relativistic formulation of Chaplygin cosmology*

The starting point is to consider the relativistic Chaplygin action in a flat spacetime, as given by Jackiw et.al. [2],

$$\mathcal{A}_{Ch} = \int d^4x[-\dot{\psi}\rho - \sqrt{(\rho^2 + a^2)(m^2 + \partial^\mu\psi\partial_\mu\psi)}]. \tag{4}$$

We propose a natural generalization of the above action in FRW background by including the metric:

$$\mathcal{A}_{Ch} = \int d^4x\sqrt{-g}[-\dot{\psi}\rho - \sqrt{(\rho^2+a^2)(m^2+\vec{\partial}\psi^2)}]. \quad (5)$$

We are using notations,

$$\partial\psi^2 \equiv g_{\mu\nu}\partial^\mu\psi\partial^\nu\psi \ , \ \vec{\partial}\psi^2 \equiv g_{ij}\partial^i\psi\partial^j\psi \ , \ \dot{\psi} \equiv \frac{d\psi}{dt}.$$

In the flat background, it has a non-relativistic (large $m$) limit for small $a^2$,

$$\mathcal{L}_{Ch} = -\dot{\psi}\rho - m\rho\sqrt{(1+\frac{a^2}{\rho^2})(1+\frac{\vec{\partial}\psi^2}{m^2})}$$

$$\approx -(\rho\dot{\psi}_{NR} + \rho\frac{\vec{\partial}\psi^2_{NR}}{2m} + \frac{ma^2}{2\rho}), \quad (6)$$

which is the standard Lagrangian of the Chaplygin fluid model with the identification of the Chaplygin coupling $\lambda \equiv \frac{ma^2}{2}$ and the mapping [2] $\psi \equiv -mt + \psi_{NR}$. It is important to note that the matter (Chaplygin) action is in the Eulerian form in co-moving coordinates, that is compatible with the FRW equations.

The Chaplygin equations of motion from (5) are,

$$\frac{d}{dt}(\sqrt{-g}\rho) + \partial^j[\frac{\sqrt{-g}\sqrt{\rho^2+a^2}g_{ij}\partial^i\psi}{\sqrt{(m^2+\vec{\partial}\psi^2)}}] = 0 \ , \quad (7)$$

$$\dot{\psi} + \rho\sqrt{\frac{m^2+\vec{\partial}\psi^2}{\rho^2+a^2}} = 0 \ . \quad (8)$$

One can eliminate $\rho$ by using (8),

$$\rho = -\frac{a\dot{\psi}}{\sqrt{m^2+\partial\psi^2}} \ . \quad (9)$$

This reduces the Chaplygin Lagrangian $\mathcal{L}_{Ch}$ (5) to the Born-Infeld Lagrangian in curved background when $\rho$ is eliminated,

$$\mathcal{L}_{BI} = -a\sqrt{-g}\sqrt{m^2+\partial\psi^2} \ . \quad (10)$$

The equation of motion for $\psi$ obtained from (10) is,

$$\partial^\mu[\frac{\sqrt{g}g_{\mu\nu}\partial^\nu\psi}{\sqrt{m^2+\partial\psi^2}}] = 0. \quad (11)$$

This agrees with the equation of motion for $\psi$ that one gets from (7) and (8) after substituting $\rho$ from (9).

The utility of introducing the Born-Infeld form of the action is the following. We wish to compute the Energy-Momentum Tensor (EMT) $T_{\mu\nu}$ that is required in the Einstein equation.

However, at the same time we wish to include the $\rho$-field to make contact with the Chaplygin cosmology. Now, the form of the action containing $\rho$ in (5) is not in a manifestly covariant form and so to get the EMT in a covariant form from (5) is awkward. On the other hand, it is straightforward to get $T_{\mu\nu}$ from the Born-Infeld model in (10) but it does not contain $\rho$. However, in $T_{\mu\nu}$ obtained from (10), $\rho$ can be reintroduced by exploiting (7)-(8). We have already checked the consistency of this procedure (see comment below (11)). We will follow this route.

$T_{\mu\nu}$ obtained from (10) is

$$T^{\mu\nu} = -a[\sqrt{m^2 + \partial\psi^2} g^{\mu\nu} - \frac{\partial^\mu\psi\partial^\nu\psi}{\sqrt{m^2 + \partial\psi^2}}]. \tag{12}$$

In particular we find,

$$T^{00} = -\frac{a}{\sqrt{m^2 + \partial\psi^2}}[(m^2 + \partial\psi^2)g^{00} - (\partial^0\psi)^2] = \sqrt{(\rho^2 + a^2)(m^2 + \vec{\partial}\psi^2)}. \tag{13}$$

Note that this form of energy density agrees with the one that can be read off directly from the Chaplygin model (5), the latter being first order in time derivative. This again ensures that the expression of the EMT from Born-Infeld and its subsequent form involving $\rho$ is consistent. Obviously we are interested in the latter as this will involve the density $\rho$ directly.

The other components of the EMT are,

$$T^{i0} = T^{0i} = -\rho\partial^i\psi \ , \tag{14}$$

$$T^{ij} = -a^2\sqrt{\frac{m^2 + \vec{\partial}\psi^2}{\rho^2 + a^2}} g^{ij} + \sqrt{\frac{\rho^2 + a^2}{m^2 + \vec{\partial}\psi^2}} \partial^i\psi\partial^j\psi \ . \tag{15}$$

Notice that the $\rho, \psi$-system, comprising of (5,7,8) is not manifestly Lorentz or general co-ordinate invariant. Hence, as a necessary and sufficient condition for the Poincare invariance of the theory as well as the consistent definition of the EMT (as given in (13,14,15) obtained from (12)), we have explicitly checked the validity of the Schwinger conditions,

$$\{T^{00}(\vec{x}), T^{00}(\vec{y})\} = (T^{0i}(\vec{x}) + T^{0i}(\vec{y}))\partial^i_{(x)}\delta(\vec{x} - \vec{y}),$$
$$\{T^{00}(\vec{x}), T^{0i}(\vec{y})\} = (T^{00}(\vec{x}) + T^{00}(\vec{y}))\partial^i_{(x)}\delta(\vec{x} - \vec{y}). \tag{16}$$

The above is computed by exploiting the (equal time) Poisson Brackets,

$$\{\psi(\vec{x}), \rho(\vec{y})\} = \delta(\vec{x} - \vec{y}) \ , \ \ \{\psi(\vec{x}), \psi(\vec{y})\} = 0 \ , \ \ \{\rho(\vec{x}), \rho(\vec{y})\} = 0 \ , \tag{17}$$

that can be read off from the symplectic structure of the Lagrangian in (5).

For completeness let us quickly derive the FRW equations from the total action

$$\mathcal{A} = \mathcal{A}_{Ch} + \mathcal{A}_{Gravity}. \tag{18}$$

We consider a FRW universe with the metric $g_{\mu\nu}$ identified below,

$$-d\tau^2 = -dt^2 + R^2(t)(\frac{dr^2}{1 - kr^2} + r^2 d\theta^2 + r^2 sin^2\theta d\phi^2). \tag{19}$$

In the Einstein equation

$$R_{\mu\nu} = -8\pi G S_{\mu\nu} \ , \ S_{\mu\nu} = T_{\mu\nu} - \frac{1}{2} g_{\mu\nu} T^\sigma_\sigma, \tag{20}$$

the Einstein tensor $R_{\mu\nu}$ for FRW spacetime is,

$$R_{00} = \frac{3\ddot{R}}{R} \ , \ R_{ij} = -(R\ddot{R} + 2\dot{R}^2 + 2k) g_{ij} \ , \ R_{0i} = 0. \tag{21}$$

In fact the elegance of our formalism will be manifest now since *everything*, including $\rho = \rho(R)$ (given in (3)) valid for Chaplygin model, will be derived from the Einstein equation (20) for the present case. One should remember that in the analysis of [6, 7, 8, 9, 10, 11, 12, 13, 14] the analogus relation (32) was obtained separately, by incorporating the *nonrelativistic* Chaplygin equation of state (1) by hand.

We start by noting that in our case the $(0i)$ component of Einstein equation is *not* vacuous. It yields

$$S_{0i} = -\rho \partial_i \psi = 0. \tag{22}$$

Since $\rho \neq 0$, this in turn indicates $\partial_i \psi = 0$ ensuring homogeneity of $\psi$. This also makes $\rho$ uniform since $\rho$ is related to $\psi$ by (9). This restriction considerably simplifies the rest of the source terms and using (13,14,15) we find,

$$S_{00} = \frac{m(\rho^2 - 2a^2)}{2\sqrt{\rho^2 + a^2}} \ , \ S_{ij} = \frac{m(\rho^2 + 2a^2)}{2\sqrt{\rho^2 + a^2}} g_{ij} \ . \tag{23}$$

From the (00) and $(ij)$ component of the Einstein equation, after eliminating $\ddot{R}$, we find

$$\frac{\dot{R}^2}{R^2} = \frac{8\pi G m}{3} \sqrt{\rho^2 + a^2} - \frac{k}{R^2}. \tag{24}$$

Notice that the matter contribution in (24) is structurally different from the corresponding equation in [6].

The last task is to obtain $\rho$ as a function of $R$. For this we put $\partial_i \psi = 0$ back in the equation of motion (7), and get

$$\frac{d}{dt}(\sqrt{-g}\rho) = 0. \tag{25}$$

Let us try to obtain a solution of the differential equation (25) directly. For FRW metric we compute $\frac{d}{dt}(\sqrt{-g}) = 3\sqrt{-g}\frac{\dot{R}}{R}$ and substitute this in (25) to find

$$\dot{\rho} = -3\rho \frac{\dot{R}}{R}, \tag{26}$$

the solution of which is trivially obtained,

$$\rho = \frac{b}{R^3},$$

with $b$ being an integration constant, as presented in (3). This is the central result of our paper.

Clearly, as stated before, the above form of density in (3) is completly different from the functional form of $\rho(R) \sim \sqrt{A + BR^{-6}}$ mentioned in (32) that was obtained in [6]. Substituting (3) in (24) we obtain the evolution equation,

$$\frac{\dot{R}^2}{R^2} = \frac{8\pi Gm}{3}\sqrt{a^2 + \frac{b^2}{R^6}} - \frac{k}{R^2}. \tag{27}$$

Coincidentally this is the same equation that was obtained in [6].

*Section II: Discussions*

Let us now come to our relativistic scheme. As we have discussed in *Section I* in detail, in our formulation, the density function is derived naturally starting from the well defined total relativistic action (18). This function is induced solely from Chaplygin matter dynamics. It is derived from (7) ( or (25)) which are dynamical equations of motion of the Chaplygin matter. We emphasise that this feature of taking into account the Chaplygin matter dynamics is completely absent in the conventional analysis [6].

Alternatively, one can also exploit the generic energy conservation principle,

$$d(energy\ density \times volume) = -pressure \times d(volume) \tag{28}$$

to derive the energy function, provided the proper relativistic generalizations of the terms appearing in (28) are considered. In the present case the pressure of the relativistic Chaplygin gas is known to be (2). In fact, the general form of pressure is obtained by identifying the Lagrangian with pressure in the relativistic theory [2],

$$p \equiv \mathcal{L}_{BI} = -a\sqrt{m^2 + \partial\psi^2} = -a^2\sqrt{\frac{m^2 + \vec{\partial}\psi^2}{a^2 + \rho^2}} = -\frac{ma^2}{\sqrt{a^2 + \rho^2}}. \tag{29}$$

For simplicity we have dropped $g$ from the Born-Infeld Lagrangian (10) and the last equality follows from the uniformity of $\psi$ obtained in (22). On the other hand, the relativistic energy density $T^{00}$ is derived from (13):

$$T^{00} = m\sqrt{\rho^2 + a^2}. \tag{30}$$

Finally, a simple algebra will show that an identical form of density (3), computed in *Section I* is recovered by using the pressure function (2) or (29) in the relativistic energy conservation equation,

$$d(T^{00}R^3) = -p\ d(R^3), \tag{31}$$

thereby proving our assertion.

We may mention that in the conventional formulation of Chaplygin cosmology [6], the Universe evolves smoothly from a dust dominated to de-Sitter model, with the help of a single fluid. This has been discussed in a large number of theoretical [7, 8, 9, 10, 11, 12, 13, 14] and observational [15, 16, 17, 18, 19] studies. The result follows from the particular form of energy density,

$$\rho = \sqrt{A + \frac{B}{R^6}}, \tag{32}$$

with $A$ and $B$ being a constant parameter and an integration constant respectively. The above relation is obtained by using the equation of state for Chaplygin gas [6], in conjunction with energy conservation equation (28). Subsequently, this form of density is substituted in the Friedman-Robertson-Walker (FRW) equation to generate the evolution equation for the scale factor.

As we have shown this is not a unique way of interpreting Chaplygin cosmology since conclusions might differ if a fully relativistic formulation is used. This indeed happens in the present case where, on the contrary, we find only a dust dominated Universe for all times.

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