The Berry phase in ferromagnetic spin systems and anomalous Hall Effect

B. Basu[*] and P. Bandyopadhyay[†]
Physics and Applied Mathematics Unit
Indian Statistical Institute
Kolkata-700108

We have shown that the study of topological aspects of the underlying geometry in a ferromagnetic spin system gives rise to an intrinsic Berry phase. This real space Berry phase arises due to the spin rotations of conducting electrons which can be manifested as a further contribution in anomalous Hall effect.

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In ferromagnetic metals with a spontaneous broken time reversal symmetry, besides the ordinary Hall effect linear dependence of the off diagonal resistivity on the appilied magnetic field, there exists a contribution proportional to the magnetization of the ferromagnet. The transverse resistivity ρ_H in ferromagnets consists of the two contributions,

$$\rho_H \;=\; R_0 B + R_S M \tag{1}$$

where B, M, R_0 and R_S are magnetic induction, magnetization, ordinary Hall coefficient and anomalous Hall coefficient. The second term, which is proportional to the magnetization, represents the anomalous Hall effect (AHE). Conventionally, the AHE is ascribed to spin-orbit interaction which involves a coupling of orbital motion of electrons to the spin polarization of conduction electrons [1] or to the asymmetric skew scattering of conduction electrons by the fluctuation of localised moments [2].

In a recent paper Haldane, [3] has pointed out that the intrinsic AHE in metallic ferromagnets is controlled by Berry phases accumulated by adiabatic motion of quasiparticles on the Fermi surface in the presence of broken inversion or time -reversal symmetry. This theory was in support of a report of Fang *et. al*[4] where they have shown that the AHE is related to the Berry phase acquired by the Bloch wave functions when the magnetic "monopole" is in the crystal momentum space. The topological features associated with the AHE have also been studied by other authors [5, 6]. It has been realized that under appropriate conditions a chirality contribution shows up in AHE [5, 6]. In case of canonical spin glass chirality contribution to the AHE was examined by Tatara and Kawamura [6, 7]. However, in order to get a net topological field (chirality) the spin-orbit coupling must be invoked. Besides, we have problems related to the 2D Kagome lattice or 3D Pyrochlore lattice where the net topological field vanishes though a nonvanishing topological Hall effect may be obtained [6, 8]. Breno, Dugaev and Taillefumier [9]have shown that a net topological field can be obtained by means of some external parameter. The analysis of this topological Hall effect does not depend on spin-orbit coupling but arises solely from the Berry phase acquired by an electron moving in a smoothly varying magnetization.

Recently, the AHE has been studied in a series of AuFe samples [10]. It has been observed that below a critical Fe concentration the alloys are spin glasses while for higher concentrations the alloys have been dubbed 're-entrant'; on cooling one first encounters a ferromagnetic ordering temperature T_c and then a second canting temperature T_k which is signalled by a dramatic drop in the low field ac susceptibility. Below T_k the system still has an overall ferromagnetic magnetization but the individual Fe spins become canted locally with respect to the global magnetization axis. The experimental data demonstrate that the degree of canting strongly modifies the AHE. This is a physically distinct Berry phase contribution occurring in real space when the spin configuration is topologically nontrivial. The present paper is an attempt to address some aspects associated with the geometry of this type of ferromagnetic system and show that the contribution of AHE in this system is a topological effect which arises due to the Berry curvature accumulated by spin rotations of moving electrons.

We start with a model which represents the effective ferromagnetic interaction between electrons on different sites. The Hamiltonian is given by

$$H = \sum_{i,j} t c_i^\dagger c_j \; - \; \frac{J_H}{2} \sum_i \mathbf{S}_i . [c_i{}^\dagger \vec{\sigma} c_j] \tag{2}$$

where (i, j) runs the nearest neighbour sites, $c_i(c_i^\dagger)$ is the annihilation (creation) operator at the site i and $\mathbf{S}_i$ is the classical spin localised at the site i and $\vec{\sigma}$ are Pauli matrices. This describes an electron hopping from site i to site j coupled to a spin at each site with a Hund coupling J_H. When J_H is strong enough the spin of the hopping electron is forced to align parallel to $\mathbf{S_i}$ and $\mathbf{S_j}$ at each site.

For a ferromagnetic system which allows a ferromagnetic ordering below T_c as well as canting of local spins below T_k we shall focus only on the position dependent magnetization direction

$$\mathbf{n} = \frac{\mathbf{M(r)}}{M} \tag{3}$$

with a fixed magnitude $|\mathbf{M}|$=M. The magnetization direction $\mathbf{n}(\mathbf{r}, t)$ satisfies the equation of motion

$$\partial_t \mathbf{n}(\mathbf{r}, t) = -\gamma \mathbf{n}(\mathbf{r}, t) \times H_{eff}(\mathbf{r}) \tag{4}$$

where γ is (minus) the gyromagnetic ratio $H_{eff}(\mathbf{r})$ is the so-called $effective\ magnetic\ field$ depending on the magnetization distribution $\mathbf{M(r)}$. Indeed, we can use a gauge transformation $U(\mathbf{r})$ which makes the quantization axis oriented along the vector $\mathbf{n(r)}$ at each point so that we can write

$$U^\dagger(\mathbf{r})[\vec{\sigma}.\mathbf{n}(\mathbf{r})]U(\mathbf{r}) = \sigma_z \tag{5}$$

In generalized spin systems we may assume that the direction of the vector $\mathbf{n(r)}$ is arbitrary at different sites. To view the rotation of the magnetization vector such that the vector $\mathbf{n(r)}$ is oriented along the quantization axis we may consider the vector $\mathbf{n(r)}$ is rotating with an angular velocity ω_0 around the z-axis under an angle ν so that at any instant of time, the magnetization vector is at a position which makes an arbitrary angle ν with the quantization axis. Then we can write the unit vector in terms of the variables ν and t as

$$\mathbf{n(r)} \to \mathbf{n}(\nu, t) \tag{6}$$

where

$$\mathbf{n}(\nu, t) = \begin{pmatrix} \sin\nu & \cos(\omega_0 t) \\ \sin\nu & \sin(\omega_0 t) \\ \cos\nu & \end{pmatrix} \tag{7}$$

It may be added here that there may be an additional phase factor in a multi-spin system where $\cos(\omega_0 t)$should be replaced by $\cos(\omega_0 t + \phi)$. However as this will not change the physics we are considering, we may omit it here. The instantaneous eigenstates of a spin operator in direction $\mathbf{n}(\nu, t)$ expanded in the σ_z-basis are given by

$$\begin{aligned} |\chi(\uparrow))_n; t> &= \quad \cos\frac{\nu}{2}|\uparrow_z> + \ \sin\frac{\nu}{2} e^{i\omega_0 t}|\downarrow_z> \\ |\chi(\downarrow))_n; t> &= -\sin\frac{\nu}{2}|\uparrow_z> + \ \cos\frac{\nu}{2} e^{i\omega_0 t}|\downarrow_z> \end{aligned} \tag{8}$$

For the time evolution from $t = 0$ to $t = \tau$ where $\tau = \dfrac{2\pi}{\omega_0}$ each eigenstate will pick up a geometric phase (Berry phase) apart from the dynamical phase which is of the form

$$|\chi(\uparrow)_n; t = 0 > \to |\chi(\uparrow)_n; t = \tau >= e^{i\gamma_+(\nu)}\ e^{i\theta_+}|\chi(\uparrow)_n; t = 0 >$$

$$|\chi(\downarrow)_n; t = 0 > \to |\chi(\downarrow)_n; t = \tau >= e^{i\gamma_-(\nu)}\ e^{i\theta_-}|\chi(\downarrow)_n; t = 0 > \tag{9}$$

where $\gamma_\pm$ is the Berry phase which is half of the solid angle $\frac{1}{2}\Omega$ swept out by the magnetization vector and $\theta_\pm$ is the dynamical phase. The dynamical phase can be eliminated by the spin-echo method [11].

However, in the context of AHE we are only concerened with the Berry phase. Geometrically we may get the explicit values of the Berry phase given by $\gamma_{\pm}$ as

$$\begin{aligned} \gamma_{+}(\nu) &= -\pi(1-\cos\nu) \\ \gamma_{-}(\nu) &= -\pi(1+\cos\nu) = -\gamma_{+}(\nu) - 2\pi \end{aligned} \tag{10}$$

This equation helps us to note that in any arbitrary spin system the Berry phase aquired by a spin up eigenstate in direction $\mathbf{n}(\nu)$ is given by

$$\gamma_{+}(\nu) \ = -\pi(1-\cos\nu) \tag{11}$$

It is known that a free polarized spin can be represented by a two-component spinor which is achieved when a scalar particle is attached with one magnetic flux quantum. Now in unit of magnetic flux quantum $\frac{hc}{|e|}$ the corresponding magnetic (monopole) strength is given by $\mid \mu \mid = \frac{1}{2}$. So the phase acquired by a free polarized spin encircling a closed path may be represented by the phase acquired by a scalar particle encircling a magnetic flux quantum which is given by $e^{i2\pi\mu}$ with $\mu = 1/2$. Indeed this phase $e^{i\pi}$ corresponds to the phase acquired by a fermion after 2π rotation.

From eqn.(11) we note that in a multi-spin system, the Berry phase acquired by a spin-up eigenstate in direction $\mathbf{n}(\nu)$ can be written in terms of μ as

$$e^{i\gamma_{+}(\nu)} = e^{i2\pi\mu(1-cos\nu)} \tag{12}$$

with $\mu = 1/2$

This helps us to note that in a spin system there is a deviation of the phase factor acquired by a spin eigenstate in direction $\mathbf{n}(\nu)$from that of a free polarized spin. The deviation of the phase factor $\mid \Delta\gamma \mid$ from the free spin case is given by

$$\mid \Delta\gamma \mid = \frac{1}{2}\cos\nu \tag{13}$$

Now for a ferromagnetic system where all the spins are aligned along the quantization axis ($\nu = 0$) we have the intrinsic phase factor acquired by a spin eigenstate

$$\mid \Delta\gamma \mid = \frac{1}{2} \tag{14}$$

This intrinsic phase factor acquired by the spin eigenstate in a ferromagnetic system, is the manifestation of some inherent magnetic field in these types of systems. This fictitous magnetic field is present due to the effect of the Berry phase of the localised spins and conduction electrons move in this field when there is strong Hund coupling.

Now from eqn.(13) we observe that we will have variation of $|\Delta\gamma|$ depending on the angle ν which is associated with the degree of canting of the local spins. In fact, the system will still have an overall ferromagnetic magnetization but the degree of canting of individual spins will now change the inherent magnetic field associated with $|\Delta\gamma|$ inducing a modification of the AHE as observed in experinents [10] We introduce a gauge potential corresponding to this magnetic field

$$\mathbf{A}(\mathbf{r},t) = -2\pi i\phi_0 \ U^{\dagger}(\mathbf{r},t)\partial_i \ U(\mathbf{r},t) \tag{15}$$

where $\phi_0 = \frac{hc}{|e|}$ is the flux quantum and $i = x, y$. Now if we neglect the spin-flip transitions and assume that the system is in the spin up (or, spin down)subspace only , we can substitute the matrix gauge field by an Abelian gauge field $\mathbf{a}(\mathbf{r},t)$. In fact, we can choose $\mathbf{a}(\mathbf{r},t)$ as

$$\mathbf{a}(\mathbf{r},t) = i < \mathbf{n}(\mathbf{r},t) \mid \nabla\mathbf{n}(\mathbf{r},t) > \tag{16}$$

The Berry phase for a closed path Γ is given by

$$\Gamma = expi \oint_{\Gamma} \mathbf{a}(\mathbf{r})\mathbf{dr} \tag{17}$$

If we assume conduction electrons as to represent $2D$ electron gas we can consider the continuum limit. Writing

$$a_i(\mathbf{r},t) = \frac{n_x \partial_i n_y - n_y \partial_i n_x}{1+n_z} \tag{18}$$

the corresponding field B takes the form

$$B = \frac{1}{4\pi}\epsilon_{\mu\nu\lambda} n_\mu (\partial_x n_\nu)(\partial_y n_\lambda) \tag{19}$$

The integral of this topological field over an area enclosed by an arbitrary contour is proportional to the Berry phase.

The topological current can be defined as

$$J_\mu = \frac{1}{8\pi}\epsilon_{\mu\nu\lambda}\mathbf{n}(\partial_\nu \mathbf{n} \times \partial_\lambda \mathbf{n}) \tag{20}$$

In terms of the vector potential a_μ the topological current is given by

$$J_\mu = \epsilon_{\mu\nu\lambda}\partial_\nu a_\lambda \tag{21}$$

This is generated due to the Berry phase accumulated by the spin rotations of moving electrons when the background magnetization is not uniform in space.

In terms of the Abelian gauge field, the Hamiltonian for a $2D$ electron gas may be expressed as

$$H = -\frac{\hbar^2}{2m}[\frac{\partial}{\partial \mathbf{r}} - ie\mathbf{a}(\mathbf{r})]^2 - gM\sigma_z \tag{22}$$

and we can define the covariant momentum operator

$$\pi_\mu = -i\frac{\partial}{\partial x_\mu} - ea_\mu(\mathbf{r}) \tag{23}$$

This leads to the noncomutativity of the momentum components

$$[\pi_x, \pi_y] = ie(\partial_x a_y - \partial_y a_x) = ieB_z \tag{24}$$

This (B_z) is the *inherent magnetic field* we were talking about. We may add here that a new form of non-commuatative space can be formulated where noncommutativity in the momentum space induces a singular type of magnetic field in the real space [12, 13].

To conclude, we may say that for a spatially varying magnetization in a ferromagnetic spin system a topological current is generated due to the topological properties associated with the underlying geometry of the system. The inherent magnetic type of behavior is caused by the Berry curvature which arises due to the spin rotations of conducting electrons and is the effect of noncommutativity in momentum space.

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