On functions that preserve M-matrices and inverse M-matrices

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In earlier works, authors such as Varga, Micchelli and Willoughby, Ando, and Fiedler and Schneider have studied and characterized functions which preserve the M-matrices or some subclasses of the M-matrices, such as the Stieltjes matrices. Here we characterize functions which either preserve the inverse M-matrices or map the inverse M-matrices to the M-matrices. In one of our results we employ the theory of Pick functions to show that if A and B are inverse M-matrices such that $B^{-1} \leq A^{-1}$, then $(B+tI)^{-1} \leq (A+tI)^{-1}$, for all $t \geq 0$.

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1. Introduction

Several papers examine the question of functions that preserve the entire class or a subclass of the M-matrices or map them into the nonnegative matrices. We mention four such papers. One paper is by Varga [12] who shows that completely monotonic functions map the M-matrices to the nonnegative matrices. A second paper is due to Micchelli and Willoughby [11] in which the authors characterize all functions that preserve the $n \times n$ Stieltjes matrices and all functions which preserve the $n \times n$ symmetric nonnegative matrices. A third paper is due to Ando [2] who shows that all Pick functions which are positive on $(0, \infty)$, preserve the entire class of the M-matrices. Ando also derives several results studying the monotonicity of the Pick functions on the class of M-matrices. A fourth paper is by Fiedler and Schneider [9] who characterize all functions which preserve the M-matrices. They show that a function has this property if and only if it is positive on $(0, \infty)$ and f' is completely monotonic.

Let $\mathcal{N}, \mathcal{P}, \mathcal{M}$, and $\mathcal{M}^{-1}$ denote the classes of nonnegative matrices, positive matrices, nonsingular M-matrices, and inverse M-matrices, respectively.

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Recall that if a function $f(z)$ is defined and is analytic on a domain containing all eigenvalues of a matrix A, then the matrix $f(A)$ is defined by the Cauchy formula

$$f(A) = \frac{1}{2\pi i}\int_{\Gamma} f(z)(zI - A)^{-1}dz, \tag{1.1}$$

where Γ is any rectifiable contour inside the domain surrounding all eigenvalues of A.

We now recall some other definitions that will be required. A function $f : (0,\infty) \to R$ is *completely monotonic* if $(-1)^k f^{(k)}(x) \geq 0$, for all $x \in (0,\infty)$ and for all $k \geq 0$. (Here $f^{(0)}$ is defined to be f.) We shall denote by $f[\lambda_1, \ldots, \lambda_n]$ the divided difference of $f(x)$ at $\lambda_1, \ldots, \lambda_n$.

In section 2, we shall prove the first result of this article which serves as a converse to Varga's result mentioned above, namely:

THEOREM 1.1 *Let f be defined and analytic on the domain, obtained by deleting all nonpositive reals from the complex plane. Then, f maps $\mathcal{M}$ into $\mathcal{N}$ only if the restriction of f to $(0,\infty)$ is completely monotonic.*

In section 2, we shall also prove a result concerning submatrix monotonicity properties satisfied by the completely monotonic functions on $\mathcal{M}$.

We shall use the results of Varga, Micchelli and Willoughby, Ando, Fiedler and Schneider, as well as Theorem 1.1 to develop the main theme of this article which concerns *functions which map the class of inverse M-matrices to the inverse M-matrices, or which map the inverse M-matrices to the M-matrices*. Our main results in this direction can be summarized as follows:

THEOREM 1.2 *Let f be defined and analytic on the domain, obtained by deleting all nonpositive reals from the complex plane. Then*:

(i) *f maps $\mathcal{M}^{-1}$ into $\mathcal{N}$ if and only if $f(1/x), x \in (0,\infty)$ is completely monotonic.*
(ii) *f maps $\mathcal{M}^{-1}$ into $\mathcal{M}^{-1}$ if and only if $f(x) > 0$, for $x \in (0,\infty)$, and $(xf(1/x))^{-2}f'(1/x)$ is completely monotonic.*
(iii) *f maps $\mathcal{M}^{-1}$ into $\mathcal{M}$ if and only if $f(x) > 0$, for $x \in (0,\infty)$, and $f'(1/x)$ is completely monotonic.*

By way of examples, we shall show that Pick functions f which satisfy that $f(x) > 0$, for $x \in (0,\infty)$, map $\mathcal{M}^{-1}$ to $\mathcal{M}^{-1}$ and they also satisfy a form of monotonicity on $\mathcal{M}^{-1}$. In a further result we shall prove that:

$$A, B \in \mathcal{M}^{-1} \quad \text{and} \quad B^{-1} \leq A^{-1} \Longrightarrow (B + tI)^{-1} \leq (A + tI)^{-1}, \quad \forall t \geq 0. \tag{1.2}$$

2. Functions on the class of M-matrices

We recall the following identity for a matrix function. Suppose $A \in C^{n,n}$ has eigenvalues $\lambda_1, \ldots, \lambda_n$. If f is analytic in a domain containing all eigenvalues of A, then

$$f(\lambda_1)I - f(A) = \sum_{m=2}^{n}(-1)^m f[\lambda_1, \ldots, \lambda_m]\prod_{r=1}^{m-1}(\lambda_r I - A). \tag{2.1}$$

In the introduction we mentioned that Varga [12, Theorem 2] has shown that if f is completely monotonic, then $f : \mathcal{M}^{-1} \to \mathcal{N}$. In the first result of this article we shall show that the converse of this result is also true. The technique of proof is similar to that in [11], with some simplifications.

THEOREM 2.1 *Let f be defined and analytic on the domain obtained by deleting all non-positive reals from the complex plane. Then a necessary condition for f to map $\mathcal{M}$ into $\mathcal{N}$ is that f, restricted to $(0, \infty)$, be completely monotonic.*

Proof Suppose that $f(A) \in \mathcal{N}$ for any $A \in \mathcal{M}$. Then, by considering a 1×1 matrix, we conclude that $f(x) \geq 0$ for $x > 0$. Suppose now that f is *not* completely monotonic. Then $(-1)^{r-1} f^{(r-1)}(x_0) < 0$ for some integer $r \geq 2$ and $x_0 > 0$. Then there exist positive numbers $\lambda_1 > \cdots > \lambda_r$ (in a neighborhood of x_0) such that $(-1)^r f[\lambda_1, \ldots, \lambda_r] > 0$.

Let T be the $r \times r$ tridiagonal matrix with 1's on the super- and sub-diagonals and zeros elsewhere. For $\epsilon > 0$, set $B = \text{diag}(\lambda_1, \ldots, \lambda_r) - \epsilon T$ and let $\mu_1 \geq \cdots \geq \mu_r$ be the eigenvalues of B. By choosing ϵ to be sufficiently small we may assume that $\mu_1 > \mu_2 > \mu_r$ and $(-1)^r f[\mu_1, \ldots, \mu_r] > 0$. In view of (2.1) we have

$$f(\mu_1)I - f(B) = \sum_{m=2}^{r} (-1)^m f[\mu_1, \ldots, \mu_m] \prod_{k=1}^{m-1} (\mu_k I - B). \tag{2.2}$$

Since B is an M-matrix, by the hypothesis on f, $f(B) \in \mathcal{N}$. As B is tridiagonal, the $(1, r)$-entry of $\prod_{k=1}^{m-1}(\mu_k I - B)$ is zero for $2 \leq m < r$. Let u be an eigenvector of B corresponding to μ_r. We may assume that $u > 0$ since B is an irreducible M-matrix. Observe that $\prod_{k=1}^{r-1}(\mu_k I - B) = \alpha u u^T$ for some $\alpha > 0$ and hence the $(1, r)$-entry of $\prod_{k=1}^{r-1}(\mu_k I - B)$ is positive. In view of the preceding observations, if we compare the $(1, r)$-entry of both sides in (2.2), we conclude that it is nonpositive in case of $f(\mu_1)I - f(B)$ but positive in case of the matrix on the right hand side. This is a contradiction and hence f must be completely monotonic. ■

The function $f(x) = e^{1/x}$ clearly maps $\mathcal{M}$ into $\mathcal{N}$. We deduce from Theorem 2.1 that the function must be completely monotonic. We now consider a further example. If $\alpha = (\alpha_1, \ldots, \alpha_m)$ and $\beta = (\beta_1, \ldots, \beta_m)$ are m-tuples of nonincreasing real numbers, then α is said to be majorized by β (see [10]) if $\sum_{i=1}^{k} \alpha_i \leq \sum_{i=1}^{k} \beta_i$, $k = 1, 2, \ldots, m-1$, and $\sum_{i=1}^{m} \alpha_i = \sum_{i=1}^{m} \beta_i$. It is shown by Ball [3] that if α and β are m-tuples of nonnegative numbers such that α is majorized by β, then the rational function

$$f(x) = \frac{(x + \alpha_1) \cdots (x + \alpha_m)}{(x + \beta_1) \cdots (x + \beta_m)} \tag{2.3}$$

is completely monotonic. We now deduce this fact using Theorem 2.1. First, consider the case $m = 2$. If $\alpha = (\alpha_1, \alpha_2)$ is majorized by $\beta = (\beta_1, \beta_2)$, then it is well-known that $\alpha_1\alpha_2 \geq \beta_1\beta_2$ and hence for any $n \times n$ real matrix A,

$$(A + \alpha_1 I)(A + \alpha_2 I) \geq (A + \beta_1 I)(A + \beta_2 I). \tag{2.4}$$

If $A \in \mathcal{M}$, then $(A + \beta_1 I)^{-1} \in \mathcal{N}$, $(A + \beta_2 I)^{-1} \in \mathcal{N}$ and it follows from (2.4) that

$$(A + \alpha_1 I)(A + \alpha_2 I)(A + \beta_1 I)^{-1}(A + \beta_2 I)^{-1} \geq I \geq 0. \tag{2.5}$$

Thus, in view of Theorem 2.1, the function $f(x) = ((x+\alpha_1)(x+\alpha_2))/((x+\beta_1)(x+\beta_2))$ is completely monotonic. In general, if $\alpha = (\alpha_1, \ldots, \alpha_m)$ is majorized by $\beta = (\beta_1, \ldots, \beta_m)$, then we may construct a sequence $\alpha = \gamma_0, \gamma_1, \ldots, \gamma_\ell = \beta$ such that γ_i is majorized by γ_{i+1}, and γ_i and γ_{i+1} differ at most in two coordinates, $i = 0, 1, 2, \ldots, \ell - 1$. A repeated application of the preceding observation for $m = 2$ shows that (2.3) is completely monotonic.

We next consider some monotonicity properties related to submatrices of the matrix $f(A)$, where f is completely monotonic. In this connection we recall Bernstein's theorem, see, for example, [6, p. 148], that states that a function f is completely monotonic on $(0, \infty)$ if and only if there exists some positive measure μ such that

$$f(x) = \int_0^\infty e^{-tx} d\mu(t). \tag{2.6}$$

THEOREM 2.2 *Let $A \in \mathcal{M}$ and partition A into*

$$A = \begin{bmatrix} A_{1,1} & A_{1,2} \\ A_{2,1} & A_{2,2} \end{bmatrix}, \tag{2.7}$$

with $A_{1,1}$ and $A_{2,2}$ being square matrices and then suppose that f is completely monotonic on $(0, \infty)$. Then:

(i) $(f(A))_{1,1} \geq f(A_{1,1}) \geq 0$.
(ii) *Let $A/A_{2,2} = A_{1,1} - A_{1,2}(A_{2,2})^{-1}A_{2,1}$ be the Schur complement of $A_{2,2}$ in A. Then $f(A/A_{2,2}) \geq f(A_{1,1}) \geq 0$.*

Proof

(i) Since A is an M–matrix, $A = sI - B$ for some nonnegative matrix B and a scalar $s > \rho(B)$. But then, on partitioning the matrix e^{-A} in conformity with (2.7), we have that

$$\left(e^{-A}\right)_{1,1} = \left(e^{-(sI-B)}\right)_{1,1} = \left(e^{-s}e^{B}\right)_{1,1} \geq e^{-(sI-B_{1,1})} = e^{-A_{1,1}}.$$

The result now follows via Bernstein's characterization given in (2.6).

(ii) It is well-known from the results of Crabtree [7] that $A/A_{2,2}$ is an M-matrix which allows us to represent $A/A_{2,2} = rI - C_{1,1}$, with $r > \rho(C_{1,1})$. Now since $A/A_{2,2} \leq A_{1,1}$ we can write that

$$e^{-A/A_{2,2}} = e^{-(rI-C_{1,1})} = e^{-r}e^{C_{1,1}} \geq e^{-r}e^{B_{1,1}} = e^{-(rI-B_{1,1})} = e^{-A_{1,1}}.$$

The result follows again by Bernstein's theorem quoted above. ■

3. Functions of inverse M-matrices

In this section, we shall mainly consider functions which map the inverse M-matrices to the nonnegative matrices, that is functions from $\mathcal{M}^{-1}$ to $\mathcal{N}$, and functions which map the inverse M-matrices to the inverse M-matrices, that is functions from $\mathcal{M}^{-1}$ to $\mathcal{M}^{-1}$.

We begin with the following result.

THEOREM 3.1 *Let f be defined and analytic on the domain obtained by deleting all nonpositive reals from the complex plane. Then f maps $\mathcal{M}^{-1}$ to $\mathcal{N}$ if and only if $f(1/x)$, $x \in (0, \infty)$ is completely monotonic.*

Proof Clearly, f maps the inverse M-matrices to the nonnegative matrices if and only if the function $g(x) = f(1/x)$ maps the M-matrices to the nonnegative matrices. By Theorem 2.1 a necessary and sufficient condition for this to occur is that $g(x)$ is completely monotonic. This completes the proof. ■

As examples of functions which satisfy the requirements of Theorem 3.1, we mention $f(x) = x^2$ and $f(x) = \sqrt{x}$. We see that if $A \in \mathcal{N}$ but $A \notin \mathcal{M}^{-1}$, then it is *not necessarily* true that $f(A) \in \mathcal{N}$ as the following example shows. Let

$$A = \begin{bmatrix} 3.1 & 2.1 & 2.1 & 1.1 & 1.1 \\ 2.1 & 3.1 & 1.1 & 2.1 & 0.10 \\ 2.1 & 1.1 & 3.1 & 0.10 & 1.1 \\ 1.1 & 2.1 & 0.10 & 2.1 & 0.10 \\ 1.1 & 0.10 & 1.1 & 0.10 & 1.1 \end{bmatrix}$$

$$\approx \begin{bmatrix} 1.9167 & -1.9167 & 0.0000 & 1.0000 & -1.8333 \\ -1.9167 & 3.9167 & -1.0000 & -3.0000 & \boxed{2.8333} \\ 0.0000 & -1.0000 & 1.0000 & 1.0000 & -1.0000 \\ 1.0000 & -3.0000 & 1.0000 & 3.0000 & -2.0000 \\ -1.8333 & 2.8333 & -1.0000 & -2.0000 & 3.6667 \end{bmatrix}^{-1} \notin \mathcal{M}^{-1}$$

and we see that

$$A^{1/2} \approx \begin{bmatrix} 1.4630 & 0.62072 & 0.59007 & 0.25016 & 0.40446 \\ 0.62072 & 1.4333 & 0.28372 & 0.75025 & \boxed{-0.13072} \\ 0.59007 & 0.28372 & 1.5897 & -0.10146 & 0.36582 \\ 0.25016 & 0.75025 & -0.10146 & 1.2084 & 0.064046 \\ 0.40446 & -0.13072 & 0.36582 & 0.064046 & 0.88397 \end{bmatrix} \notin \mathcal{N}.$$

We next come to functions which preserve the inverse M-matrices.

THEOREM 3.2 *Let f be defined and analytic on the domain obtained by deleting all non-positive reals from the complex plane and suppose that* $f:(0,\infty)\to(0,\infty)$. *Then, f maps* $\mathcal{M}^{-1}$ *to* $\mathcal{M}^{-1}$ *if and only if the function*:

$$h(x) := \frac{1}{x^2(f(1/x))^2} f'(1/x) \tag{3.1}$$

is completely monotonic.

Proof Clearly, f preserves the inverse M-matrices if and only if $g(x) = \left(f\left(x^{-1}\right)\right)^{-1}$ preserves the M-matrices. By the result of Fiedler and Schneider [9] mentioned in the introduction, a necessary and sufficient condition for this to happen is that $g'(x)$ is completely monotonic. But $g'(x) = h(x)$, where $h(x)$ is given in (3.1). This completes the proof. ■

We now turn to examples of functions which preserve the inverse M-matrices. Let us begin with the following function: $f(x) = x^r$, where $0 < r < 1$. Then $h(x)$ of (3.1) is given by $h(x) = rx^{r-1}$ which is easily checked to be completely monotonic. Actually $f(x) = x^r$ is known to be an example of a Pick function. An analytic function defined on $H_+ = \{z|\ \mathrm{Im}(z) > 0\}$ with range in the closed half plane $\{z|\ \mathrm{Im}(z) \geq 0\}$ is called a *Pick function*. It is known that a Pick function f admits the representation

$$f(z) = \alpha + \beta z + \int_{-\infty}^{\infty} \left[\frac{1}{\lambda - z} - \frac{\lambda}{\lambda^2 + 1}\right] d\nu(\lambda), \tag{3.2}$$

with $\alpha \in \mathbb{R}$ and $\beta \geq 0$ and ν a positive measure on the real line. For further general references on Pick functions and completely monotonic functions, see Alzer and Berg [1], Bapat and Raghavan [4], Bhatia [6], Berman and Plemmons [5], Donoghue [8] and Varga [12].

In Ando [2, Thm 3.1], the author shows that if f is a Pick function such that $f(t) > 0$ for all $t > 0$, then f preserves the M-matrices. Now suppose that f is such a Pick function. Then as is known, see Ando [2, p. 304], $g(t) = (f(1/t))^{-1}$ is also a Pick function which is positive on $(0, \infty)$. Hence $g(t)$ preserves the M-matrices and so, in addition to $g(t) > 0$ on $(0, \infty)$, $g'(t)$ is completely monotonic. But $g'(t) = f'(1/t)/[t^2(f(1/t))^2]$ and so, on setting $h(t) = g'(t)$, it follows from Theorem 3.2 that if f is a Pick function such that $f(t) > 0$ for all $t \in (0, \infty)$, then f preserves the inverse M-matrices.

Continuing, Ando shows that if f is a Pick function, such that $f(t) > 0$ for $t \in (0, \infty)$, then not only is $f : \mathcal{M} \to \mathcal{M}$, but f is also monotonic on $\mathcal{M}$, namely if $C, D \in \mathcal{M}$ with $C \leq D$, then $f(C) \leq f(D)$. We now define an order relation on $\mathcal{M}^{-1}$ as follows. For two matrices $A, B \in \mathcal{M}^{-1}$ we shall write that $A \preceq_f B$ if $B^{-1} \leq A^{-1}$. Note that if $A \preceq_f B$, then $A \leq B$, but the converse is not necessarily true.

Now, suppose that $A, B \in \mathcal{M}^{-1}$ with $A \preceq_f B$ and that f is a Pick function, such that $f(x) > 0$ for $x \in (0, \infty)$ so that, as mentioned above, $g(x) = (f(1/x))^{-1}$ is also a Pick function. But then $(f(B))^{-1} = g(B^{-1}) \leq g(A^{-1}) = (f(A))^{-1}$ showing that on $\mathcal{M}$, f preserves the precedence order, namely:

$$A \preceq_f B,\ f \text{ a Pick function with } \quad f(x) > 0,\ x \in (0, \infty) \ \Rightarrow\ f(A) \preceq_f f(B). \tag{3.3}$$

We now have the following interesting inequality for inverse M-matrices.

THEOREM 3.3 *Let A and B be inverse M-matrices, such that $B^{-1} \leq A^{-1}$. Then*

$$(B+tI)^{-1} \leq (A+tI)^{-1}, \quad \forall t \geq 0. \tag{3.4}$$

Proof For an arbitrary, but fixed, $t \geq 0$, consider the function $f_t(x) = t + x$ which is a Pick function satisfying that $f_t(x) > 0$, for all $x \in (0, \infty)$. Then the function

$$g_t(x) := \frac{1}{f(t+1/x)}$$

is a Pick function which satisfies that $g_t(x) > 0$ for all $x \in (0, \infty)$. Hence by Ando's result mentioned earlier, if $B^{-1} \leq A^{-1}$, then

$$(B+tI)^{-1} = g_t(B^{-1}) \leq g_t(A^{-1}) = (A+tI)^{-1},$$

and our proof is done. ■

We comment that $A, B \in \mathcal{M}^{-1}$ with $A \leq B$ does not imply that $B^{-1} \leq A^{-1}$, i.e. it does not imply that $A \preceq_f B$. However, we can expect there to be examples for which $A, B \in \mathcal{M}^{-1}$, $A \not\preceq_f B$, but that there exists $t > 0$, such that $A + tI \preceq_f B + tI$. Indeed, on taking

$$A = \begin{bmatrix} 2 & 1/2 & 1/3 & 1/4 & 1/5 \\ 1/2 & 4/3 & 1/4 & 1/5 & 1/6 \\ 1/3 & 1/4 & 6/5 & 1/6 & 1/7 \\ 1/4 & 1/5 & 1/6 & 8/7 & 1/8 \\ 1/5 & 1/6 & 1/7 & 1/8 & 10/9 \end{bmatrix} \tag{3.5}$$

and

$$B = \begin{bmatrix} 2.5 & 1 & 1 & 1 & 1 \\ 1 & 2 & 1 & 1 & 1 \\ 1 & 1 & 2 & 1 & 1 \\ 1 & 1 & 1 & 2 & 1 \\ 1 & 1 & 1 & 1 & 2 \end{bmatrix},$$

it is easily checked that $A, B \in \mathcal{M}^{-1}$, $A \leq B$, and $B^{-1} \not\leq A^{-1}$. However, on taking $t = 5$, we find that

$$(A+5I)^{-1} - (B+5I)^{-1} = \begin{bmatrix} 0.003429 & 0.003249 & 0.007001 & 0.008842 & 0.009931 \\ 0.003249 & 0.007806 & 0.009629 & 0.01074 & 0.01149 \\ 0.007001 & 0.009629 & 0.01067 & 0.01140 & 0.01193 \\ 0.008842 & 0.01074 & 0.01140 & 0.01191 & 0.01230 \\ 0.009931 & 0.01149 & 0.01193 & 0.01230 & 0.01260 \end{bmatrix}.$$

Our next result is a characterization of functions which map $\mathcal{M}^{-1}$ to $\mathcal{M}$.

THEOREM 3.4 *Let f be defined and analytic on the domain obtained by deleting all non-positive reals from the complex plane and suppose that* $f:(0,\infty)\to(0,\infty)$. *Then, f maps* $\mathcal{M}^{-1}$ *to* $\mathcal{M}$ *if and only if* $f'(1/x)$ *is completely monotonic.*

Proof Clearly, f maps $\mathcal{M}^{-1}$ to $\mathcal{M}$ if and only if $f(1/x)$ maps $\mathcal{M}$ to $\mathcal{M}$. Therefore the result follows from the theorem of Fiedler and Schneider [9] mentioned in the introduction. ■

We finally mention that the sets of all functions f which map the interval $(0,\infty)$ into itself and which satisfy that (i) f is completely monotonic on $(0,\infty)$, or (ii) $f(1/x)$ is completely monotonic on $(0,\infty)$, or (iii) $f'(1/x)/(xf(1/x))^2$ is completely monotonic on $(0,\infty)$, or (iv) $f'(1/x)$ is completely monotonic on $(0,\infty)$ all form a convex cone, namely, if α, $\beta\geq 0$ and if f, g are functions which belong to one of the classes, (i)–(iv), then $\alpha f+\beta g$ also belongs to the same class. As an example, consider the functions $f(x)=\log(1+x)$ and $g(x)=x$. Both functions satisfy the requirements of class (iii). Thus, if $A\in\mathcal{M}^{-1}$ and $h(x)=\log(1+x)+x$, then $h(A)=\log(I+A)+A\in\mathcal{M}^{-1}$. For instance, take A to be the inverse M–matrix in (3.5). Then

$$h(A)=\begin{bmatrix} 3.070 & 0.6807 & 0.4507 & 0.3365 & 0.2683 \\ 0.6807 & 2.152 & 0.3450 & 0.2762 & 0.2302 \\ 0.4507 & 0.3450 & 1.971 & 0.2323 & 0.1995 \\ 0.3365 & 0.2762 & 0.2323 & 1.893 & 0.1756 \\ 0.2683 & 0.2302 & 0.1995 & 0.1756 & 1.850 \end{bmatrix}$$

$$=\begin{bmatrix} 0.3628 & -0.09700 & -0.05814 & -0.04037 & -0.03044 \\ -0.09700 & 0.5150 & -0.05844 & -0.04708 & -0.03925 \\ -0.05814 & -0.05844 & 0.5399 & -0.04384 & -0.03837 \\ -0.04037 & -0.04708 & -0.04384 & 0.5509 & -0.03586 \\ -0.03044 & -0.03925 & -0.03837 & -0.03586 & 0.5574 \end{bmatrix}^{-1}.$$

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