Optimal Designs for Estimation of Ratio of Variance Components in Diallel Crosses

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Abstract

The results on optimal designs for diallel crosses are presently available for the standard fixed effects linear model. In some cases, however, parental lines may be randomly selected from a population of lines resulting in a random effects model. Ghosh and Das (2003) discussed A-optimal designs in this context for estimation of heritability. In this paper we first propose an unbiased estimator of the ratio of the variance components which has a one-to-one relation with heritability. We then obtain an expression of the variance of this unbiased estimator of the ratio of variance components. Through minimization of the variance we obtain optimal designs and show certain connections with the optimization problem under the fixed effects model.

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1 Introduction

Plant breeders frequently need overall information on average performance of individual inbred lines in crosses known as general combining ability. For this purpose diallel crossing techniques are employed. Griffing (1956) defines a model for diallel crosses in terms of genotypic values where the breeding value of the cross (i, j) is expressed as the sum of general combining abilities for the two lines. In certain contexts, specific combining ability effects representing the interaction between lines i and j in a cross (i, j) are also included in the model; see Kempthorne (1969) and Mayo (1980) for details.

Accordingly the analysis of the observations arising out of n crosses involving p lines is carried out by postulating a model

$$Y_{ijl} = \mu + g_i + g_j + e_{ijl} \; ; \; i < j \tag{1.1}$$

where Y_{ijl} is the observation arising out of the l-th replication of the cross (i, j), g_i is the i-th line effect with $E(g_i) = 0, Var(g_i) = \sigma_g^2 \geq 0, Cov(g_i, g_j) = 0$, μ is the general mean and e_{ijl} is the random error component, uncorrelated with g_i, with expectation zero and variance $\sigma_e^2 > 0$, $1 \leq i < j \leq p$. Here μ, σ_e^2 and σ_g^2 are unknown parameters. Also, the specific combining ability effects are assumed to be negligible and have been absorbed in the error component. In the model, (1.1), μ is a fixed effect while g_i, g_j $(i < j)$ and e_{ijl} are random effects.

Our primary interest is in *heritability*, h^2, which is defined as $h^2 = 4\sigma_g^2/(2\sigma_g^2 + \sigma_e^2)$. Such a measure expresses the extent to which individual's phenotypes are determined by genotypes. In order to get a good estimator of h^2 we propose optimal designs for unbiased estimation of σ_g^2/σ_e^2 since $h^2 = \frac{4\sigma_g^2}{2\sigma_g^2+\sigma_e^2} = \frac{4(\sigma_g^2/\sigma_e^2)}{2(\sigma_g^2/\sigma_e^2)+1}$. Let T be an unbiased estimator of σ_g^2/σ_e^2. Then an estimator of h^2 is $4T/(2T+1)$. Hence an unbiased estimator of σ_g^2/σ_e^2 will lead to a asymptotically unbiased estimator of h^2.

The results on optimal designs for diallel crosses are presently available for the standard fixed effects linear model. In some cases, however, parental lines may be randomly selected from a population of lines resulting in a random effects model. In this context, under a random effects model Ghosh and Das (2003) obtained an estimator of the ratio of the variance components. In order to address the issue of optimal designs they considered the A-optimality criteria for the estimation of heritability in the sense that the designs minimize the sum of the variances of the estimators of the variance components.

In this paper we first propose an unbiased estimator T of σ_g^2/σ_e^2. We then obtain an expression of the variance of T. The large sample variance of $4T/(2T+1)$ is proportional to the variance of T, the proportionality constant being a function of σ_g^2/σ_e^2. Through minimization of the variance of T we obtain optimal designs and show certain connections with the optimization problem under the fixed effects model.

2 Unbiased Estimation of Ratio of Variance Components and their Variances

Consider a diallel cross experiment carried out using a design d with p lines and b blocks each having k crosses. Our model is

$$Y = \mu 1_n + D_2'\beta + D_1'g + e \tag{2.1}$$

where Y is the vector of bk $(= n)$ observations, μ is the general mean, g is the $p \times 1$ vector of general combining ability effects with $\mathbb{E}(g) = 0$ and $\mathbb{D}(g) = \sigma_g^2 I$, β is the fixed effect due to blocks and e is the error vector with $\mathbb{E}(e) = 0$ and $\mathbb{D}(e) = \sigma_e^2 I$. Also, $D_1 = (d_{uv}^{(1)})$ is the $p \times n$ line versus observation incidence matrix with $d_{uv}^{(1)} = 1$ if v-th observation is out of a cross involving the u-th line and $d_{uv}^{(1)} = 0$ otherwise. Similarly, $D_2 = (d_{uv}^{(2)})$ is the $b \times n$ block versus observation incidence matrix with $d_{uv}^{(2)} = 1$ if the v-th observation arise from the u-th block and $d_{uv}^{(2)} = 0$ otherwise. Here 1_t represents a $t \times 1$ column vector of all ones and I_t denotes an identity matrix of order t. In situations where the order is evident from the context, we write respectively 1 and I instead of 1_t and I_t. Thus, $\mathbb{E}(Y) = \mu 1_n + D_2'\beta$, $\mathbb{D}(Y|\sigma_g^2, \sigma_e^2) = \sigma_g^2 D_1' D_1 + \sigma_e^2 I_n$. We assume that $Y \sim N_n(\mu 1_n + D_2'\beta, \sigma_g^2 D_1' D_1 + \sigma_e^2 I_n)$, where $N_n(\theta, \Sigma)$ denotes n-variate normal distribution with mean vector θ and dispersion matrix Σ. Let $s = (s_1, s_2, \dots, s_p)'$ where s_i is the replication of the i-th line. Also, for $i \neq j$, let g_{ij} be the number of times cross (i,j) appears in the design, and $g_{ii} = s_i$. Then it is easy to see that $D_1 D_1' = G = (g_{ij})$ and $D_1 1 = s$. Let $N = D_1 D_2' = (n_{ij})$ be the incidence matrix with n_{ij} indicating the number of times the i-th line occurs in the j-th block. For such a design d, let $C_d = G - k^{-1}NN'$. C_d is also called the C-matrix of the design d.

Following Das and Ghosh (2003) and Searle et. al. (1992), expected value of sum of squares due to lines (SSL) and the expected value of sum of squares due to error (SSE) have

$$\mathbb{E}\begin{bmatrix} SSL \\ SSE \end{bmatrix} = L \begin{pmatrix} \sigma_g^2 \\ \sigma_e^2 \end{pmatrix} \tag{2.2}$$

where

$$L = \begin{pmatrix} \operatorname{tr}(C_d) & p-1 \\ 0 & n-b-p+1 \end{pmatrix}$$

and for a square matrix A, $\operatorname{tr}(A)$ stands for the trace. Also,

$$\mathbb{D}\begin{pmatrix} SSL \\ SSE \end{pmatrix} = \begin{pmatrix} 2\{\sigma_g^4 \operatorname{tr}(C_d^2) + 2\sigma_e^2\sigma_g^2 \operatorname{tr}(C_d) + \sigma_e^4 (p-1)\} & 0 \\ 0 & 2(n-b-p+1)\,\sigma_e^4 \end{pmatrix}. \tag{2.3}$$

Then,

$$\mathbb{D}\begin{pmatrix} \hat{\sigma}_g^2 \\ \hat{\sigma}_e^2 \end{pmatrix} = L^{-1}\mathbb{D}\begin{pmatrix} SSL \\ SSE \end{pmatrix}(L^{-1})' = 2\begin{pmatrix} t_{11} & t_{12} \\ t_{21} & t_{22} \end{pmatrix} \tag{2.4}$$

where

$$t_{11} = \{(n-b-p+1)(\sigma_g^4 \operatorname{tr}(C_d^2) + 2\sigma_e^2\sigma_g^2 \operatorname{tr}(C_d) + \sigma_e^4) + \sigma_e^4 (p-1)^2\} \\ /\{(n-b-p+1) \operatorname{tr}^2(C_d)\},$$

$$t_{12} = t_{21} = -\sigma_e^4 (p-1)/\{(n-b-p+1) \operatorname{tr}(C_d)\},$$

and $t_{22} = \sigma_e^4/(n-b-p+1)$.

THEOREM 2.1. *For a design d, an unbiased estimator of σ_g^2/σ_e^2 is*

$$T = \frac{(n-b-p-1)(SSL/SSE) - p + 1}{\operatorname{tr}(C_d)}.$$

PROOF. Note that $SSE/\sigma_e^2 \sim \chi^2_{n-b-p+1}$ and is distributed independently of SSL. Furthermore, $E(1/(SSE/\sigma_e^2)) = 1/(n-b-p-1)$. Thus,

$$E(SSL/SSE) = E(SSL)E(1/SSE) = \sigma_e^{-2}E(SSL)E(1/(SSE/\sigma_e^2)) \\ = \sigma_e^{-2}E(SSL)/(n-b-p-1).$$

Now using Equation (2.2) we get

$$E(SSL/SSE) = \sigma_e^{-2}(\sigma_g^2 \operatorname{tr}(C_d) + \sigma_e^2(p-1))/(n-b-p-1) \\ = ((\sigma_g^2/\sigma_e^2)\operatorname{tr}(C_d) + (p-1))/(n-b-p-1)$$

and the theorem is established. □

We have considered the estimator T because of it being simple and unbiased. However, other estimators with better properties may exist. This estimation problem is open for further research.

THEOREM 2.2. *Let d be a design with p lines, b blocks each of size k. Then, the variance of T is*

$$V(T; d, \sigma_g^2, \sigma_e^2) = \alpha \left\{ (n-b-p-1)\sigma_g^4 \frac{tr(C_d^2)}{tr^2(C_d)} + 2t\sigma_e^2\sigma_g^2 \frac{1}{tr(C_d)} + t(p-1)\sigma_e^4 \frac{1}{tr^2(C_d)} + \sigma_g^4 \right\} \quad (2.5)$$

where $\alpha = 2/\{(n-b-p-3)\sigma_e^4\}$ *and* $t = n-b-2$.

PROOF. In view of Theorem 2.1, $Var(T) = \frac{(n-b-p-1)^2}{\text{tr}^2(C_d)} Var(SSL/SSE)$. Now,

$$Var(SSL/SSE) = E(SSL/SSE)^2 - E^2(SSL/SSE). \quad (2.6)$$

The second term in (2.6) can be obtained from Theorem 2.1 itself and is given by $E^2(SSL/SSE) = \{((\sigma_g^2/\sigma_e^2)\text{tr}(C_d) + (p-1))/(n-b-p-1)\}^2$. For the first term, since SSL and SSE are distributed independently, $E(SSL/SSE)^2 = E(SSL^2)\ E(1/SSE^2)$. Now from Equation (2.2) and (2.3),

$$\begin{aligned} E(SSL^2) &= Var(SSL) + E^2(SSL) \\ &= 2\{\sigma_g^4 \ \text{tr}(C_d^2) + 2\sigma_e^2\sigma_g^2 \ \text{tr}(C_d) + \sigma_e^4 \ (p-1)\} \\ &\quad + \{\sigma_g^2 \ \text{tr}(C_d) + \sigma_e^2 \ (p-1)\}^2. \end{aligned}$$

Also, $E(1/SSE^2) = E(1/(SSE/\sigma_e^2)^2)/\sigma_e^4$. Now, since $SSE/\sigma_e^2 \sim \chi^2_{n-b-p+1}$, it is easy to see that $E(1/(SSE/\sigma_e^2)^2) = 1/\{(n-b-p-1)(n-b-p-3)\}$. Substituting the above expressions in (2.6), we get the desired result. □

The results for estimation of σ_g^2/σ_e^2 and the corresponding variance expression under unblocked diallel cross experiments can be obtained as a special case of the above results by taking the number of blocks as one. For example, the unbiased estimator of σ_g^2/σ_e^2 under an unblocked model, using a design d_0 with p lines and n crosses, reduces to $\{(n-p-2)(SSL/SSE) - p+1\}/\text{tr}(C_{0d_0})$ where $C_{0d_0} = G - \frac{1}{n}ss'$, is the C-matrix of d_0.

3 Optimal Designs

A diallel cross experiment is said to be complete if each of the $\binom{p}{2}$ crosses appear equally often in the experiment, otherwise it is said to be a partial diallel cross experiment. Let $\mathcal{D}(p, b, k)$ be the class of diallel cross designs with p lines arranged in b blocks of k crosses each and $\mathcal{D}(p, n)$

the class of unblocked designs for diallel crosses involving p lines and n crosses. Also, for $2n/p$ an integer, $\mathcal{D}_0(p,n)$ denotes the subclass of designs in $\mathcal{D}(p,n)$ with $s_i = 2n/p$; $i = 1,\ldots,p$. In fact, among designs in $\mathcal{D}(p,n)$, only designs in the subclass $\mathcal{D}_0(p,n)$ have maximal $\mathrm{tr}(C_{0d})$. Finally, let $\mathcal{D}_0(p,b,k)$ be the subclass of designs in $\mathcal{D}(p,b,k)$ for which $\mathrm{tr}(C_d)$ is maximum.

A design d is said to be optimal if, among all designs in $\mathcal{D}$, d minimizes the variance of T. From 2.5 it then follows that an optimal design in $\mathcal{D}(p,b,k)$ minimizes $\mathrm{tr}(C_d^2)/\mathrm{tr}^2(C_d)$ and $1/\mathrm{tr}(C_d)$. In other words, from 2.5 we observe that the minimization problem addressed in Ghosh and Das (2003) is analogus to the minimization of variance of T. Thus, A-optimal designs obtained in Ghosh and Das (2003) are also optimal for the minimization of the variance of T. Analogous results hold in the unblocked situation since in order to minimize the variance of T, within the class of designs $\mathcal{D}(p,n)$, it is sufficient to minimize $\mathrm{tr}(C_{0d}^2)/\mathrm{tr}^2(C_{0d})$ and $1/\mathrm{tr}(C_{0d})$.

In view of the above, A-optimal designs in Ghosh and Das (2003) are optimal for the estimation of σ_g^2/σ_e^2. Thus we have the following results on optimal designs for estimation of σ_g^2/σ_e^2.

(i) Complete diallel cross designs in $\mathcal{D}(p,n)$ are optimal.

(ii) The existence of a nested balanced incomplete block (NBIB) design d with parameters $v = p, b_1 = b, b_2 = bk, k_1 = 2k, k_2 = 2$ yields an optimal incomplete block design d^* for diallel crosses. The construction methods and elaborate tables of NBIB designs are available in a review paper by Morgan et. al. (2001). The tables in their paper provide solutions to our optimal diallel cross designs within the parametric range $2k < p < 16$, $bk \leq 15p$. The case $2k = p$ is dealt in Gupta and Kageyama (1994). The NBIB designs have been extended to nested balanced block designs and a series of designs, optimal under our setup, is given in Das et al. (1998).

(iii) Das et al. (1998) gave two general methods of construction of block designs for partial diallel crosses. Their designs belong to $\mathcal{D}_0(p,b,k)$ with $2k/p$ an integer. The designs are optimal in $\mathcal{D}_0(p,b,k)$.

(iv) For unblocked designs, partial diallel cross designs in which every line appears $2n/p$ times and each cross appears either $\lambda = [2n/\{p(p-1)\}]$ or $\lambda + 1$ times are optimal. A common way to construct such a design is to form crosses between the two treatments in each block of a conventional binary incomplete block design with p treatments each occurring $2n/p$ times,

n distinct blocks of size 2 each and treatment concurrences λ and $\lambda+1$. This includes the M-designs of Singh and Hinkelmann (1995), the first series of designs of Mukerjee (1997), and some designs listed in Das et al. (1998).

(v) As a result of the very nature of the derived objective function under the random effects model that is being minimized, every previously known MS-optimal design under the fixed effects model (Das et al., 1998) would also be optimal under our set-up.

Optimal partial diallel crosses listed above necessarily have orthogonal blocks. A diallel cross design is said to be orthogonally blocked if each line occurs in every block r/b times where r is the constant replication number of the lines in the design; see Gupta et. al.(1995). In general, for non-orthogonal block designs, Mukerjee (1997) has provided some methods for constructing efficient partial diallel crosses. Let $p = n_1 n_2$ where $n_1 \geq 2, n_2 \geq 3$. Partition the set $\{1, 2, \ldots, p\}$ into n_1 mutually exclusive and exhaustive subsets $\{S_1, S_2, \ldots, S_{n_1}\}$ each of cardinality n_2. Let

$$d_0^{**} = \{(i, j) : 1 \leq i < j \leq p \text{ and } i, j \in S_u \text{ for some } u\}. \tag{3.1}$$

In the construction of the block design d^{**} with p $(= n_1 n_2)$ lines, the lines are denoted by a_j^u, $1 \leq u \leq n_1$, $0 \leq j \leq n_2 - 1$. In 3.1 take $S_u = \{a_0^u, a_1^u, \ldots, a_{n_2-1}^u\}$. Then from Equation (3.1) d_0^{**} consists of $n_1 n_2(n_2-1)/2$ crosses where in a cross the two lines are from the same S_u. For n_2 (≥ 5) odd, Mukerjee's general approach for grouping the crosses in d_0^{**} into blocks is now given.

Let M_1 be the incidence matrix of a general block design d_1 involving n_1 treatments and n_2 blocks such that each block has size n_1 and each treatment is replicated n_2 times. For $1 \leq u \leq n_1$, $0 \leq l \leq n_2 - 1$, in the l-th occurence of treatment u in d_1, replace treatment u by the $(n_2-1)/2$ crosses $\{(a_{j+l}^u, a_{n_2-j+l}^u) : 1 \leq j \leq (n_2-1)/2\}$, where $j+l$ and $n_2 - j + l$ are reduced mod n_2. The resulting block design, d^{**}, for diallel crosses belongs to $\mathcal{D}(n_1 n_2, n_2, n_1(n_2-1)/2)$ and represents a partitioning of the crosses in d_0^{**} into blocks.

EXAMPLE. Let $n_1 = 3, n_2 = 5$. Then $p = 15, b = 5, k = 6$ and the design with rows as blocks is:

$$[(a_1^1, a_4^1), (a_2^1, a_3^1), (a_1^2, a_4^2), (a_2^2, a_3^2), (a_1^3, a_4^3), (a_2^3, a_3^3)]$$

$$[(a_2^1, a_0^1), (a_3^1, a_4^1), (a_2^2, a_0^2), (a_3^2, a_4^2), (a_2^3, a_0^3), (a_3^3, a_4^3)]$$

$$[(a_3^1, a_1^1), (a_4^1, a_0^1), (a_3^2, a_1^2), (a_4^2, a_0^2), (a_3^3, a_1^3), (a_4^3, a_0^3)]$$

$$[(a_4^1, a_2^1), (a_0^1, a_1^1), (a_4^2, a_2^2), (a_0^2, a_1^2), (a_4^3, a_2^3), (a_0^3, a_1^3)]$$

$$[(a_0^1, a_3^1), (a_1^1, a_2^1), (a_0^2, a_3^2), (a_1^2, a_2^2), (a_0^3, a_3^3), (a_1^3, a_2^3)]$$

.

Let, for a block design d, $\lambda_{d1} \leq \lambda_{d2} \leq \cdots \leq \lambda_{d(p-1)}$ be the non-zero eigenvalues of the information matrix C_d. Then, after some algebra, we find that for $i = 1, \ldots, n_2 - 1$, $\lambda_{d^{**}i} = n_2 - 2 - \frac{2}{n_2 - 1}$, for $i = n_2, \ldots, p - n_1$, $\lambda_{d^{**}i} = n_2 - 2$ and for $i = p - n_1 + 1, \ldots, p - 1$, $\lambda_{d^{**}i} = 2(n_2 - 1)$ with $d^{**} \in \mathcal{D}(p, n_2, \frac{n_1(n_2-1)}{2})$.

We now give lower bounds to efficiency of non-orthogonal block designs d^{**} for estimating ratio of variance components. For $p = n_1n_2, b = n_2, k = n_1(n_2-1)/2$, we first obtain the lower bounds to efficiency with respect to the competing classes $\mathcal{D}_0(p, b, k)$. Note that for a block design $d \in \mathcal{D}_0(p, b, k)$ for diallel crosses, $\mathrm{tr}(C_d) = 2b(k-1)$. Mukerjee (1997) has shown that $\lambda_{d_01} \leq n_2 - 2$ where λ_{d_01} is the minimum nonzero eigenvalue of C_{0d_0} with $d_0 \in \mathcal{D}(n_1n_2, n_1n_2(n_2-1)/2)$. Thus, it follows that, $\lambda_{d1} \leq \lambda_{d_01} \leq n_2 - 2$ where $d \in \mathcal{D}(n_1n_2, n_2, n_1(n_2-1)/2)$ and d_0 is the design ignoring the block classification of the design d. Also, we know that for $d \in \mathcal{D}_0(p, b, k)$, $\mathrm{tr}(C_d^2)$ is minimum when $\lambda_{di} = 2b(k-1)/(p-1)$, $i = 1, \ldots, p-1$. However, $2b(k-1)/(p-1) = n_2 - 2 + \{n_2(n_1-1) - 2\}/(p-1)$. Thus, in our design setup, for block designs $d \in \mathcal{D}_0(p, b, k)$, with $n_1 > 1$, it follows that $\mathrm{tr}(C_d^2) = \Sigma_{i=1}^{p-1}\lambda_{di}^2 \geq (n_2-2)^2 + \{2b(k-1) - (n_2-2)\}^2/(p-2) = W_0$, say. Thus, for a design $d \in \mathcal{D}_0(p, b, k)$ with $n_1, n_2, \sigma_g^2, \sigma_e^2$ fixed, from 2.5, a lower bound to $V(T; d, \sigma_g^2, \sigma_e^2)$ is obtained by substituting W_0 and $2b(k-1)$ for $\mathrm{tr}(C_d^2)$ and $\mathrm{tr}(C_d)$, respectively. We denote this lower bound by $V_0^*(T; n_1, n_2, \sigma_g^2, \sigma_e^2)$.

Thus, from 2.5, for given n_1, n_2, σ_g^2 and σ_e^2, the A-efficiency of the design $d^{**} \in \mathcal{D}_0(p, b, k)$ is at least as large as $e_{A_0}(n_1, n_2, \sigma_g^2, \sigma_e^2)$ where

$$e_{A_0}(n_1, n_2, \sigma_g^2, \sigma_e^2) = \frac{V_0^*(T; n_1, n_2, \sigma_g^2, \sigma_e^2)}{V(T; d^{**}, \sigma_g^2, \sigma_e^2)} \tag{3.2}$$

The denominator is obtained by substituting the values of $\lambda_{d^{**}i}$, in $\mathrm{tr}(C_{d^{**}}^2)$. Also, $\mathrm{tr}(C_{d^{**}}) = 2b(k-1)$.

Now, for any $d \in \mathcal{D}(p, b, k)$, where p, b, k are arbitrary but fixed, it is easy to see that $\mathrm{tr}(C_d^2)/\mathrm{tr}^2(C_d) \geq 1/(p-1) = W$, say. Also, we know that $\mathrm{tr}(C_d)$ is bounded above by $k^{-1}b\{2k(k-1-2x) + px(x+1)\}$, where x is the largest integer not exceeding $2k/p$ (see Das et. al., 1998). Thus, for a design $d \in \mathcal{D}(p,b,k)$ with $p,b,k,\sigma_g^2,\sigma_e^2$ fixed, from 2.5, a lower bound to $V(T;d,\sigma_g^2,\sigma_e^2)$ is obtained by substituting W for $\frac{\mathrm{tr}(C_d^2)}{\mathrm{tr}^2(C_d)}$ and substituting $k^{-1}b\{2k(k-1-2x)+px(x+1)\}$ for $\mathrm{tr}(C_d)$. We denote this lower bound by $V^*(T;p,b,k,\sigma_g^2,\sigma_e^2)$.

Thus, from 2.5, for given p,b,k,σ_g^2 and σ_e^2, the A-efficiency of a design $d^\# \in \mathcal{D}(p,b,k)$ is at least as large as $e_A(p,b,k,\sigma_g^2,\sigma_e^2)$ where

$$e_A(p,b,k,\sigma_g^2,\sigma_e^2) = \frac{V^*(T;p,b,k,\sigma_g^2,\sigma_e^2)}{V(T;d^\#,\sigma_g^2,\sigma_e^2)} \tag{3.3}$$

As earlier, the denominator is obtained by substituting the actual values of $\mathrm{tr}(C_{d^\#})$ and $\mathrm{tr}(C_{d^\#}^2)$ for the design $d^\#$.

Under classes $\mathcal{D}_0(p,b,k)$ and $\mathcal{D}(p,b,k)$, using 3.2 and 3.3, we obtained the lower bounds to efficiency of d^{**} for $2 \le n_1 \le 15$ and $5 \le n_2 \le 15$ with n_2 odd. Here, each of σ_g^2 and σ_e^2 has been taken within the range of $(0.1, 3.0)$, with increments of 0.1. It is observed that efficiencies are greater than 0.90 in both cases when $2 \le n_1 \le 15$, $5 < n_2 \le 15$. Furthermore, the efficiencies are greater than 0.95 for 43.9% of the parametric sets with $2 \le n_1 \le 15$, $5 < n_2 \le 15$ and both σ_g^2, σ_e^2 in the range of $(0.1, 3.0)$. For the parametric range $2 \le n_1 \le 15$ and $n_2 = 5$, (i) efficiencies are more than 0.90 when $2 \le n_1 \le 4$, (ii) 94.2% of the designs with $5 \le n_1 \le 15$ have efficiencies greater than 0.9, (iii) the efficiencies are atleast 0.893. We observe that the efficiencies are generally robust against the values of variance components, and depends only on design parameters.

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