Matrix algebraic formulae concerning some exceptional rules of two-dimensional cellular automata

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Abstract

In the past, cellular automata based models and machines [The Theory of Self-Reproducting Automata, University of Illinois Press, Urbana, 1996; Rev. Mod. Phys. 55 (1983) 601; Am. Math. Month. 97 (1990) 24; Matrix and Linear Algebra, Prentice-Hall, India, 1991; TRE Trans. Circuits CT-6 (1959) 45; Cellular Automata Machines, MIT Press, Cambridge, 1987] were proposed for simulation of physical systems, but without any analytical insight into the behaviour of the underlying simulation process. The set of papers [Int. J. Comput. Math. Appl. 33 (1997) 79; Int. J. Comput. Math. Appl. 37 (1999) 115; Matrix algebraic formulae concerning a particular rule of two dimensional cellular automata, Inf. Sci., submitted] made a significant departure from this traditional approach. In the mentioned papers, a simple and precise mathematical model using matrix algebra built on GF(2) was reported for characterising the behaviour of two-dimensional nearest neighbourhood linear cellular automata with null and periodic boundary conditions. As a sequel, in the present paper an attempt has been made to characterise a number of exceptional transformations or rules, each of which behaving uniquely, not matching with any other rules. Thus this set of exceptional rules demand special attention.

Keywords: Two-dimensional cellular automata; Linear algebra; Periodic boundary condition; Null boundary condition

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1. Introduction

The cellular automata (CA) studies the mathematical formalization of discrete dynamical system whose structure is a regular uniform array of a particular dimension. The two-dimensional CA in its simplest form consists of $m \times n$ cells arranged in m rows and n columns, where each cell takes one of the values of 0 or 1. A configuration of the system is an assignment of states to all the cells. Every configuration determines a next configuration via a linear transition rule that is local in the sense that the state of a cell at time $t+1$ depends only on the states of some of its neighbours at time t using modulo 2. For 2D CA nearest neighbours, there are nine cells arranged in a 3×3 matrix centering that particular cell. Mathematically, the $(t+1)$th state of the (i,j)th cell can be written as

$$a_{i,j}(t+1) = f(a_{i-1,j-1}(t), a_{i-1,j}(t), a_{i-1,j+1}(t), a_{i,j-1}(t), a_{i,j}(t), a_{i,j+1}(t), a_{i+1,j-1}(t), a_{i+1,j}(t), a_{i+1,j+1}(t))$$

The rule according to which the content of each cell is altered is called the rule number of that particular CA. If same rule is applied in case of all the cells of m rows and n columns of one 2D CA configuration, then the CA is called uniform or regular. Otherwise it is called hybrid. Our discussion is restricted to uniform type CA. Regarding the neighbourhood of the first and last cell there are two approaches. If the first and last cells are considered to be adjacent, then it is called periodic boundary condition (P). But if the first and last cells are connected to '0' state, this condition is called Null Boundary condition (N). If in a CA the neighbourhood dependence is on EX-OR, then the CA is called an additive or linear CA. Regarding the various nomenclatures of transformations we follow the already existing ones [7–9]. If a next state of a cell is described in the form of a truth table, then the decimal equivalent of the output is called the rule number. The conventional method of defining a rule number for a linear rule in 2D CA can be explained in the following way:

64	128	256
32	1	2
16	8	4

Here the current cell is represented by the centremost box and the other boxes are 8 neighbourhoods of that cell. If the next value of the current cell is dependent on a particular cell (or cells), then the value of that cell (or cells) will be filled up by 1 and the others will be 0. The rule number will be the decimal equivalent of the output as explained in the above table. There may be $2^9 = 512$ (rule number 0 to rule number 511) rules all of which are linear.

Let us try to form the clusters of different rule numbers in the sense of similar looking behaviour. It has been done successively by taking 1 neighbour, 2 neighbours, ..., 9 neighbours at a time. It is important to note that taking 4 neighbours at a time, the rule numbers 170N and 170P and 340N and 340P, taking 5 neighbours at a time 171N and 171P and 341N and 341P, taking 8 neighbours at a time 510N and 510P and taking 9 neighbours at a time 511N and 511P are showing exceptional special behaviours that do not match with any other rule numbers. In this paper an attempt has been made to explain and characterise the odd nature of these exceptional rules. In Section 2, mathematical formulation and results are highlighted. Section 3 contains the concluding remarks.

2. Mathematical formulation and results

2.1. Characterization of rule numbers 170N and 170P

In general for any m and n the map matrix $(T_{170\mathrm{N}})_{mn\times mn}$ is of the form:

$$\begin{pmatrix} S & I & 0 & 0 & 0 & \cdots & \cdots & 0 & 0 \\ I & S & I & 0 & 0 & \cdots & \cdots & 0 & 0 \\ 0 & I & S & I & 0 & \cdots & \cdots & 0 & 0 \\ 0 & 0 & I & S & I & 0 & \cdots & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & \cdots & \cdots & 0 & I & S & I \\ 0 & 0 & 0 & \cdots & \cdots & \cdots & 0 & I & S \end{pmatrix}$$

where each partitioned matrix is of order $n \times n$. And

$$S_{n\times n} = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & \cdots & \cdots & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & \cdots & \cdots & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & \cdots & \cdots & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & \cdots & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & \cdots & \cdots & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & \cdots & \cdots & \cdots & 0 & 1 & 0 \end{pmatrix}.$$

In general for any m and n the map matrix $(T_{170\mathrm{P}})_{mn\times mn}$ is of the form

$$\begin{pmatrix} S_c & I & 0 & 0 & 0 & \cdots & \cdots & 0 & I \\ I & S_c & I & 0 & 0 & \cdots & \cdots & 0 & 0 \\ 0 & I & S_c & I & 0 & \cdots & \cdots & 0 & 0 \\ 0 & 0 & I & S_c & I & 0 & \cdots & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & \cdots & \cdots & 0 & I & S_c & I \\ I & 0 & 0 & \cdots & \cdots & \cdots & 0 & I & S_c \end{pmatrix}$$

where each partitioned matrix is of order $n \times n$. And

$$(S_c)_{n\times n} = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & \cdots & \cdots & 0 & 1 \\ 1 & 0 & 1 & 0 & 0 & \cdots & \cdots & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & \cdots & \cdots & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & \cdots & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & \cdots & \cdots & 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & \cdots & \cdots & \cdots & 0 & 1 & 0 \end{pmatrix}.$$

Dimension of kernel of these two rule matrices have been studied in the earlier papers [7,9] whose values are equal to be $\mathrm{GCD}(m+1, n+1) - 1$ and $2\mathrm{GCD}(m, n) - (m \times n \bmod 2)$, respectively.

2.2. Characterization of rule numbers 171N and 171P

The rule matrices of these two rule numbers are exactly same as those for rule numbers 170N and 170P, respectively, except only there are $S + I$ and $S_c + I$ in places of S and S_c. So proceeding as in the same way for 170N and for 170P [7,9] we have two results:

Theorem 1
(a) *The dimension of kernel of* $T_{171\mathrm{N}} = \dim$ *of* $\ker(p_m(S + I))$.
(b) *The dimension of kernel of* $T_{171\mathrm{P}}$

$$= \begin{cases} \ker(S_c + I) & \text{if } m = 1 \\ 2\ker((S_c + I)p_{m/2-1}(S_c + I)) & \text{if } m = even \\ \ker(p_{(m-1)/2}(S_c + I) + p_{(m-3)/2}(S_c + I)) & \\ \quad + \ker((S_c + I) \cdot (p_{(m-1)/2}(S_c + I) + p_{(m-3)/2}(S_c + I))) & \text{if } m = odd \end{cases}$$

where the polynomial $p_i(X)$ *is defined as* $p_{-1}(X) = 0$, $p_0(X) = I$, $p_i(X) = X \cdot p_{i-1}(X) + p_{i-2}(X)$, $i \geqslant 1$.

For the proof, the reader is referred to [7,9].

Let us now recall some well-known results [7] of this polynomial definition such as the following:

Lemma 1(a). $p_i(X) = p_{i-j}(X) \cdot p_j(X) + p_{i-j-1}(X) \cdot p_{j-1}(X)$, $j \leqslant i$.

By putting $2i + 1$ in place of i and $i + 1$ in place of j in the above formula we can get a useful result such as

Lemma 1(b). $p_{2i+1}(X) = X \cdot (p_i(X))^2$.

And by putting $2i$ in place of i and i in place of j, we can also have an another formula such as

Lemma 1(c). $p_{2i}(X) = (p_i(X) + p_{i-1}(X))^2$.

At this stage we have the following theorem.

Theorem 2. *If $m = 2^k \cdot q - 1$, where k is any +ve integer and q = any +ve odd integer, then $p_m(X) = X^{2^{k-1}-1} \cdot p_{2q-1}^{2^{k-1}}(X)$.*

Proof

$$\begin{aligned} p_m(X) &= p_{2^k \cdot q-1}(X) = p_{2(2^{k-1} \cdot q-1)+1}(X) = X \cdot p^2_{2^{k-1} \cdot q-1}(X) \\ &= X \cdot (X \cdot p^2_{2^{k-2} \cdot q-1}(X))^2 = X \cdot X^2 \cdot p^4_{2^{k-2} \cdot q-1}(X) \\ &= X \cdot X^2 \cdot X^4 \cdot p^8_{2^{k-3} \cdot q-1}(X) = \cdots = X^{1+2+4+8+\cdots+2^{k-2}} \cdot p_{2q-1}^{2^{k-1}}(X) \\ &= X^{2^{k-1}} \cdot X^{-1} \cdot p_{2q-1}^{2^{k-1}}(X) \end{aligned}$$

Hence the proof. □

Corollary 1. *Let us put $q = 1$. In this case $m = 2^k - 1$, then $p_m(X) = X^{2^{k-1}-1} \cdot p_1^{2^{k-1}}(X) = X^{2^{k-1}-1} \cdot X^{2^{k-1}} = X^{2^k-1} = X^m$.*

Corollary 2. *Let us consider that $m = 2^k \cdot q - 1$, where q = any +ve odd number other than 1, let $q = 2t + 1$, $t = 1, 2, 3, \ldots$ In this case $p_m(X) = X^{2^{k-1}-1} \cdot p_{2q-1}^{2^{k-1}}(X) = X^{2^{k-1}-1} \cdot p_{4t+1}^{2^{k-1}}(X) = X^{2^{k-1}-1} \cdot p_{2 \cdot 2t+1}^{2^{k-1}}(X) = X^{2^{k-1}-1} \cdot (X \cdot p^2_{2t}(X))^{2^{k-1}} = X^{2^{k-1}-1} \cdot X^{2^{k-1}} \cdot p_{2t}^{2^k}(X) = X^{2^k-1} \cdot p_{2t}^{2^k}(X)$.*

To proceed further let us study the rank of $S + I$ and $S_c + I$.

Lemma 2. rank$(S + I)$ = *full if order $n = 3t$ or $3t + 1$ and is $n - 1$ if $n = 3t + 2$, $t = 0, 1, 2, 3, \ldots$*

Proof. The characteristic polynomial of $S_{n\times n}$ in λ can be written as [7] $p_n(\lambda) = |S + \lambda \cdot I| = \lambda \cdot p_{n-1}(\lambda) + p_{n-2}(\lambda)$.

$$|S + I + \lambda \cdot I| = |S + (\lambda + 1) \cdot I| = (\lambda + 1) \cdot p_{n-1}(\lambda + 1) + p_{n-2}(\lambda + 1).$$

Putting $\lambda = 0$ in the second equation

$$\begin{aligned} p_n(1) = |S + I| &= p_{n-1}(1) + p_{n-2}(1) = p_{n-2}(1) + p_{n-3}(1) + p_{n-2}(1) \\ &= p_{n-3}(1) = p_{n-6}(1) = p_{n-9}(1) \\ &= \cdots = \begin{cases} p_0(1) & \text{if } n = 3t, \\ p_1(1) & \text{if } n = 3t + 1, \\ p_2(1) & \text{if } n = 3t + 2,\ t = 0, 1, 2, 3, \ldots \end{cases} \end{aligned}$$

Now, since $p_{-1}(\lambda) = 0$, $p_0(\lambda) = 1$, $p_1(\lambda) = \lambda$, $p_2(\lambda) = \lambda^2 + 1$, therefore, $p_0(1) = 1$, $p_1(1) = 1$, $p_2(1) = 0$.

So, $|S + I| = 1$ if $n = 3t$ and $n = 3t + 1$, i.e. $\text{rank}(S + I) = \text{full}$. But, $|S + I| = 0$, if $n = 3t + 2$. A minor of $|S + I|$ of order $n \times n$ is $|S + I|$ of order $(n - 1) \times (n - 1)$. Since $n - 1 = 3t + 1$, therefore $|S + I|_{n-1} = 1$ and so $\text{rank}((S + I)_{n\times n}) = n - 1$, if $n = 3t + 2$. □

Lemma 3. $\text{rank}(S_c + I) =$ *full if order* $n = 3t + 1$ *or* $3t + 2$ *and is* $n - 2$ *if* $n = 3t$, $t = 0, 1, 2, 3, \ldots$

Proof. $\ker((S_c + I)_{n\times n}) = \ker(P + P^T + I) = \text{GCD}(3, n) - 1$, where P is the permutation matrix [9].

Therefore $\ker((S_c + I)_{n\times n}) = 0$ iff $\text{GCD}(3, n) = 1$, i.e. if $n = 3t + 1$ or $3t + 2$, $t = 0, 1, 2, 3, \ldots$ Else if $n = 3t$, $t = 1, 2, 3, \ldots, \ker((S_c + I)_{n\times n}) = \text{GCD}(3, 3t) - 1 = 3 - 1 = 2$.

So, we see that $(S + I)_{n\times n}$ is of full rank if $n = 3t$ or $3t + 1$. but $(S_c + I)_{n\times n}$ is of full rank if $n = 3t + 1$ or $3t + 2$. □

Corollary 3. *It follows from the results of Theorem* 1(*a*) *and Corollary* 1 *that if* $m = 2^k - 1$, *where* $k = 1, 2, 3, \ldots$, *and* $n = 3t$ *or* $3t + 1$, $t = 1, 2, 3, \ldots$ *i.e. whenever* $S_c + I$ *matrix is of full rank, in these cases,* $\ker(T_{171N}) = 0$.

Corollary 4. *It follows from the results of Theorem* 1(*b*) *and Corollary* 1 *that if* $m = 2^k$, *where* $k = 1, 2, 3, \ldots$, *and* $n = 3t + 1$ *or* $3t + 2$, $t = 0, 1, 2, 3, \ldots$ *i.e. whenever* $S_c + I$ *matrix is of full rank, in these cases,* $\ker(T_{171P}) = 0$.

Let us define two new matrices assuming the invertibility of $(S + I)_{n\times n}$ and $(S_c + I)_{n\times n}$. Let $T = S(S + I)^{-1}$ for $n = 3t$ or $3t + 1$, so that $\text{rank}(T) = \text{rank}(S)$. And $T_c = S_c(S_c + I)^{-1}$ for $n = 3t + 1$ or $3t + 2$, so that $\text{rank}(T_c) = \text{rank}(S_c)$.

2.3. Characterization of rule numbers 510N and 510P

In general for any m and n the map matrix $(T_{510\mathrm{N}})_{mn\times mn}$ is of the form

$$\begin{pmatrix} S & S+I & 0 & 0 & 0 & \cdots & \cdots & 0 & 0 \\ S+I & S & S+I & 0 & 0 & \cdots & \cdots & 0 & 0 \\ 0 & S+I & S & S+I & 0 & \cdots & \cdots & 0 & 0 \\ 0 & 0 & S+I & S & S+I & 0 & \cdots & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & \cdots & \cdots & 0 & S+I & S & S+I \\ 0 & 0 & 0 & \cdots & \cdots & \cdots & 0 & S+I & S \end{pmatrix}.$$

If we assume the value of $n=3t$ or $3t+1$ the its dimension of kernel is equal to that of the matrix

$$\begin{pmatrix} T & I & 0 & 0 & 0 & \cdots & \cdots & 0 & 0 \\ I & T & I & 0 & 0 & \cdots & \cdots & 0 & 0 \\ 0 & I & T & I & 0 & \cdots & \cdots & 0 & 0 \\ 0 & 0 & I & T & I & 0 & \cdots & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & \cdots & \cdots & 0 & I & T & I \\ 0 & 0 & 0 & \cdots & \cdots & \cdots & 0 & I & T \end{pmatrix}.$$

Proceeding analogously as that of $(T_{170\mathrm{N}})_{mn\times mn}$ we see the next result

Theorem 3. $\ker((T_{510\mathrm{N}})_{mn\times mn}) = \ker((p_m(T))_{n\times n})$ for $n=3t$ or $3t+1$, $t=0,1,2,3,\ldots$

Corollary 5. *For even values of* n, *since* $\mathrm{rank}(S)=\mathrm{full}$, *using the results of Theorem* 2 *and Corollary* 1, *we can write that for* $m=2^k-1$, $k=1,2,3,\ldots$, $\mathrm{rank}(T_{510\mathrm{N}})=$ *full, i.e. kernel = zero.*

In general for any m and n the map matrix $(T_{510\mathrm{P}})_{mn\times mn}$ is of the form

$$\begin{pmatrix} S_c & S_c+I & 0 & 0 & 0 & \cdots & \cdots & 0 & S_c+I \\ S_c+I & S_c & S_c+I & 0 & 0 & \cdots & \cdots & 0 & 0 \\ 0 & S_c+I & S_c & S_c+I & 0 & \cdots & \cdots & 0 & 0 \\ 0 & 0 & S_c+I & S_c & S_c+I & 0 & \cdots & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & \cdots & \cdots & 0 & S_c+I & S_c & S_c+I \\ S_c+I & 0 & 0 & \cdots & \cdots & \cdots & 0 & S_c+I & S_c \end{pmatrix}.$$

If we assume the value of $n = 3t + 1$ or $3t + 2$ the its dimension of kernel is equal to that of the matrix

$$\begin{pmatrix} T_c & I & 0 & 0 & 0 & \cdots & \cdots & 0 & I \\ I & T_c & I & 0 & 0 & \cdots & \cdots & 0 & 0 \\ 0 & I & T_c & I & 0 & \cdots & \cdots & 0 & 0 \\ 0 & 0 & I & T_c & I & 0 & \cdots & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & \cdots & \cdots & 0 & I & T_c & I \\ I & 0 & 0 & \cdots & \cdots & \cdots & 0 & I & T_c \end{pmatrix}.$$

Proceeding analogously as that of $(T_{170P})_{mn \times mn}$ we see the next result under the above stated restricted values of n

Theorem 4

$$\ker((T_{510P})_{mn \times mn}) = \begin{cases} \ker(T_c) & \textit{if } m = 1 \\ 2\ker((T_c)p_{m/2-1}(T_c)) & \textit{if } m = \textit{even} \\ \ker(p_{(m-1)/2}(T_c) + p_{(m-3)/2}(T_c)) & \\ \quad + \ker((T_c) \cdot (p_{(m-1)/2}(T_c) + p_{(m-3)/2}(T_c))) & \textit{if } m = \textit{odd} \end{cases}$$

So using the results of Theorem 2 and Corollary 1 for $m = 2^k$, where $k =$ any positive integer, $\ker(T_{510P}) = 2\ker((T_c)p_{m/2-1}(T_c)) = 2\ker(T_c^{2^{k-2}} \cdot p_1^{2^{k-2}}(T_c) = 2\ker(T_c^{2^{k-2}} \cdot T_c^{2^{k-2}}) = 2\ker(T_c^{2^{k-1}}) = 2\ker(S_c^{2^{k-1}}) = 2\ker((P + P^{\mathrm{T}})^{2^{k-1}}) = 2\ker(P^{2^{k-1}} + P^{\mathrm{T}^{2^{k-1}}}) = 2\mathrm{GCD}(2 \cdot 2^{k-1}, n)$ [9]. $= 2\mathrm{GCD}(2^k, n) = 2\mathrm{GCD}(m, n)$.

Corollary 6. *Let us consider that $m = 2^k \cdot q$, where $q =$ any +ve odd number other than* 1, *let $q = 2t + 1$, $t = 1, 2, 3, \ldots$. In this case using the result of Corollary* 2

$$T_c \cdot p_{m/2-1}(T_c) = T_c \cdot p_{2^{k-1} \cdot q-1}(T_c) = T_c^{2^{k-1}} \cdot p_{2t}^{2^{k-1}}(T_c).$$

We have not been able to arrive at a neat formula which gives a closed form expression for the required kernel and further research in this area may throw more light into the problem. However, given some particular values of t, from the above formulation, it is quite easy to come out with the values of the required kernel.

For example, if we assume that $t = 1$, i.e. $q = 3$, i.e. the values of $m = 2^k.3$, then in this case $\ker(T_{510P}) = 2\mathrm{GCD}(m/3, n)$.

2.4. Characterization of rule numbers 340N, 340P, 511N, 511P

Theorem 5. *Using the usual notation of Kronecker product of two matrices* $\operatorname{rank}(A \otimes B) = \operatorname{rank}(A) \cdot \operatorname{rank}(B)$.

Proof. Let us assume that the basis of the null space of $A \otimes B$ to be constructed of the vectors of the form $X \otimes Y$ i.e. $(A_{m\times n} \otimes B_{n\times n})(X \otimes Y) = 0$. i.e. $(AX) \otimes (BY) = 0$.

It implies and is implied by that at least one of the two equations $AX = 0$ and $BY = 0$ must hold. Let $S_1 m \times m$ and $S_2 n \times n$ be the null spaces of A and B, respectively, and p and q are their nullities. The vector space R^m and R^n can be written as $R^m = S_1 \cup S_1^{\perp}$ and $R^n = S_2 \cup S_2^{\perp}$. Then the null space of $A \otimes B$ is $(S_1 \otimes R^n) \cup (R^m \otimes S_2) - S_1 \otimes S_2$. So its dimension is $= p \times n + m \times q - p \times q$. Therefore $\operatorname{rank}(A \otimes B) = mn - np - mq + pq = (m-p)(n-q) = \operatorname{rank}(A) \cdot \operatorname{rank}(B)$. □

Applying the above result we can have some more formulas of kernels of the following rule numbers.

Theorem 6

$$\operatorname{rank}(T_{340\mathrm{N}}) = \operatorname{rank}(S_m \otimes S_n) = \operatorname{rank}(S_m) \cdot \operatorname{rank}(S_n).$$
$$\operatorname{rank}(T_{340\mathrm{P}}) = \operatorname{rank}(S_{cm} \otimes S_{cn}) = \operatorname{rank}(S_{cm}) \cdot \operatorname{rank}(S_{cn}).$$
$$\operatorname{rank}(T_{511\mathrm{N}}) = \operatorname{rank}((S_m + I_m) \otimes (S_n + I_n)) = \operatorname{rank}((S_m + I_m)) \cdot \operatorname{rank}((S_n + I_n)).$$
$$\operatorname{rank}(T_{511\mathrm{P}}) = \operatorname{rank}((S_{cm} + I_m) \otimes (S_{cn} + I_n)) = \operatorname{rank}((S_{cm} + I_m)) \cdot \operatorname{rank}((S_{cn} + I_n)).$$

2.5. Characterization of rule numbers 341N and 341P

In general for any m and n the map matrix $(T_{341\mathrm{N}})_{mn\times mn}$ is of the form

$$\begin{pmatrix} I & S & 0 & 0 & 0 & \cdots & \cdots & 0 & 0 \\ S & I & S & 0 & 0 & \cdots & \cdots & 0 & 0 \\ 0 & S & I & S & 0 & \cdots & \cdots & 0 & 0 \\ 0 & 0 & S & I & S & 0 & \cdots & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & \cdots & \cdots & 0 & S & I & S \\ 0 & 0 & 0 & \cdots & \cdots & \cdots & 0 & S & I \end{pmatrix}.$$

We know that the matrix S is of full rank if the order is even, otherwise its kernel $= 1$. So for even values of n, the kernel of $T_{341\mathrm{N}}$ is equal to that of the following matrix:

$$\begin{pmatrix} S^{-1} & I & 0 & 0 & 0 & \cdots & \cdots & 0 & 0 \\ I & S^{-1} & I & 0 & 0 & \cdots & \cdots & 0 & 0 \\ 0 & I & S^{-1} & I & 0 & \cdots & \cdots & 0 & 0 \\ 0 & 0 & I & S^{-1} & I & 0 & \cdots & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & \cdots & \cdots & 0 & I & S^{-1} & I \\ 0 & 0 & 0 & \cdots & \cdots & \cdots & 0 & I & S^{-1} \end{pmatrix}.$$

Following as in the same procedure of $T_{170\mathrm{N}}$ its kernel $= \ker(p_m((S_n)^{-1}))$.

Corollary 7. *Again using the results of Theorem* 4 *and Corollary* 1, *we can write that if* $m = 2^k - 1$, $k = 1, 2, 3, \ldots$

$$p_m((S_n)^{-1}) = ((S_n)^{-1})^{2^k-1}$$

Since $(S_n)^{-1}$ *is of full rank, so* $\ker(p_m((S_n)^{-1})) = 0$, *for* $m = 2^k - 1, k = 1, 2, 3, \ldots$

In general for any m and n the map matrix $(T_{341\mathrm{P}})_{mn\times mn}$ is of the form

$$\begin{pmatrix} I & S_c & 0 & 0 & 0 & \cdots & \cdots & 0 & S_c \\ S_c & I & S_c & 0 & 0 & \cdots & \cdots & 0 & 0 \\ 0 & S_c & I & S_c & 0 & \cdots & \cdots & 0 & 0 \\ 0 & 0 & S_c & I & S_c & 0 & \cdots & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & \cdots & \cdots & 0 & S_c & I & S_c \\ S_c & 0 & 0 & \cdots & \cdots & \cdots & 0 & S_c & I \end{pmatrix}.$$

We have not been able to analyse this matrix. Experiments are being continued to solve the nature of this matrix. So far computer output shows the results of zero kernel for the values of m equal to any integral powers of 2. Hence before closing this section we include the following conjecture.

Conjecture. *When* $m =$ *integral powers of two, the dimension of kernel of* $T_{341\mathrm{P}}$ *is zero for all* n.

3. Concluding remarks

Of the six exceptional rules viz. 170, 171, 340, 341, 510, 511 both for null and periodic boundary conditions, rule 170 has been completely solved [3,7,9]. In the present paper, the rules 340 and 511 have been dealt with satisfactorily giving complete solutions regarding the dimensionality of the kernel. Also the results for 171N, 171P, 510N and 510P considered above are essentially incomplete in the sense that only with some specific set of values of m and n we can come out with results. The main difficulties in arriving at a complete solution for all of them seems to be in sufficiently advancing the results of $p_m(S+I)$ and $p_m(S_c+I)$ for T_{171N} and T_{171P}, respectively, and also the results of Theorems 3 and 4 in general for T_{510N} and T_{510P}, respectively. Our next effort is being concentrated along this direction as well as the complete solution of 341N and 341P.

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