ON A SIMPLIFIED METHOD OF EXPRESSING THE COMPONENTS OF THE SECOND ORDER INTERACTION IN A 3^3 FACTORIAL DESIGN

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In a $3 \times 3 \times 3$ factorial design, it is well-known that the eight degrees of freedom for the second order interaction are given by the four contrasts, each yielding two degrees of freedom and being between the three groups of treatments in each of the following four cyclically generated sets:

Table 1. The Four Cyclically Generated Sets.

I			II			III			IV		
1	2	3	1	3	2	1	2	3	1	3	2
2	3	1	3	2	1	3	1	2	2	1	3
3	1	2	2	1	3	2	3	1	3	2	1
2	3	1	2	1	3	2	3	1	2	1	3
3	1	2	1	3	2	1	2	3	3	2	1
1	2	3	3	2	1	3	1	2	1	3	2
3	1	2	3	2	1	3	1	2	3	2	1
1	2	3	2	1	3	2	3	1	1	3	2
2	3	1	1	3	2	1	2	3	2	1	3

Of the other infinite number of ways of partitioning the degrees of freedom for the second order interaction, the one of most significance in fertilizer trials is that of splitting these up into unitary elements corresponding to the interaction of the three linear responses, etc., obtained when the two degrees of freedom for each of the three main effects have been split up into two single degrees of freedom, corresponding to the linear response and the curvature of the response curve. These eight components are, therefore, $N_1K_1P_1$, $N_1K_1P_2$, $N_1K_2P_1$, $N_1K_2P_2$, $N_2K_1P_1$, $N_2K_1P_2$, $N_2K_2P_1$, and $N_2K_2P_2$. It is of some interest and importance to obtain the appropriate compounds for each of these interactions in terms of the four cyclic sets given above, for that would considerably facilitate the computation of one or more of these components in a partially confounded (balanced or otherwise) $3 \times 3 \times 3$ factorial arrangement.

Let the treatments in the design be all combinations of three levels of nitrogen, three of potash and three of superphosphate. Fisher[1] has given the suitable compound for the linear component $N_1\ K_1\ P_1$, but has not given them for the remaining seven, nor has he given a general method of obtaining them. It is the purpose of the present note to formulate a general rule for obtaining the requisite expressions in the various cases and giving these in each case.

Considering any one of the components, say $N_2 K_2 P_2$, we have

$$N_2 K_2 P_2 = (n_3 - 2n_2 + n_1)(k_3 - 2k_2 + k_1)(p_3 - 2p_2 + p_1).$$

The expressions for the quadratic responses within each bracket on the right-hand side should next be broken up into the difference between the excess of the third level over the second and that of the second level over the first. We then, have, on expansion,

$$\begin{aligned} N_2 K_2 P_2 &= [(n_3 - n_2) - (n_2 - n_1)]\,[(k_3 - k_2) - (k_2 - k_1)]\,[(p_3 - p_2) - (p_2 - p_1)] \\ &= (n_3 - n_2)(k_3 - k_2)(p_3 - p_2) - (n_3 - n_2)(k_3 - k_2)(p_2 - p_1) \\ &\quad - (n_3 - n_2)(k_2 - k_1)(p_3 - p_2) + (n_3 - n_2)(k_2 - k_1)(p_2 - p_1) \\ &\quad - (n_2 - n_1)(k_3 - k_2)(p_3 - p_2) + (n_2 - n_1)(k_3 - k_2)(p_2 - p_1) \\ &\quad + (n_2 - n_1)(k_2 - k_1)(p_3 - p_2) - (n_2 - n_1)(k_2 - k_1)(p_2 - p_1). \end{aligned}$$

Now, each of these expressions or any other with eight treatment combinations obtained in this manner, four of which are with a positive and the other four with a negative sign, can be expressed as a compound of eight of the Latin Squares suitably chosen from the aggregate of twelve, two being taken from each cyclic set. The rule is straightforward and general. Mark out on the 3×3 Latin Square (see Fig. 1) of which the rows represent the three levels of nitrogen and columns the three levesls of potash, the four points which give the four combinations of nitrogen and potash of the given expression. Then select out those Latin Squares from the aggregate of twelve of which the two levels of superphosphate in the given expression constitute either of the two cross-joins of the above four points. For instance, let us obtain the required compound for

$$(n_3 - n_2)(k_3 - k_2)(p_3 - p_2).$$

The four points have been marked in Fig. 1 by crosses:

	k_1	k_2	k_3
n_1			
n_2		×	×
n_3		×	×

FIG. 1.

Then the following eight squares have the numbers 2 or 3 (levels of superphosphate in the expression) at the two cross-joins:

TABLE 2. THE EIGHT SQUARES.

2	3	1	3	1	2	2	1	3	3	2	1
3	1	2	1	2	3	1	3	2	2	1	3
1	2	3	2	3	1	3	2	1	1	3	2
2	3	1	3	1	2	2	1	3	3	2	1
1	2	3	2	3	1	3	2	1	1	3	2
3	1	2	1	2	3	1	3	2	2	1	3

Knowing the signs of the eight treatment combinations in the above expression, we immediately obtain, by comparing coefficients of the treatments, the required signs of the Latin Squares in order as + − + − − + − +.

Each of the 8 treatment combinations is seen to be repeated thrice and the treatment combinations other than these eight get cancelled out. Hence we have

$$3\,(n_3 - n_2)\,(k_3 - k_2)\,(p_1 - p_3)$$

$$= \begin{vmatrix} 2 & 3 & 1 \\ 3 & 1 & 2 \\ 1 & 2 & 3 \end{vmatrix} - \begin{vmatrix} 3 & 1 & 2 \\ 1 & 2 & 3 \\ 2 & 3 & 1 \end{vmatrix} + \begin{vmatrix} 2 & 1 & 3 \\ 1 & 3 & 2 \\ 3 & 2 & 1 \end{vmatrix} - \begin{vmatrix} 3 & 2 & 1 \\ 2 & 1 & 3 \\ 1 & 3 & 2 \end{vmatrix}$$

$$- \begin{vmatrix} 2 & 3 & 1 \\ 1 & 2 & 3 \\ 3 & 1 & 2 \end{vmatrix} + \begin{vmatrix} 3 & 1 & 2 \\ 2 & 3 & 1 \\ 1 & 2 & 3 \end{vmatrix} - \begin{vmatrix} 2 & 1 & 3 \\ 3 & 2 & 1 \\ 1 & 3 & 2 \end{vmatrix} + \begin{vmatrix} 3 & 2 & 1 \\ 1 & 3 & 2 \\ 2 & 1 & 3 \end{vmatrix}$$

Similarly for others; and, finally, the required expression for $N_2K_2P_3$ may be obtained by taking the algebraic sum of the appropriate compounds for all the eight expressions above.

Adopting this procedure all through, we finally obtain the following:

i. $3\,N_1\,K_1\,P_3$

$$= - \begin{vmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \\ 3 & 1 & 2 \end{vmatrix} - \begin{vmatrix} 2 & 3 & 1 \\ 3 & 1 & 2 \\ 1 & 2 & 3 \end{vmatrix} + 2 \begin{vmatrix} 3 & 1 & 2 \\ 1 & 2 & 3 \\ 2 & 3 & 1 \end{vmatrix} + 2 \begin{vmatrix} 1 & 3 & 2 \\ 3 & 2 & 1 \\ 2 & 1 & 3 \end{vmatrix}$$

$$- \begin{vmatrix} 2 & 1 & 3 \\ 1 & 3 & 2 \\ 3 & 2 & 1 \end{vmatrix} - \begin{vmatrix} 3 & 2 & 1 \\ 2 & 1 & 3 \\ 1 & 3 & 2 \end{vmatrix} + \begin{vmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \\ 2 & 3 & 1 \end{vmatrix} - 2 \begin{vmatrix} 2 & 3 & 1 \\ 1 & 2 & 3 \\ 3 & 1 & 2 \end{vmatrix}$$

$$+ \begin{vmatrix} 3 & 1 & 2 \\ 2 & 3 & 1 \\ 1 & 2 & 3 \end{vmatrix} + \begin{vmatrix} 1 & 3 & 2 \\ 2 & 1 & 3 \\ 3 & 2 & 1 \end{vmatrix} - 2 \begin{vmatrix} 2 & 1 & 3 \\ 3 & 2 & 1 \\ 1 & 3 & 2 \end{vmatrix} + \begin{vmatrix} 3 & 2 & 1 \\ 1 & 3 & 2 \\ 2 & 1 & 3 \end{vmatrix}$$

ii. $3\,N_1\,K_2\,P_1$

$$= \begin{vmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \\ 3 & 1 & 2 \end{vmatrix} + \begin{vmatrix} 2 & 3 & 1 \\ 3 & & 2 \\ 1 & 2 & 3 \end{vmatrix} - 2 \begin{vmatrix} 3 & 1 & 2 \\ 1 & 2 & 3 \\ 2 & 3 & 1 \end{vmatrix} + 2 \begin{vmatrix} 1 & 3 & 2 \\ 2 & 3 & 1 \\ 3 & 1 & 2 \end{vmatrix}$$

$$- \begin{vmatrix} 2 & 1 & 3 \\ 1 & 3 & 2 \\ 3 & 2 & 1 \end{vmatrix} - \begin{vmatrix} 3 & 2 & 1 \\ 2 & 1 & 3 \\ 1 & 3 & 2 \end{vmatrix} \quad \begin{vmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \\ 2 & 3 & 1 \end{vmatrix} + 2 \begin{vmatrix} 2 & 3 & 1 \\ 1 & 2 & 3 \\ 3 & 1 & 2 \end{vmatrix}$$

$$- \begin{vmatrix} 3 & 1 & 2 \\ 2 & 3 & 1 \\ 1 & 2 & 3 \end{vmatrix} + \begin{vmatrix} 1 & 3 & 2 \\ 2 & 1 & 3 \\ 3 & 2 & 1 \end{vmatrix} - 2 \begin{vmatrix} 2 & 1 & 3 \\ 3 & 2 & 1 \\ 1 & 3 & 2 \end{vmatrix} + \begin{vmatrix} 3 & 2 & 1 \\ 1 & 3 & 2 \\ 2 & 1 & 3 \end{vmatrix}$$

iii. $N_1\,K_2\,P_2$

$$= - \begin{vmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \\ 3 & 1 & 2 \end{vmatrix} + \begin{vmatrix} 2 & 3 & 1 \\ 3 & 1 & 2 \\ 1 & 2 & 3 \end{vmatrix} + \begin{vmatrix} 2 & 1 & 3 \\ 1 & 3 & 2 \\ 3 & 2 & 1 \end{vmatrix} - \begin{vmatrix} 3 & 2 & 1 \\ 2 & 1 & 3 \\ 1 & 3 & 2 \end{vmatrix}$$

$$- \begin{vmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \\ 2 & 3 & 1 \end{vmatrix} + \begin{vmatrix} 3 & 1 & 2 \\ 2 & 3 & 1 \\ 1 & 2 & 3 \end{vmatrix} + \begin{vmatrix} 1 & 3 & 2 \\ 2 & 1 & 3 \\ 3 & 2 & 1 \end{vmatrix} - \begin{vmatrix} 3 & 2 & 1 \\ 1 & 3 & 2 \\ 2 & 1 & 3 \end{vmatrix}$$

iv. $3\,N_2 K_1 P_1$

$$= \begin{vmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \\ 3 & 1 & 2 \end{vmatrix} + \begin{vmatrix} 2 & 3 & 1 \\ 3 & 1 & 2 \\ 1 & 2 & 3 \end{vmatrix} - 2\begin{vmatrix} 3 & 1 & 2 \\ 1 & 2 & 3 \\ 2 & 3 & 1 \end{vmatrix} + 2\begin{vmatrix} 1 & 3 & 2 \\ 3 & 2 & 1 \\ 2 & 1 & 3 \end{vmatrix}$$

$$- \begin{vmatrix} 2 & 1 & 3 \\ 1 & 3 & 2 \\ 3 & 2 & 1 \end{vmatrix} - \begin{vmatrix} 3 & 2 & 1 \\ 2 & 1 & 3 \\ 1 & 3 & 2 \end{vmatrix} + \begin{vmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \\ 2 & 3 & 1 \end{vmatrix} - 2\begin{vmatrix} 2 & 3 & 1 \\ 1 & 2 & 3 \\ 3 & 1 & 2 \end{vmatrix}$$

$$+ \begin{vmatrix} 3 & 1 & 2 \\ 2 & 3 & 1 \\ 1 & 2 & 3 \end{vmatrix} - \begin{vmatrix} 1 & 3 & 2 \\ 2 & 1 & 3 \\ 3 & 2 & 1 \end{vmatrix} + 2\begin{vmatrix} 2 & 1 & 3 \\ 3 & 2 & 1 \\ 1 & 3 & 2 \end{vmatrix} - \begin{vmatrix} 3 & 2 & 1 \\ 1 & 3 & 2 \\ 2 & 1 & 3 \end{vmatrix}$$

v. $N_2 K_1 P_2$

$$= - \begin{vmatrix} 1 & 2 & 3 \\ 3 & 3 & 1 \\ 3 & 1 & 2 \end{vmatrix} + \begin{vmatrix} 2 & 3 & 1 \\ 3 & 1 & 2 \\ 1 & 2 & 3 \end{vmatrix} + \begin{vmatrix} 2 & 1 & 3 \\ 1 & 3 & 2 \\ 3 & 2 & 1 \end{vmatrix} - \begin{vmatrix} 3 & 2 & 1 \\ 2 & 1 & 3 \\ 1 & 3 & 2 \end{vmatrix}$$

$$+ \begin{vmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \\ 2 & 3 & 1 \end{vmatrix} - \begin{vmatrix} 3 & 1 & 2 \\ 2 & 3 & 1 \\ 1 & 2 & 3 \end{vmatrix} - \begin{vmatrix} 1 & 3 & 2 \\ 2 & 1 & 3 \\ 3 & 2 & 1 \end{vmatrix} + \begin{vmatrix} 3 & 2 & 1 \\ 1 & 3 & 2 \\ 2 & 1 & 3 \end{vmatrix}$$

vi. $N_2 K_2 P_1$

$$= \begin{vmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \\ 3 & 1 & 2 \end{vmatrix} - \begin{vmatrix} 2 & 3 & 1 \\ 3 & 1 & 2 \\ 1 & 2 & 3 \end{vmatrix} + \begin{vmatrix} 2 & 1 & 3 \\ 1 & 3 & 2 \\ 3 & 2 & 1 \end{vmatrix} - \begin{vmatrix} 3 & 2 & 1 \\ 2 & 1 & 3 \\ 1 & 3 & 2 \end{vmatrix}$$

$$- \begin{vmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \\ 2 & 3 & 1 \end{vmatrix} + \begin{vmatrix} 3 & 1 & 2 \\ 2 & 3 & 1 \\ 1 & 2 & 3 \end{vmatrix} - \begin{vmatrix} 1 & 3 & 2 \\ 2 & 1 & 3 \\ 3 & 2 & 1 \end{vmatrix} + \begin{vmatrix} 3 & 2 & 1 \\ 1 & 3 & 2 \\ 2 & 1 & 3 \end{vmatrix}$$

vii. $N_2 K_2 P_2$

$$= \begin{vmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \\ 3 & 1 & 2 \end{vmatrix} + \begin{vmatrix} 2 & 3 & 1 \\ 3 & 1 & 2 \\ 1 & 2 & 3 \end{vmatrix} - 2\begin{vmatrix} 3 & 1 & 2 \\ 1 & 2 & 3 \\ 2 & 3 & 1 \end{vmatrix} - 2\begin{vmatrix} 1 & 3 & 2 \\ 3 & 2 & 1 \\ 2 & 1 & 3 \end{vmatrix}$$

$$+ \begin{vmatrix} 2 & 1 & 3 \\ 1 & 3 & 2 \\ 3 & 2 & 1 \end{vmatrix} + \begin{vmatrix} 3 & 2 & 1 \\ 2 & 1 & 3 \\ 1 & 3 & 2 \end{vmatrix} + \begin{vmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \\ 2 & 3 & 1 \end{vmatrix} - 2\begin{vmatrix} 2 & 3 & 1 \\ 1 & 2 & 3 \\ 3 & 1 & 2 \end{vmatrix}$$

$$+ \begin{vmatrix} 3 & 1 & 2 \\ 2 & 3 & 1 \\ 1 & 2 & 3 \end{vmatrix} + \begin{vmatrix} 1 & 3 & 2 \\ 2 & 1 & 3 \\ 3 & 2 & 1 \end{vmatrix} - 2\begin{vmatrix} 2 & 1 & 3 \\ 3 & 2 & 1 \\ 1 & 3 & 2 \end{vmatrix} + \begin{vmatrix} 3 & 2 & 1 \\ 1 & 3 & 2 \\ 2 & 1 & 3 \end{vmatrix}$$

REFERENCE

1. FISHER, R. A.: *The Design of Experiments.* Edinburgh: Oliver & Boyd, 2nd Edition, 1937.