TABLE OF CONFIDENCE INTERVAL FOR THE MEDIAN IN SAMPLES FROM ANY CONTINUOUS POPULATION

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Introduction

When the parent population is of known form but involves unknown parameters, cases arise where sampling distributions of certain statistics calculated from a random sample involves only one of the unknown parameters. Here it is possible to lay down in terms of the observations, and at any level, the confidence interval for the unknown population parameter which we often seek to estimate. This idea originated with R. A. Fisher[2] and is also otherwise expressed as 'fiducial inference', 'interval estimate', etc. Recently W. R. Thompson[3] and S. R. Savur[4] have independently obtained the Confidence Interval for the Median without reference to the form of the population. This is important work in view of the fact that in small samples it is not easy to test whether the assumption of normality holds good. There is perhaps only one other instance where the Confidence Interval of a population parameter (mean) has been determined (that too only approximately) without reference to population form. This is got from Anderson's inversion of Tchebycheff's inequality and depends on the sample estimates of the mean and standard deviation[1]. The Confidence Interval for the Median involves little calculations and possesses elegant properties, *viz.*, non-dependence on population form, determinateness and extreme simplicity.

When the population form is known it may be possible to work out the sampling distribution of the median in random samples from that population. The determination of the Confidence Interval of the Median in this case is very difficult especially when parameters other than the median of the population enters into the distribution function. In fact the exact distribution of the median has been worked out only for the rectangular and exponential populations.[5] For populations for which the Median is an efficient statistic in estimating the centre of location, a confidence range for the latter worked out straight from the sampling distribution of the Median statistic, can be expected to be smaller than the one obtained by Thompson and Savur. Apart from theories of Estimation and of Testing of Hypotheses, which have been developed only for samples coming from populations of known form, the Confidence Range of the Median discussed in the papers of Thompson and Savur has as its greatest merit the non-dependence on the population form and hence the freedom for the statistician from assuming a form for the population which cannot be reasonably justified.

There is a small discrepancy in the results obtained by Thompson and Savur. There is variation also in method of approach and in scope of the tests of significance of two samples using the median statistic, proposed by the two authors. The Table prepared in this paper confines itself to the Confidence Intervals for the Median as obtained by Thompson. The opportunity is taken to compare the points of view of both authors.

THOMPSON'S RESULTS

Thompson has set himself the problem of getting Confidence Intervals for the Median in random samples $[x_i]$ of n from a continuous population such that x_i is not equal to x_j. Thompson's method may be summarised as follows:—

Let $f(x)$ be the unknown population form such that $f(x)$ is continuous and remains positive in the range $a<x<b$ and $\int_a^b f(x)dx = 1$ and $f(x) = 0$ outside the range $a<x<b$. If p be defined as

$$p = \int_a^x f(x)\, dx$$

p is known as the probability integral of x and it is known that the distribution of p is according to the rectangular law, $\phi(p) = 1$, in the range $0<p<1$.

In a random sample of n observations $x_1, x_2, \ldots\ldots\ldots x_k, \ldots\ldots\ldots x_n$ arranged in ascending order of magnitude the probability that the k-th observation lies between x and $x+dx$ is

$$\left(\int_a^x f(x)\, dx\right)^{k-1} \quad f(x)dx. \quad \left(\int_x^b f(x)dx\right)^{n-k} = p^{k-1}.dp.(1-p)^{n-k}$$

The expression on the right hand side is the probability that the probability integral of the k-th individual lies between p and $p+dp$.

If p_k is the probability integral of x_k

$$P(x<x_k) = P(p<p_k) = \int_p^1 p^{k-1}\,(1-p)^{n-k}dp = I_{1-p}\,(n-k+1,\,k)$$

where $I_x\,(p,\,q)$ is the function tabulated in the Incomplete Beta Funtion Tables.

If M be the median of the population the probability integral corresponding to it by definition is $\frac{1}{2}$.

Therefore,

$$P(M<x_k) = P\,(\tfrac{1}{2}<p_k) = I_{0\cdot 5}\,(n-k+1,\,k)$$

It can also be seen that

$$P(M<x_k) = P(M>x_{n-k+1})$$

Hence
$$P(x_k<M<x_{n-k+1}) = 1 - 2\, I_{0\cdot 5}\,(n-k+1,\,k)$$

It can easily be demonstrated that the integral

$$\int_p^1 p^{k-1}\,(1-p)^{n-k}\,dp$$

is equivalent to the sum of the first k terms of the expansion of the binomial $(q+p)^n$. This can be seen directly in our problem because $P\,(p<p_k)$ is the sum of the probabilities that exactly 0, 1, 2, . . . $k-1$ observations have probability integral less than p.

The expression

$$P\,(x_k<M<x_{n-k+1}) = 1 - 2I_{\,0\cdot 5}\,(n-k+1,\,k)$$

tells us that we can be confident that the unknown population median will lie in the range between the k-th and $(n-k+1)$-th observations in $100\,[1-2\,I_{0\cdot 5}\,(n-k+1,\,k)]$ per cent of cases.

The Confidence Interval of any percentile can be obtained by this method; only it is not symmetrically placed with respect to the observations. Thus if Q is the lower quartile

$$P(Q<x_k)=P(1/4<p_k)=I_{0.75}(n-k+1,\ k)$$

If an integer l can be found such that

$$P(Q<x_k)=P(Q>x_l)$$

then it follows that

$$P(x_k<Q<x_l)=1-2\ I_{0.75}(n-k+1,\ k)$$

which gives a Confidence Interval for the lower quartile Q.

Savur's Results

Savur approaches the problem straight, by stating that the chance that in a random sample of n values there are not more than k values below M is

$$P=\frac{1}{2^n}(1+{}_nC_1+{}_nC_2+\ldots\ldots+{}_nC_k)=I_{0.5}(n-k,\ k+1)$$

This according to Thompson will be the chance that M is less than x_{k+1}. Savur views it as the chance that M is less than or *equal* to x_k. Savur must naturally be assuming that there is a finite probability for an individual observation to coincide with M. This is true only for discontinuous populations.

Also it is clear, on a little reflection, that the confidence interval for the Median of discontinuous populations cannot be solved without reference to the form of the population. For discontinuous populations the separate probabilities of an individual observation becoming less than, equal to, or greater than the median depend on the form of the population and on the parameters involved. It is not possible therefore to get the Confidence range for the Median of a discontinuous population as easily or in the same form as that for continuous distributions.

Table of Confidence Interval

Accepting therefore the results of Thompson and confining our attention to continuous populations as he has done, a Table of Confidence Intervals for the Median can be easily prepared, with the help of the Incomplete Beta Function Tables, for all values of n which these Tables permit. Usually the Confidence Coefficient is fixed at 0·95 or 0·99. This amounts to finding k such that given n

$$I_{0.5}(n-k+1,\ k)=0.025 \text{ or } 0.005.$$

In our problem since k can have only integral values it is not possible to fix the Confidence Coefficient exactly at 0·95 or 0·99 for all values of n. The best we can do is to choose such values for k which brings the Confidence Coefficient,

$$1-2\ I_{0.5}(n-k+1,\ k)$$

nearest to (and greater than) the conventional values 0·95 or 0·99. Thus for a sample consisting of 30 observations, we can say, with a confidence coefficient of 95·72 per cent., that the population median will be between the 10th and 21st ranked observations, and can say, with a confidence coefficient of 99·48 per cent., that the population median will be between the 8th and 23rd ranked observations.

It is obvious from Table I that, following this rule, for samples of size less than 6 and of size less than 8 no definite confidence interval can be obtained for coefficients in the neighbourhood of 0·95 and 0·99 respectively. This limitation may well be noted. Also for some of the smaller values of n, included in the Table, the Confidence Interval remains the same whether the Confidence Coefficient is 0·90 or 0·95. The interval is therefore very inelastic for small values of n.

For still larger values of n than have been included in Table I, an approximate method of calculating the value of k is available, as pointed out by Savur. For large values of n, the binomial $(\frac{1}{2}+\frac{1}{2})^n$ will tend to a normal distribution with mean $n/2$ and standard deviation $\sqrt{n/2}$. Corresponding to the partial sum $I_{0.5}(n-k+1,k)$ of the binomial we will then have the tail area $\frac{1}{2}(1-\alpha)$ of the normal curve beyond x, the relative deviate, given by

$$x=\frac{(n/2)-k}{(\sqrt{n}/2)}=\frac{n-2k}{\sqrt{n}}$$

as given in Table II of Pearson's Tables for Statisticians and Biometricians, Part I.

For a given confidence coefficient $\alpha=0·95$ or 0·99 the corresponding value of x can be read from this Table, and the values of k readily obtained from the relation

$$k=\frac{n-x\sqrt{n}}{2}$$

The integer nearest to the value of this expression should be chosen for the value of k. This means that corresponding to the value of k finally accepted α will be automatically adjusted to slightly different values in the neighbourhood of the original value of α.

Thus for a sample of 400 observations, $n=400$, and at the 95 percent Confidence Coefficient $x=1·96$. Therefore $k=(400-1·96\times 20)/2=180·4$. The median of the population lies in the range between the 180th and 221st observations, but with a confidence coefficient slightly above 95 percent.

As an illustration, of the closeness of this approximate method to the actual values obtained in Table I, let us get the confidence interval for, say, $n=64$. Here $k=(64-1·96\times 8)/2=24·16$. Therefore the median of the population lies in the range between the 24th and 41st observations, with a confidence coefficient slightly above 95%. We have given in Table 1, the same confidence interval, but the exact value of the confidence coefficient is 96·72 per cent. The approximate method can therefore be relied on to give reasonably accurate results with even moderately large values of n.

Test of Significance of Two Samples

There is a fundamental difference in the tests suggested by Thompson and Savur to judge whether two given samples belong to populations with different medians. Thompson's approach is as follows:—

Let $x_1, x_2, \ldots\ldots x_n$ and $x_1', x_2', \ldots\ldots x'_{n'}$ be the two samples and let k_m denote the number of values of the first sample that is less than x'_m, $m=1, 2, \ldots\ldots n'$.

If M and M' denote the medians in the population of the two samples, Thompson then proves the inequality :—

$$\sum_{m=1}^{n'} ({}_{n'}C_m)\, I_{0.5}\,(n-k_m+1,\, k_m) < 2^{n'}.\ P(M<M') < 1+\sum_{m=1}^{n'} ({}_{n'}C_{m-1})\, I_{.5}\,(n-k_m,\, k_m+1)$$

This according to Thompson provides the best upper and lower bounds for $P(M<M')$. We can make rigorous tests of significance at a definite probability level of (say) 0·95 or 0·99 by agreeing to accept that M and M' are different if $P(M<M')$ exceeds 0·95 or 0·99.

The hypothesis tested here is just the opposite of what is usually done. According to the usual method the null hypothesis is whether the two medians are equal and the probability of getting a pair of samples as or more divergent than the observed pair is to be obtained. The two samples are declared significantly different if this probability is less than 0·05 or 0·01. Thompson has not considered the possibility of getting a precise test of such a null hypothesis.

Savur proceeds to test the significance of two samples by arbitrarily laying down the rule: "The medians of two samples are significantly different from each other if the intervals (on our limit for random chance)* for the corresponding M's do not have a common part. If the higher end of one of the intervals be the same as the lower end of the other the corresponding M's can be considered to be just significantly different from each other" (p. 569 of (4)).

Evidently some confusion has entered in Savur's rule, by mixing up the problem of estimation and the problem of tests of significance. Savur does not state, with his rule, on what probability level the test of significance is made. There is however evidence that he is believing that when this rule is adopted the test of significance is being made at the same "5 per cent limit for random chance" as used in finding the intervals.

This point can be made clear by comparing Thompson's and Savur's tests. Suppose we are given two samples of 6 each. It can be seen from Table 1 that the 95 per cent. confidence intervals of the median in this case extends from the smallest to the largest observation. Suppose that the ranges of the two samples do not overlap. According to Savur's criterion this gives a significant result. To apply Thompson's criterion we see that $k_1=k_2=......=k_6=6$ and the limits of $P(M<M')$ are given by

$$(1-2^{-6})^2 = 0{\cdot}9690 < P(M<M') < 1$$

The Confidence Intervals for M and M' used for Savur's test strictly correspond to a confidence coefficient of $(1-2\cdot2^{-6})$. According to Thompson's test we can declare significance at the level $1-2\cdot2^{-6}+(2^{-6})^2$. In general for two samples of equal size n (say) if confidence intervals of the median are obtained for the coefficient $1-2\cdot2^{-n}$, in which case the interval coincides with the range of the sample, Savur's test amounts to testing at a level equal to $(1-2^{-n})^2$. For other values of the confidence coefficient it is not easy to find the relation between the confidence coefficient and the level of significance for testing two samples by Savur's method. It can be seen however that Savur's test is more stringent than Thompson's test though an exact measure of the stringency

* Savur's 5 per cent limit for random chance corresponds to 0·90 confidence coefficient, as he is leaving out 5 per cent at each tail. He is not justified to compare his median tests on 5 per cent limit for random chance with Fisher's test when for the latter the usual 5 per cent level amounts in Savur's terminology to 2·5 per cent limit for random chance.

Table I. Table of Confidence Interval for the Median in Samples from any Continuous Population

Sample size	Confidence coefficient $= P[x_k < M < x_{n-k+1}] = 1 - 2I_{0.5}(n-k+1, k)$						Sample size
	≧ 0·95			≧ 0·99			
n	n	$n-k+1$	$I_{0.5}(n-k+1, k)$	k	$n-k+1$	$I_{0.5}(n-k+1, k)$	n
6	1	6	·0156	...	...	...	6
7	1	7	·0078	...	...	...	7
8	1	8	·0039	1	8	·0039	8
9	2	8	·0195	1	9	·0020	9
10	2	9	·0107	1	10	·0010	10
11	2	10	·0059	1	11	·0005	11
12	3	10	·0193	2	11	·0032	12
13	3	11	·0112	2	12	·0017	13
14	3	12	·0065	2	13	·0009	14
15	4	12	·0176	3	13	·0037	15
16	4	13	·0106	3	14	·0021	16
17	5	13	·0245	3	15	·0012	17
18	5	14	·0154	4	15	·0038	18
19	5	15	·0096	4	16	·0022	19
20	6	15	·0207	4	17	·0013	20
21	6	16	·0133	5	17	·0036	21
22	6	17	·0085	5	18	·0022	22
23	7	17	·0173	5	19	·0013	23
24	7	18	·0113	6	19	·0033	24
25	8	18	·0216	6	20	·0020	25
26	8	19	·0145	7	20	·0017	26
27	8	20	·0096	7	21	·0030	27
28	9	20	·0178	7	22	·0010	28
29	9	21	·0121	8	22	·0041	29
30	10	21	·0214	8	23	·0026	30
31	10	22	·0147	8	24	·0017	31
32	10	23	·0100	9	24	·0035	32
33	11	23	·0175	9	25	·0023	33
34	11	24	·0122	10	25	·0043	34
35	12	24	·0205	10	26	·0030	35
36	12	25	·0144	10	27	·0020	36
37	13	25	·0235	11	27	·0038	37
38	13	26	·0168	11	28	·0023	38
39	13	27	·0119	12	28	·0047	39
40	14	27	·0192	12	29	·0032	40
41	14	28	·0138	12	30	·0022	41
42	15	28	·0218	13	30	·0040	42
43	15	29	·0158	13	31	·0027	43
44	16	29	·0244	14	31	·0048	44
45	16	30	·0178	14	32	·0033	45
46	16	31	·0129	14	33	·0023	46
47	17	31	·0200	15	33	·0040	47
48	17	32	·0147	15	34	·0028	48
49	18	32	·0222	16	34	·0047	49
50	18	33	·0164	16	35	·0033	50
51	19	33	·0244	16	36	·0023	51
52	19	34	·0182	17	36	·0039	52
53	19	35	·0135	17	37	·0027	53
54	20	35	·0201	18	37	·0043	54
55	20	36	·0150	18	38	·0032	55

in the former case is totally lacking. In view of the simplicity of operation of Savur's test it is to be commended in all cases except where his test may declare insignificant a result which may turn out significant according to Thompson's test.

The Problem of r Samples

Thompson does not put forward any single test whereby differences among the medians of r (>2) samples can be detected, corresponding to the analysis of variance for testing the r means. Savur's rule quoted above for the case of two samples seems to be meant to cover the general case as well. But in view of what I have remarked in the previous section, Savur's rule will get more and more stringent, at the same time, not providing an exact measure of this stringency as r increases.

Table I.—(*Continued*)

Sample size	Confidence coefficient $= P(x_k < M < x_{n-k+1}) = 1 - 2\, I_{0.5}(n-k+1, k)$						Sample size
	≧ 0·95			≧ 0·99			
n	k	$n-k+1$	$I_{0.5}(n-k+1, k)$	k	$n-k+1$	$I_{0.5}(n-k+1, k)$	n
56	21	36	·0220	18	39	·0023	56
57	21	37	·0166	19	39	·0038	57
58	22	37	·0240	19	40	·0027	58
59	22	38	·0182	20	40	·0018	59
60	22	39	·0137	20	41	·0031	60
61	23	39	·0198	21	41	·0019	61
62	23	40	·0150	21	42	·0036	62
63	24	40	·0215	21	43	·0026	63
64	24	41	·0164	22	43	·0041	64
65	25	41	·0232	22	44	·0030	65
66	25	42	·0178	23	44	·0046	66
67	26	42	·0249	23	45	·0034	67
68	26	43	·0192	23	46	·0025	68
69	26	44	·0147	24	46	·0038	69
70	27	44	·0207	24	47	·0028	70
71	27	45	·0160	25	47	·0043	71
72	28	45	·0222	25	48	·0032	72
73	28	46	·0172	26	48	·0018	73
74	29	46	·0237	26	49	·0033	74
75	29	47	·0185	26	50	·0020	75
76	29	48	·0143	27	50	·0040	76
77	30	48	·0198	...	...	...	77
78	30	49	·0154	...	...	...	78
79	31	49	·0211	...	...	...	79
80	81	50	·0165	...	...	...	80
81	32	50	·0224	...	...	...	81

Thus it appears that whereas the works of Thompson and Savur afford a brilliant example in the problem of interval estimation of the centre of location when the population form is unspecified, it may still be contended that they have not yet succeeded in supplying satisfactory tools for tests of significance in general of various statistical hypotheses based on the Median.

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