A Berry phase approach towards newly observed fractional quantum Hall states

B. Basu *, P. Bandyopadhyay

Physics and Applied Mathematics Unit, Indian Statistical Institute, Kolkata 700108, India

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Abstract

We have studied here the newly observed families of fractional quantum Hall states in the framework of Berry phase. It has been shown that this approach embraces in a unified way the whole spectrum of quantum Hall states with their various characteristic features. The newly observed states can be well accommodated within the primary sequence of FQH states and need not be considered as *second generation* FQH states.

1. Introduction

The recent experiments [1] on high mobility samples at very low temperatures predicted the existence of fractional quantum Hall (FQH) states at some unusual filling factors such as $\nu = 4/11$, $5/13$, $5/17$, $6/17$, $4/13$ and $7/11$ [2]. For the states $\nu = 4/11$ and $5/13$ a deep minimum in ρ_{xx} and also a respectable plateau in ρ_{xy} was observed. But for $\nu = 6/17$, $4/13$, $5/17$ and $7/11$ states there was no plateau in ρ_{xy} and also minimum in ρ_{xx} was not pronounced. It has been observed that the new state at $\nu = 4/11$ is a fully polarized FQH state. A weak state was also observed in $\nu = 3/8$ and at $\nu = 3/10$. These sequences of fractions do not fit into the standard series [3] of integral quantum Hall effects (IQHE) of composite fermions (CF) at $\nu = \frac{p}{2mp\pm 1}$, but the states $\nu = 4/11$ or $4/13$ appear in the hierarchy of quasiparticle condensates [4,5]. However, others such as $\nu = 3/8$ or $3/10$ do not belong to this hierarchy and the origin of their incompressibility seems to be puzzling. It has been proposed [1] that these states may be regarded as FQHE of composite fermions attesting to residual interactions between these composite particles. Finite size diagonalization of small clusters of electrons by Mandal and Jain [6] and Chang et al. [7] suggest that the state at

* Corresponding author.

E-mail addresses: banasri@isical.ac.in (B. Basu), pratul@isical.ac.in (P. Bandyopadhyay).

$\nu = 4/11$ is partially polarized which differs from the experimental findings of Pan et al. [1].

A significant analysis has been done by Quinn et al. [8,9] in which they interpret these states as a novel family of FQH states involving pairing correlations among the quasiparticles (QP). The correlations depend upon the behaviour of the QP–QP pseudopotential $V_{\mathrm{QP}}(L')$ where the interaction energy of a pair depends on the angular momentum. These pairs are proposed to have Laughlin correlations with one another and to form condensed states at a sequence which include all new fractions found in experiments.

Besides, it is also proposed [10] that the new composite particles consist each of a composite fermion of the first generation and a vortex like excitation which is based on the framework of Hamiltonian theory of Murthy and Shankar [11]. Very recently, Lopez and Fradkin [12] have proposed that the new states may be viewed as the hierarchical Jain states such that these are the quasiparticles and quasiholes of the primary states of the Jain sequence. Indeed, they have constructed the FQH states observed by Pan et al. as the fully polarized hierarchical descendants of the Jain series. All these approaches suggest that these new states effectively represent the *second generation* of FQH states as these are expressed as FQH states of FQH states.

In this Letter we shall show that all these new states may be treated as the FQH states in the primary sequence when we analyze the series from the viewpoint of Berry phase. Our analysis suggests the existence of FQH states which include apart from the states observed by Pan et al. as well as all the predicted states by Quinn et al. and also by Lopez and Fradkin. Besides, in this formalism these states are found to be fully polarized and their particle–hole conjugate states are found to be unpolarized. This is consistent with the observation in experiments for the state with $\nu = \frac{4}{11}$ and its particle–hole conjugate state $\frac{7}{11}$.

In some earlier papers [13,14] we have analyzed the sequence of quantum Hall states from the viewpoint of chiral anomaly and Berry phase. In our approach, we have considered the spherical geometry which was first used by Haldane [4] as an attractive alternative to finite size studies of quantum Hall effect. Here, the electrons are confined on the surface of a sphere of large radius R with a magnetic monopole of strength μ at the centre. In this geometry, the single electron is represented as a spin S, the orientation of which indicates the point on the sphere about which the state is localized. We can now write the quantum Hall states in terms of spinor wave functions and take advantage of the analysis in terms of chiral anomaly which is associated with the Berry phase. In this geometry the angular momentum relation is given by

$$\mathbf{J} = \mathbf{r} \times \mathbf{p} - \mu \hat{\mathbf{r}},$$
$$\mu = 0, \pm 1/2, \pm 1, \pm 3/2, \ldots . \quad (1)$$

From the monopole harmonics $Y_\ell^{m,\mu}$ with $\ell = 1/2, |m| = |\mu| = 1/2$, we can introduce a two-component spinor $\theta = \binom{u}{v}$ where

$$u = Y_{1/2}^{1/2,1/2} = \sin\frac{\theta}{2} \exp\bigl[i(\phi - \chi)/2\bigr],$$
$$v = Y_{1/2}^{-1/2,1/2} = \cos\frac{\theta}{2} \exp\bigl[-i(\phi + \chi)/2\bigr]. \quad (2)$$

Here μ corresponds to the eigenvalue of the operator $i\frac{\partial}{\partial \chi}$.

Following the same arguments of Haldane [4] we can construct the N-particle wave function for the quantum Hall fluid at $\nu = \frac{1}{m}$ as [4,13]

$$\psi_N^{(m)} = \prod_{i<j} (u_i v_j - u_j v_i)^m, \quad (3)$$

m being an odd integer. Here u_i (v_j) corresponds to the ith (jth) position of the spinor such that the angle between position vectors $\mathbf{r}_i$ and $\mathbf{r}_j$ is given by $\theta_{ij} = 2 \arcsin|u_i v_j - u_j v_i|$. Apart from a phase factor $\prod_k e^{-im\chi_k}$, which is related to the Berry phase the N-particle wave function $\psi_N^{(m)}$ is identical with that proposed by Haldane [4] and hence it will not affect the dynamical calculations.

Since $\psi_N^{(m)}$ is totally antisymmetric for odd m and symmetric for even m we can identify [4] m as $m = J_i + J_j$ for the N-particle system where J_i is the angular momentum of the ith particle. It is evident from Eq. (1) that with $\mathbf{r} \times \mathbf{p} = 0$ and $\mu = \frac{1}{2}$ we have $m = 1$ which corresponds to the complete filling of the lowest Landau level. From the Dirac quantization condition $e\mu = \frac{1}{2}$, we note that this state corresponds to $e = 1$ describing the IQH state with $\nu = 1$.

The next higher angular momentum state can be achieved either by taking $\mathbf{r} \times \mathbf{p} = 1$ and $|\mu| = \frac{1}{2}$ (which implies the higher Landau level) or by taking $\mathbf{r} \times \mathbf{p} = 0$ and $|\mu_{\mathrm{eff}}| = \frac{3}{2}$ implying the ground state for the Landau level. However, with $|\mu_{\text{eff}}| = \frac{3}{2}$, we find the filling fraction $\nu = \frac{1}{3}$ which follows from the condition $e\mu = \frac{1}{2}$ for $\mu = \frac{3}{2}$. Generalizing this we can have $\nu = \frac{1}{5}$ with $|\mu_{\text{eff}}| = \frac{5}{2}$. It may be mentioned that for a quantum Hall particle the charge is given by $-\nu e$ when ν is the filling factor.

From the relation (2), we note that one can consider the geodesic projection coordinate

$$\zeta = \frac{u}{v} = \tan\frac{\theta}{2} e^{i\phi} \tag{4}$$

for the Hopf fibration $S^2 = \frac{SU(2)}{U(1)}$ and the base space turns out to be a Kähler manifold. The symplectic structure is given by [15]

$$\Omega = 2i\frac{d\zeta \wedge d\bar{\zeta}}{(1+|\zeta|^2)^2} = 2i\frac{\partial^2 K}{\partial\zeta\partial\bar{\zeta}} d\zeta \wedge d\bar{\zeta}, \tag{5}$$

where $K = \ln(1+|\zeta|^2)$ is the Kähler potential. The Hilbert space $\mathcal{H}_N$ on this Hopf fibration S^2 is composed by the $N = 2J+1$ one particle wave functions ψ_m^J around the Dirac monopole $\mu(J = |\mu|)$. When we have $\mu = n/2$ with n an odd integer, we can consider this as an n-particle system with each particle having $\mu = 1/2$ and the geodesic projection coordinate is given by a $\zeta' = (u/v)^n$. Hence for $\mu = n/2$, n odd and $n \neq 1$, we have a correlated system where the symplectic structure of the phase space is modified as $\Omega' = n\Omega$. The deformation causes to increase the occupied area of an electron and hence changes the electron density to lie at certain fractions of the density corresponding to the density of the completely filled Landau level. When these electrons condense into an incompressible fluid we observe fractional quantum Hall effect at the particular filling factors.

It may be remarked that as μ here corresponds to the monopole strength, we can relate this with the Berry phase. Indeed $\mu = \frac{1}{2}$ corresponds to one flux quantum and the flux through the sphere when there is a monopole of strength μ at the centre is 2μ. The Berry phase of a fermion of charge q, when it moves in a closed path, is given by $e^{i\phi_B}$ with $\phi_B = 2\pi q N_\phi$ where N_ϕ = number of flux quanta enclosed by the loop traversed by the particle.

If μ is an integer, we can have a relation of the form

$$\mathbf{J} = \mathbf{r} \times \mathbf{p} - \mu\hat{\mathbf{r}} = \mathbf{r}' \times \mathbf{p}' \tag{6}$$

which indicates that the Berry phase associated with μ may be unitarily removed to the dynamical phase. Evidently, the average magnetic field may be considered to be vanishing in these states. The attachment of $2m$ vortices (m an integer) to an electron effectively leads to the removal of Berry phase to the dynamical phase. So, FQH states with $2\mu_{\text{eff}} = 2m+1$ can be viewed as if one vortex line is attached to the electron. Now we note that for a higher Landau level we can consider the Dirac quantization condition $e\mu_{\text{eff}} = \frac{1}{2}n$, with n being a vortex of strength $2\ell + 1$. This can generate FQH states having the filling factor of the form $\frac{n}{2\mu_{\text{eff}}}$ where both n and $2\mu_{\text{eff}}$ are odd integers. Indeed, we can write the filling factor as [13,14]

$$\nu = \frac{n}{2\mu_{\text{eff}}} = \frac{1}{\frac{2\mu_{\text{eff}} \mp 1}{n} \pm \frac{1}{n}} = \frac{n}{2mn \pm 1}, \tag{7}$$

where $2\mu_{\text{eff}} \mp 1$ is an even integer given by $2mn$.

In this scheme, the FQH states with ν having the form

$$\nu = \frac{n'}{2mn' \pm 1} \tag{8}$$

(n' an even integer) can be generated through particle–hole conjugate states

$$\nu = 1 - \frac{n}{2mn \pm 1} = \frac{n(2m-1) \pm 1}{2mn \pm 1} = \frac{n'}{2mn' \pm 1}. \tag{9}$$

Now, from Eq. (7),

$$\frac{n}{2\mu_{\text{eff}}} = \frac{n}{2m' \pm 1} = \frac{n}{2mn \pm 1} \tag{10}$$

we note that the integer m' has been taken to be the product mn. However, not all m' can be written in the product form mn. So we can write in a generalized way,

$$m' = mn \pm \tilde{m}, \quad \tilde{m} \text{ being an integer.}$$

Indeed replacing n by q where q is any integer we can express the generalized relation as

$$\frac{q}{2\mu_{\text{eff}}} = \frac{q}{2m' \pm 1} = \frac{q}{2mq \pm (2\tilde{m} \pm 1)}, \tag{11}$$

where we have taken $m' = mq \pm \tilde{m}$, with $\tilde{m}$ being an integer. The relation (7) is now modified as

$$\nu = \frac{1}{2m \pm (\frac{2\tilde{m} \pm 1}{q})} = \frac{1}{2m \pm p/q}, \tag{12}$$

where $p \geqslant 1$ an odd integer. It is noted that for $p = 1$ implying $\tilde{m} = 0$ corresponds to the Jain sequence.

The even denominator filling factors are obtained when μ_{eff} is an integer. As we have pointed out that for integer μ, the Berry phase can be removed to the dynamical phase, these states can only be observed when they appear as pair states [16,17]. This corresponds to the non-Abelian Berry phase and represents non-Abelian quantum Hall fluid. In this case, the relation (12) will take the form

$$\nu = \frac{1}{2m \pm \frac{2\tilde{n}}{q}} = \frac{1}{2m \pm p/q}, \tag{13}$$

where $p(q)$ is an even(odd) integer.

Now to have a physical interpretation of the states given by (12) and (13) we note that these correspond to the attachment of p vortices (flux quanta) in a cluster of q electrons in the lowest Landau level.

For $p > 1$ and q odd we can take

(i) $q - 2$ electrons of which each one is attached with a vortex is coupled with a residual boson composed of two electrons;
(ii) $q - 1$ electrons, each of them being attached with a vortex is coupled with the residual fermion.

In a similar way for q even, we may view the relation (12) such that $q - 1$ electrons, each having one vortex attached with it is coupled with the residual fermion.

It is noted that when an electron is attached with a magnetic flux, its statistics changes and it is transformed into a boson. These bosons condense to form a cluster which is coupled with the residual fermion or boson composed of two fermions. Indeed the residual boson or fermion will undergo a "statistical" interaction tied to a geometric Berry phase effect that winds the phase of the particle as it encircles the vortices. This suggests that for odd q, p takes the values $q - 2$ and $q - 1$ and for even q, p is given by $q - 1$. Also we observe that the attachment of vortices to electrons in a cluster will make the fluid an incompressible one. Indeed as two vortices cannot be brought very close to each other, there will be a hard core repulsion in the system which accounts for the incompressibility of the quantum Hall fluid.

Following Wojs, Yi and Quinn [18] we may consider here the pseudopotential which is a function of the relative angular momentum of a pair of electrons each attached with a magnetic flux quantum. It has been pointed out by these authors that Laughlin correlations among interacting fermions confined to a spherical surface can be achieved when the largest values of the pair angular momentum L' or smallest values of the relative angular momentum $R = 2l - L'$, l being the single particle angular momentum are maximally avoided. It has recently been demonstrated that the adiabatic addition of flux automatically gives rise to Laughlin correlations [19]. In fact from the angular momentum relation (1) we note that the addition of a magnetic flux quantum ($\mu = 1/2$) to an electron suggests that the single fermion has angular momentum $l = 1/2$ and so for a pair of such fermions we have $L' = 2l - 1$ implying $R = 1$. Laughlin correlations occur if and only if the pseudopotential $V(R)$ is superharmonic, that is, rises with increasing L' faster than $L'(L' + 1)$ as the avoided values of L' is approached. The pseudopotential $V(R)$ for electrons in the lowest Landau level ($n = 0$) is superharmonic at all values of L'. In our present scheme, of the $(q - 1)$ or $(q - 2)$ fermions each attached with one magnetic flux quantum even number of fermions will form bosonic pairs each with relative angular momentum $R = 1$ and $L' = 0$. These bosonic pairs along with the single boson formed by an electron attached with a magnetic flux quantum (when $(q - 1)$ or $(q - 2)$ is odd) condense to form a cluster which is coupled with the residual fermion or boson through a "statistical interaction" caused by the Berry phase effect when the latter encircles the vortices.

It should be mentioned here that as for the filling fraction given by Eq. (12) the mother relation is the same as in the conventional filling fraction given by Eq. (7), i.e., $\nu = \frac{n}{2\mu_{\mathrm{eff}}}$, the wave function for these states will be of the same Laughlin-type as in Eq. (3) with $m = 1/\nu$. The incompressibility of the quantum Hall fluid having these new filling factors is implicit in this Laughlin-type wave function.

Now we consider some specific cases (Tables 1 and 2). Values with bold faces in Tables 1 and 2 have been reported in Ref. [1]. The values $\nu = 4/5$ and $\nu = 5/7$ have been observed by Du et al. [20] which appear as weak depressions in the longitudinal resistivity. In a similar way, we can carry on with other q and p-values.

In this context we may add that the states we have obtained include all the states predicted by Quinn et

Table 1
q odd, $p > 1$ and $p = q - 1$ or $q - 2$

q	p	General form of ν	Values of m	ν
3	2	$\frac{1}{2m\pm 2/3}$	1	**3/8**, 3/4
			2	3/14, **3/10**
5	3	$\frac{1}{2m\pm 3/5}$	1	**5/13**, 5/7
			2	5/23, **5/17**
	4	$\frac{1}{2m\pm 4/5}$	1	5/14, 5/6
			2	5/24, 5/16
7	5	$\frac{1}{2m\pm 5/7}$	1	7/19, 7/9
			2	7/33, 7/23
	6	$\frac{1}{2m\pm 6/7}$	1	7/20, 7/8
			2	7/34, 7/22

Table 2
q even, $p > 1$ and $p = q - 1$

q	p	General form of ν	Values of m	ν
4	3	$\frac{1}{2m\pm 3/4}$	1	**4/11**, 4/5
			2	4/19, **4/13**
6	5	$\frac{1}{2m\pm 5/6}$	1	**6/17**, 6/7
			2	6/29, 6/19

al. [8,9]. Indeed, their classification scheme suggests the relations

$$\nu^{-1} = 2\tilde{p} + 1 \pm (2 + \tilde{q}/2)^{-1}, \tag{14}$$

where $\tilde{p}$ and $\tilde{q}$ are integers.

As mentioned earlier, the even denominator states are expected to appear as paired states. Indeed, in this case μ_{eff} is an integer and hence can be removed to the dynamical phase. These states can only be observed in paired states. This suggests that the newly observed states $\nu = 3/8$ and $\nu = 3/10$ should appear in paired states which has also been suggested by some other authors [8,9,21]. From our analysis, it appears that these states correspond to non-Abelian Berry phase and represent non-Abelian quantum Hall fluid.

In this formalism it is evident that for the filling fraction ν the charge of the quasiparticle is $-\nu e$. This result is consistent with the experimental results and is identical with that of others in case of the FQH states with $\nu = 1/m$ (m being an odd integer). But there is a controversy regarding the charge of quasiparticles in the FQH states with the filling factor $\nu = n/m$ (m an odd integer). In the composite fermion model, the predicted charge of the quasiparticles having $\nu = n/m$ is always $1/m$ which is supposed to be supported by experiments at a bit higher temperature [22] but is in contrary to the experimental result [23] where it is shown that the charges of the quasiparticles are $e/3$, $2e/5$ and $\sim 3e/7$ at $\nu = 1/3, 2/5$ and $3/7$ at extremely low temperature. In this context, we may add that the Dirac quantization condition which is a consequence of quantum field theory at $T = 0$ (no finite temperature effect is taken into account) and so our result is expected to be valid in the close vicinity of $T = 0$. However at higher temperature it may happen that for quasiparticles with $\nu = n/m$, n being related to higher Landau level, the system is dissociated into n copies of quasiparticles each with charge $e^* = 1/m$.

In a recent paper [24] we have analyzed the polarization of quantum Hall states in the framework of the hierarchical analysis in terms of the Berry phase. There it is observed that the states in the lowest Landau level corresponding to the filling factors $\nu = 1$ and $\nu = \frac{1}{2m+1}$ with m an integer correspond to the fully polarized states. Indeed, in such a system even number of vortices are gauged away and the attachment of one vortex (magnetic flux) to an electron leads to the fully polarized state. However, in the higher Landau level we have the filling factor given by

$$\nu = \frac{n}{2\mu_{\text{eff}}} = \frac{1}{\frac{2\mu_{\text{eff}}\mp 1}{n} \pm \frac{1}{n}} = \frac{1}{2m \pm \frac{1}{n}} = \frac{n}{2mn \pm 1},$$

where n is an odd integer corresponding to a vortex of strength $2l + 1$. As this effectively corresponds to the attachment of $\frac{1}{n}$ flux unit attached to an electron, the state will be partially polarized. Again the particle–hole conjugate states given by

$$\nu = 1 - \frac{n}{2mn \pm 1},$$

n being an odd integer will correspond to unpolarized states. This analysis suggests that the newly observed states as reported in [1] which appear in the lowest Landau level will correspond to fully polarized states. Indeed, the states corresponding to $\nu = 4/11$ has been observed to be fully polarized and the particle–hole conjugate state $\nu = 7/11$ is found to be unpolarized [1].

Finally, we may mention here that the hierarchical interpretation of the Haldane–Halperin scheme was questioned [25] because of the specific form of the QP–QP interaction. Besides, as mentioned earlier, the newly observed states $\nu = 3/8$ and $\nu = 3/10$ do not belong to this hierarchy. On the other hand, the Jain classification scheme effectively reveals a fundamental connection between IQHE and FQHE as the FQH states are considered to be IQH states of composite fermions. However, to interpret the newly observed states this fundamental concept has to be abandoned and we have to take into account the residual CF–CF interaction. A specific form of this interaction [8,9] based on the pseudopotentials such that the pair interaction energy depends on the relative angular momentum nicely explains the new states. But some other states observed by Du et al. [20] which appear as weak depression in the longitudinal resistivity cannot be accommodated in this scheme. Our present analysis suggests that these *second generation* FQH states are not truly second generation states, rather these appear in the primary sequence of the FQH states. This classification scheme can explain all the states observed to date [1,20]. This also predicts the polarization of the states $\nu = 4/11$ and $\nu = 7/11$ consistent with experiment.

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