MISCELLANEOUS

A FACTORIAL APPROACH TO CONSTRUCTION OF TRUE COST OF LIVING INDEX AND ITS APPLICATION IN STUDIES OF CHANGES IN NATIONAL INCOME

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SUMMARY. A new formula for the *true index* of cost of living has been evolved through a new formulation. Incidentally, it has been shown how the price and the quantity components of value change in National Income can be evaluated.

1. Introduction

A new formula has been evolved for the *true index* of cost of living through a factorial approach and this has been shown to possess all the properties of Wald's true index (1939) as well as those of Frisch's true index (1936) obtained by the double-expenditure method especially in so far as it lies between Staehle's limits (1935).

2. New formulation

An indifference-defined price index denotes the ratio of expenditures between two situations ensuring the same standard of living. Similarly, a quantity index also stands for the ratio of the two standards of living. As two points on an indifference surface are *equivalent*, the quantity index between such a pair will be unity. Conversely, if the quantity index be equal to unity, it implies that the standards of living in the two situations are the same. Moreover, the quantity index between two equivalent points q_1 and q_0 on $I(q)$ will also be unity where $I(q)$ stands for the choice indicator describing the scale of preference of an individual. If, therefore, a quantity q_1 could be found (by any formulation), in situation 1 such that the quantity index κ_{01}, between q_0 and q_1, is unity, then q_1 can be regarded as equivalent to q_0. The price index, π_{01}, found by such a joint formulation, will represent the true index of cost of living of the period 1 compared to the period 0.

As an illustration of our formulation, we may consider Fisher's ideal formulae for price and quantity :

$$F_p = \sqrt{\frac{V_{10}}{V_{00}} \frac{V_{11}}{V_{01}}} \quad \text{... (2.1)}$$

and

$$F_q = \sqrt{\frac{V_{01}}{V_{00}} \frac{V_{11}}{V_{10}}}, \quad \text{... (2.2)}$$

where

$$V_{jk} = \sum_{i=1}^{n} p_j^i q_k^i \quad (j, k = 0, 1) \quad \text{... (2.3)}$$

is the value of $q_k = (q_k^1, q_k^2, ..., q_k^n)$ calculated at prices $p_j = (p_j^1, p_j^2, ..., p_j^n)$; F_p and F_q are Fisher's ideal indexes for price and quantity respectively. If now $F_q = 1$, the price index reduces to V_{11}/V_{00}.

The condition $F_q = 1$ implies

$$V_{11}\ V_{01} = V_{10}\ V_{00}, \qquad \dots \quad (2.4)$$

a condition which was independently worked out by Frisch in his double-expenditure approach. (Frisch, 1936).

3. Factorial approach

Let p_0^i and q_0^i be the price and quantity of the i-th commodity in the situation 0, and let p_1^i and q_1^i be the corresponding figures in the situation 1. From these two sets of entries we obtain four values for the i-th item : V_{00}^i, V_{01}^i, V_{10}^i, and V_{11}^i where, of course, $V_{jk}^i = p_j^i q_k^i$ $(j, k = 0, 1)$. In analogy with a factorial experiment involving two factors at two levels each, we have the price and quantity components as follows :

$$P^i = \tfrac{1}{2}[V_{11}^i - V_{01}^i + V_{10}^i - V_{00}^i] = \tfrac{1}{2}V_{00}^i(\pi^i-1)(\kappa^i+1) \qquad \dots \quad (3.1)$$

$$Q^i = \tfrac{1}{2}[V_{11}^i - V_{10}^i + V_{01}^i - V_{00}^i] = \tfrac{1}{2}V_{00}^i(\pi_i+1)(\kappa^i-1), \qquad \dots \quad (3.2)$$

where $\pi_i = \dfrac{p_1^i}{p_0^i}$ and $\kappa^i = \dfrac{q_1^i}{q_0^i}$ are the price and quantity indexes respectively of the i-th item. It is easy to see that the value change, $V_{11}^i - V_{00}^i$, can simply be written as the sum of the price and quantity components. That is,

$$V_{11}^i - V_{00}^i = P^i + Q^i, \qquad \dots \quad (3.3)$$

which when summed over all the n commodities yields the following :

$$V_{11} - V_{00} = P + Q, \qquad \dots \quad (3.4)$$

where

$$P = \sum_{i=1}^{n} P^i,\ Q = \sum_{i=1}^{n} Q^i \ \text{ and } \ V_{jj} = \sum_{i=1}^{n} V_{jj}^i,\ (j = 0, 1). \qquad \dots \quad (3.5)$$

If now we define π and k as the *general* price and quantity indexes respectively we can, by analogy of identities (3.1) and (3.2), write

$$P = \tfrac{1}{2}V_{00}(\pi-1)(\kappa+1), \qquad \dots \quad (3.6)$$

and

$$Q = \tfrac{1}{2}V_{00}(\pi+1)(\kappa+1). \qquad \dots \quad (3.7)$$

Following Stuvel (1957) we may equate these to ΣP^i and ΣQ^i, where P^i and Q^i are defined by (3.1) and (3.2). That is,

$$\tfrac{1}{2}\ V_{00}(\pi-1)(\kappa+1) = \tfrac{1}{2}(V_{11} - V_{01} + V_{10} - V_{00}) \qquad \dots \quad (3.8)$$

$$\tfrac{1}{2}\ V_{00}(\pi+1)(\kappa-1) = \tfrac{1}{2}(V_{11} - V_{10} + V_{01} - V_{00}) \qquad \dots \quad (3.9)$$

or, alternatively, after removing the factor $\tfrac{1}{2}V_{00}$ from both sides,

$$(\pi-1)(k+1) = \frac{V_{11}}{V_{00}} - L_q + L_p - 1 \qquad \dots \quad (3.10)$$

$$(\pi+1)(k-1) = \frac{V_{11}}{V_{00}} - L_p + L_q - 1, \qquad \dots \quad (3.11)$$

where $\frac{V_{11}}{V_{00}}$ is the value index, L_p and L_q are the Laspeyres' price and quantity indexes, i.e. $L_p = V_{10}/V_{00}$, $L_q = V_{01}/V_{00}$. From (3.8) and (3.9) we have :

$$\pi - k = (L_p - L_q) \qquad \dots \quad (3.12)$$

$$\pi k = \frac{V_{11}}{V_{00}} \qquad \dots \quad (3.13)$$

from which the price and quantity indexes are obtained as

$$\pi = \tfrac{1}{2}(L_p - L_q) + \sqrt{[\tfrac{1}{2}(L_p - L_q)]^2 + \frac{V_{11}}{V_{00}}} \qquad \dots \quad (3.14)$$

$$k = \tfrac{1}{2}(L_q - L_p) + \sqrt{[\tfrac{1}{2}(L_p - L_q)]^2 + \frac{V_{11}}{V_{00}}}. \qquad \dots \quad (3.15)$$

On account of the symmetry in the relationship of π and k in (3.14) and (3.15), the indexes π and k satisfy certain fundamental tests, such as the time reversal and the factor reversal tests and are, therefore, preferred to other known formulae (Stuvel).

The condition of equivalence in our formulation can be worked out by equating the quantity index to unity. If this were done, i.e., if $\kappa = 1$, it would follow from (3.11) that the quantity component in the value change is zero, i.e.,

$$V_{11} - V_{10} + V_{01} - V_{00} = 0 \qquad \dots \quad (3.16)$$

and also from (3.13),

$$\pi = \frac{V_{11}}{V_{00}}. \qquad \dots \quad (3.17)$$

Relation (3.16) provides the necessary condition for determining the point q_1 such that it is equivalent to q_0. This condition has a striking similarity with that of Frisch. In the latter case, however, the condition may be written in terms of logarithms, i.e.,

$$\log V_{11} - \log V_{10} + \log V_{01} - \log V_{00} = 0 \qquad \dots \quad (3.18)$$

which is, in fact, similar to (3.16).

The condition of equivalence could also be worked out from formulae other than Stuvel's. In fact such conditions have already been derived in the case of Fisher's ideal index which is rather a special case of the generalised indexes of Stuvel (Banerjee). In particular cases of the generalised index, however, the conditions of equivalence take complicated expressions.

4. The new index compared with those of Wald and Frisch

We next compare our index obtained in (3.17) with Frisch's true index as also with Wald's true index. Suppose the Engel curves C_0 and C_1 are linear and are given by the following equations.

The equations of

$$C_0 : q_0^i = \alpha_0^i V + \beta_0^i$$
$$C_1 : q_1^i = \alpha_1^i V + \beta_1^i$$
$$(i = 1, 2, \ldots, n). \qquad \ldots \quad (4.1)$$

Let $$\sum_{i=1}^{n} \alpha_j^i p_k^i = a_{jk} \text{ and } \sum_{i}^{n} \beta_j^i p_k^i = b_{jk}, (j, k = 0, 1). \qquad \ldots \quad (4.2)$$

Then

$$a_{jj} = 1, \quad b_{jj} = 0 \quad (j = 0, 1).$$

Also $$\sum_{i=1}^{n} p_0^i q_0^i = V_{00}, \; V_{01} = a_{10} V_{11} + b_{10}, \; V_{10} = a_{01} V_{00} + b_{01}.$$

In terms of the above coefficients, Frisch's index is given by

$$F = \frac{-b_{10} + \sqrt{b_{10}^2 + 4a_{10}a_{01} V_{00}^2 + 4a_{10} b_{01} V_{00}}}{2a_{10} V_{00}} \qquad \ldots \quad (4.3)$$

and, Wald's true index is given by

$$W = \sqrt{\frac{a_{01}}{a_{10}}} + \frac{1}{V_{00}} \; \frac{b_{01} - \frac{b_{10}}{a_{10}} \sqrt{a_{01} a_{10}}}{1 + \sqrt{a_{10} a_{01}}} \qquad \ldots \quad (4.4)$$

while our index, obtained through the condition (3.16) of equivalence, is obtained as

$$B = \frac{1 + a_{01}}{1 + a_{10}} + \frac{1}{V_{00}} \; \frac{(b_{01} - b_{10})}{1 + a_{10}}. \qquad \ldots \quad (4.5)$$

Following the same line of argument, as adopted by Wald, it can be shown that our index B also lies between the limits of Staehle (see Wald).
That is,

$$\frac{V_{11}}{V_{01}} \leqslant B \leqslant \frac{V_{10}}{V_{00}}, \qquad \ldots \quad (4.6)$$

where $V_{01} = V_{00}$. Or, in terms of the coefficients of the Engel curves, the new index lies between,

$$\frac{1}{a_{10}} - \frac{1}{V_{00}} \; \frac{b_{10}}{a_{10}} \quad \text{and} \quad a_{01} + \frac{b_{01}}{V_{00}}. \qquad \ldots \quad (4.7)$$

When, however, $a_{10} = a_{01}$, we see $W = B$. If, in addition, $b_{10} = b_{01}$, we see $F = W = B = 1$.

Although formulae (4.3)—(4.5) are exact in terms of the coefficients of the Engel curves, yet in practice, the indexes have to be computed from *estimates* of the coefficients of the Engel curves which are subject to errors of estimation. The error of these indexes can, of course, be determined taking p_0 and p_1 as given, since the standard errors of the regression coefficients are known.

Even if the point q_1 is such that Q, i.e., the quantity component in the value change, is *not* exactly equal to zero, the price formula given in (3.14) may still be considered as a good approximation to the true index, provided the quantity component in the value change did not differ from zero significantly. Tests of significance similar to those applied to factorial experiments may perhaps be applied in our case also, though with caution, in view of non-experimental nature of economic data.

In the next section we consider a practical situation for which our methods could be applied with some advantages in explaining value changes in terms of price and quantity variations.

5. Price and quantity components in value change of national income

Estimates of national income are available for Indian Union in a series of papers entitled, "Estimates of National Income", published by the Central Statistical Organisation. These papers show, besides other particulars, a statement on the movement of net national output at factor cost from year to year. Comparison of the national output is made possible, as usual, through the indexes of national output provided at current prices (indexes of value) and constant prices (indexes of output or quantity). These indexes furnish, of course, a basis of comparison, but a complete picture of the movement of national income will not be available unless it is known how much of the change in value is due to change in price and how much to change in output (quantity).

We shall now apply formulae (3.4)—(3.7) to Indian national income data to explain changes in the national income in terms of changes in prices and also changes in the volume of national output.

In "Estimates of National Income", the values of L_q (quantity index) and V_{11}/V_{00} (value index) are provided, but not the values of L_p (price index). For L_p (Laspeyres' price index), we may take the wholesale price indexes* for India perhaps as close substitutes. These price indexes (base 1939 = 100) are not, of course, identical with the price indexes which would have been obtained from the statistics utilised in the estimation of national income. However, the order of disagreement between the two, is probably not so large as to vitiate the study to an unacceptable extent, or to make the computations absolutely hypothetical.

The computed price components and quantity components in value change are given in the following Table for the period 1949-50 to 1957-58. In the Table, the values of L_q (Col. 3) and V_{11}/V_{00} (Col. 4) have been taken from the papers on national income, while, the values of L_p (Col. 2), as stated before, are the wholesale price indexes obtained as simple averages of the monthly indexes for 12 months of the corresponding fiscal years; π and k (Cols. 6 and 7 respectively) are the adjusted price and quantity indexes. Column 5 shows the percentage change of value of net national output as given by

$$100\left(\frac{V_{11}}{V_{00}}-1\right). \qquad \dots \quad (5.1)$$

* Vide reports issued by the Office of the Economic Adviser, Government of India.

TABLE 1. PRICE AND QUANTITY COMPONENTS

years	L_p	L_q	$100 V_{11}/V_{00}$	$100(V_{11}/V_{00}-1)$	π	κ	P	Q
(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)
1948–49	100.0	100.0	100.0	0.0	100.0	100.0	0.0	0.0
–50	103.0	102.0	104.2	4.2	102.6	101.6	2.6	1.6
–51	109.6	102.3	110.2	10.2	108.7	101.4	8.7	1.5
–52	116.3	105.2	115.3	15.3	113.1	102.0	13.2	2.1
–53	101.8	109.4	113.5	13.5	102.8	110.4	3.0	10.5
–54	106.3	116.0	121.2	21.2	105.4	115.1	5.8	15.4
–55	101.0	118.8	111.1	11.1	96.4	114.2	–3.4	14.5
–56	96.4	121.2	115.4	15.4	95.7	120.5	–4.7	20.1
–57	107.1	127.2	130.8	30.8	104.8	124.9	5.3	25.5
–58	110.3	125.2	131.3	31.3	107.4	122.3	8.2	23.1

The percentage difference as shown in Col. 5 is obviously the sum of the components in Cols. 8 and 9.

Table 1 shows that during the years, 1954-55 and 1955-56, the price component, that is, the contribution made by price to value change, has been negative, as the corresponding price index (adjusted) is less than 100. As a result, a part of the contribution made by quantity (output) to the increase in value of national income during those years has been masked by the depressing price effect.

6. Generalisation of two-factor index numbers

Index numbers for price (p) and quantity (q) involving two factors may be regarded as two-dimensional, as only two factors, price and quantity, are involved, e.g., Laspeyres', Paasche's or Fisher's index. Multidimensional index numbers involving more than two factors could also be constructed by a direct generalisation of (3.14) and (3.15). For example, if p, q, r be the factors observed each at two levels 0 and 1, the three dimensional index for p may be written as

$$\pi = \tfrac{1}{2}(L_{p/qr}-L_{qr/p})+\sqrt{[\tfrac{1}{2}(L_{p/qr}-L_{qr/p})]^2+R_{pqr}} \qquad \ldots \quad (6.1)$$

where

$$L_{p/qr} = V_{100}/V_{000}, \quad L_{qr/p} = V_{011}/V_{000}$$
$$R_{pqr} = V_{111}/V_{000}, \qquad \dots \quad (6.2)$$

the V's being defined by $$V_{jkl} = \sum_{i=1}^{n} p_j^i q_k^i r_l^i, \quad (j, k, l = 0, 1). \qquad \dots \quad (6.3)$$

Similarly, three dimensional indexes for q and r can be defined. A mention may perhaps be made of the work of Wisniewski (1931), Siegel (1945) and Gini (1937) who extended Fisher's two-factor formula to three or more factors. Siegel suggested an interesting application of the three-dimensional index, viz., the treatment of changes in wages as a result of changes in labour cost per unit of output, man-hour productivity and man-hours worked. Wisniewski gave indexes of price, area harvested, yield rate and crop value, where crop value was taken as the product of price (per bushel), area harvested (in acres) and yield rate (bushels per acre). Gini considered an example in anthropometry, i.e., in an analysis of variations in the volume of the chest in terms of the variations in antero posterior diameter, transverse diameter of the thorax and sternal height. The generalised formula which they used appear to be much more complicated in form than those suggested in (6.1) based on analogy of 2^m-factorial experiments.

In the case of three or more factors, however, we cannot generalise the result of (3.4); for instance, for three factors

$$V_{111} - V_{000} = P + Q + R + PQR \qquad \dots \quad (6.4)$$

i.e., the value change is not equal to sum of the main components unless it is known *a priori* that the second order interaction (PQR) is negligible. Similar difficulties arise in the case of four or more factors. Those difficulties could perhaps be got over and exactly similar formulae worked out by assuming that higher order interactions (that is, interactions higher than three factor interactions) are in practice not very important and could therefore, be ignored. Moreover, in economic applications such high order interactions do not appear to have any obvious physical interpretation.

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