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Brief Note

A note on solution of the dispersion equation for small-amplitude internal waves

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This note is concerned with establishing the nature of the roots of a dispersion equation, which arises in the study of small-amplitude internal waves in two immiscible superposed fluids, wherein the upper fluid has a free surface and the lower fluid has a rigid bottom. All roots of this dispersion equation are found by considering graphs of appropriate functions, and the fact that these are the only possible roots, has been established by Rouche's theorem of complex variable theory.

Key words: Dispersion equation, internal waves, Rouche's theorem.

1. Introduction

Various classes of small amplitude surface waves in water are usually being investigated in the literature within the framework of linearised theory of water waves for the last many decades (cf. Wehausen and Laitone [8]). As gravity is the only external force, these waves are also known as surface gravity waves. The relation between the wave number and the angular frequency of a surface gravity wave is known as the dispersion equation. This terminology is derived from the fact that waves of different wave lengths, propagating at different speeds, disperse, since the speed of wave propagation is the ratio of the angular frequency and the wave number. For infinitely deep water, if k is the wave number and ω is the angular frequency of a surface gravity wave, then the dispersion equation is given by (cf. Wehausen and Laitone [8], p. 472)

$$k = \frac{\omega^2}{g}\,(\equiv K)\,, \tag{1.1}$$

where g is the acceleration due to gravity. Equation (1.1) possesses the solution $k = K$, and this corresponds to the time-harmonic progressive surface waves represented by $\phi(x,y) = e^{-Ky\pm iKx}$, where $\mathrm{Re}\{\phi(x,y)e^{-i\omega t}\}$ is the velocity potential describing the two-dimensional motion in deep water occupying the position

$y \geq 0$, the y-axis being directed vertically downwards, the plane $z = 0$ being the mean free surface, and the x-direction being the direction of wave propagation.

For water of uniform finite depth h, Eq. (1.1) modifies to the transcendental equation (cf. WEHAUSEN and LAITONE [8], p. 474)

$$k \tanh kh = K. \tag{1.2}$$

It is well known that the transcendental equation (1.2) has real roots $\pm k_0$ ($k_0 > 0$) and countably infinite number of purely imaginary roots $\pm ik_n$ ($n = 1, 2, \cdots$) where $0 < k_1 < k_2 < \cdots$, and $k_n \to n\pi/h$ as $n \to \infty$. The positive real root k_0 corresponds to progressive surface waves with wave number k_0, while the purely imaginary roots correspond to evanescent modes. The fact that the Eq. (1.2) has no other roots except $\pm k_0$ and $\pm\, ik_n$ ($n = 1, 2, \ldots$), can be proved by employing Rouche's theorem of complex variable theory (cf. CHURCHILL *et al.* [1]) to the functions

$$f(k) = k \sinh kh - K \cosh kh, \qquad g(k) = k \sinh kh$$

within a square with vertices $k = \dfrac{(2m-1)\pi}{2n}(\pm 1 \pm i)$, m being a large positive integer, in the complex k-plane as was demonstrated by RHODES–ROBINSON [5] for the dispersion equation in which the effect of surface tension at the free surface was included.

For two superposed immiscible fluids separated by a common interface, the upper fluid extending infinitely upwards and the lower fluid extending infinitely downwards, the wave number k of small amplitude interface gravity waves or internal waves is related to the angular frequency ω by the dispersion relation (cf. WEHAUSEN and LAITONE [8], p. 647)

$$k = \sigma K, \tag{1.3}$$

where $\sigma = (1+s)/(1-s)$ with $s = \rho_2/\rho_1$ ($\rho_2 < \rho_1$), ρ_1, ρ_2 being the densities of the lower and upper fluids respectively. If the upper fluid is of uniform finite height h above the mean interface and has a free surface while the lower fluid extends infinitely downwards, then the corresponding equation is (cf. LINTON and MCIVER [4])

$$(k - K)\{k(\sigma + e^{-2kh}) - k(1 - e^{-2kh})\} = 0. \tag{1.4}$$

This equation has two real roots, one is K and the other is v say, where v satisfies the equation

$$K(\sigma + e^{-2vh}) = v(1 - e^{-2vh}), \tag{1.5}$$

so that

$$K\sigma < v < K\frac{\sigma+1}{1-e^{-2K\sigma h}}. \tag{1.6}$$

Thus there exist time-harmonic progressive waves with two different wave numbers $K,\ v$. An equivalent form of the Eq. (1.5) was in fact given earlier in Art. 231 of the treatise by LAMB [2] wherein a description of some of the types of wave motion which can occur in a two-layer fluid with both a free surface and an interface, was also mentioned.

Study of wave motion in a two-layer fluid has gained importance due to a plan to construct underwater pipe bridge across the Norwegian fjords. A fjord consists of a layer of fresh water over a layer of salt water. If the lower layer is very deep, then the two-layer fluid mentioned above models a fjord. For this type of two-layer fluid, problems of interaction of small amplitude waves on the free surface as well as on the interface have been investigated by LINTON and MCIVER [4], LINTON and CADBY [3]. If the aforesaid fjord is not very deep, then it can be modelled as a two-layer fluid wherein the lower fluid is of uniform finite depth H, say, below the mean interface, and as before, the upper fluid is of height h above the interface and has a free surface. In this case, the dispersion equation (1.4) modifies to

$$k^2(1-s) - kK(\coth kh + \coth kH) + K^2(s + \coth kh \coth kH) = 0. \tag{1.7}$$

This equation is given by SHERIEF *et al.* [6, 7] while investigating forced gravity waves due to a plane and a cylindrical vertical porous wave-maker in a two-layer fluid. They simply stated without proof that the Eq. (1.6) has two real positive roots, two real negative roots and an infinite number of purely imaginary roots of the form $\pm i\lambda_n \ \ (n = 1, 2, \cdots)$. The purpose of this note is to show that the roots of the Eq. (1.6) are indeed of this form and to establish that there are no other roots by employing the Rouche's theorem of complex variable theory (cf. CHURCHILL *et al.* [1]).

Thus establishing the nature of the roots of the transcendental Eq. (1.6) has some significance in the investigation of small-amplitude wave interaction problems in a two-layer fluid modelling a Norwegian fjord.

2. Solution of the dispersion equation

Let $\mu = H/h$ and $kh = z$, then Eq. (1.6) reduces to

$$f(z) = \coth \mu z, \tag{2.1}$$

where

$$f(z) = \frac{Khz\coth z - (1-s)z^2 - s(kh)^2}{Kh(Kh\coth z - z)}. \tag{2.2}$$

The real roots of Eq. (2.1) can be obtained graphically from plots of $y = \coth \mu x$ and $y = f(x)$. Theses plots are given in Figs. 1 to 3 for three values of μ, viz. $\mu = 1 (h = H), \mu = 0.7 (h > H)$ and $\mu = 1.7 (h < H)$. We note that the asymptotes of the curve $y = f(x)$ are $x = \pm k_0 h$, where $\pm k_0$ are the only two real roots of the Eq. (1.2) (i.e. real roots of the dispersion equation for a single fluid of depth h below its mean free surface). From the Figs. 1 to 3, it is obvious that the Eq. (1.6) has four real roots, two positive and two negative. If the two positive roots are denoted by m_1, m_2 then the negative roots are $-m_1, -m_2$, since Eq. (1.6) remains unchanged if k is replaced by $-k$. Also, if $m_1 < m_2$ then $m_1 < k_0 < m_2$. If we replace k by ik in (1.6), then it becomes, after writing $z = kh$,

$$g(z) = \cot \mu z, \tag{2.3}$$

where

$$g(z) = -\frac{Khz\cot z + (1-s)z^2 - s(Kh)^2}{Kh(Kh\cot z + z)}. \tag{2.4}$$

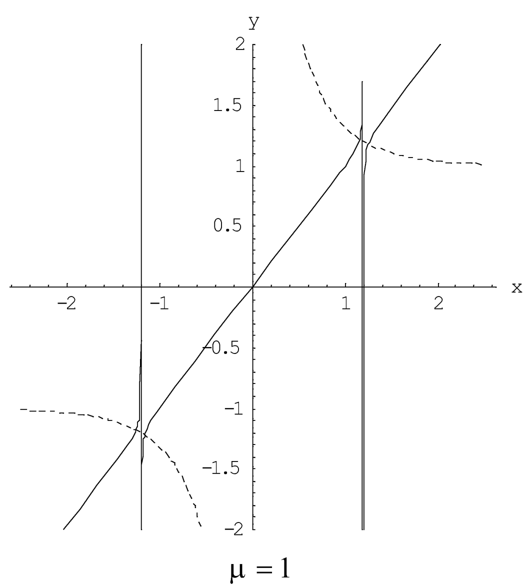
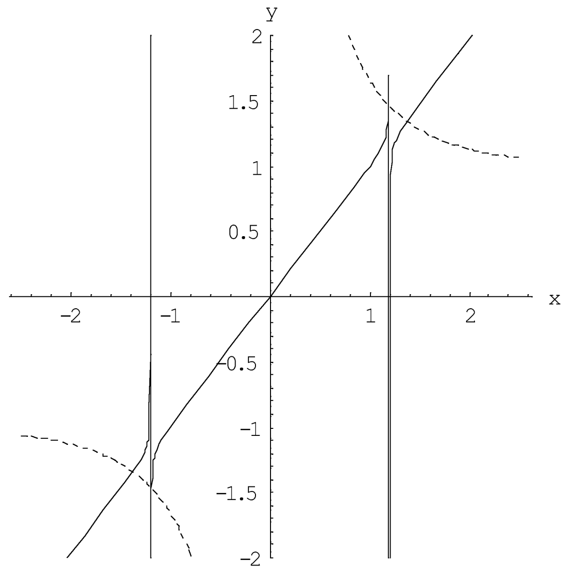
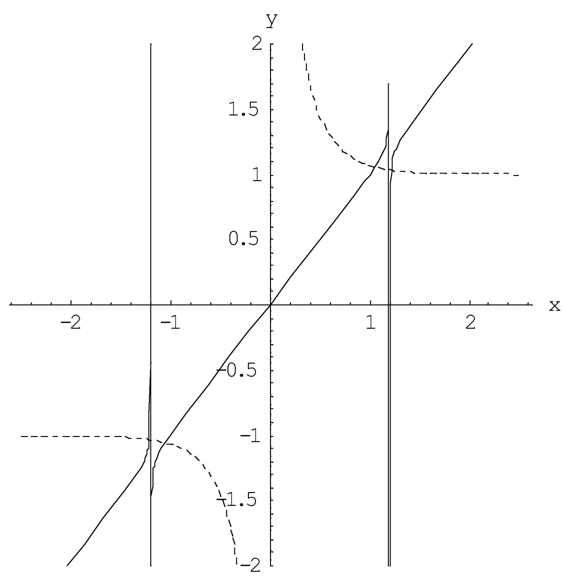


FIG. 1. – – – – – $y = \coth x$, ——— $y = \frac{xKh\coth x - (1-s)x^2 - s(Kh)^2}{Kh(Kh\coth x - x)}$



$\mu = 0.7$

FIG. 2. – – – – – $y = \coth(.7x)$, ——— $y = \frac{xKh \cot x - (1-s)x^2 - s(Kh)^2}{Kh(Kh \coth x - x)}$.



$\mu = 1.7$

FIG. 3. – – – – – $y = \cot(1.7x)$, ——— $y = \frac{xKh \cot x - (1-s)x^2 - s(Kh)^2}{Kh(Kh \coth x - x)}$.

Thus the purely imaginary roots of the Eq. (1.6) can be obtained graphically from the plots of the curves $y = \cot \mu x$ and $y = g(x)$. Let μ be expressed as $\mu = p/q$ where p and q are integers prime to each other but $p = q$ when $\mu = 1$. In Figs. 4 to 6, plots of $y = \cot \mu x$ and $y = g(x)$ are given for $\mu = 1, 0.7 (p = 7, q = 10), 1.7 (p = 17, q = 10)$. From these figures it is obvious that there exists an infinite number of purely imaginary roots of the Eq. (1.6) given by $\pm i\lambda_m$, $(m = 1, 2, \cdots)$. If m is a multiple of $(p+q)$, i.e. $m = n(p+q)$, say, then it is easy to see that $k_{nq} < \lambda_{n(p+q)} < nq\pi$, $(n = 1, 2, \cdots)$ and as n becomes large, $\lambda_{n(p+q)} \to nq\pi$ for any p.

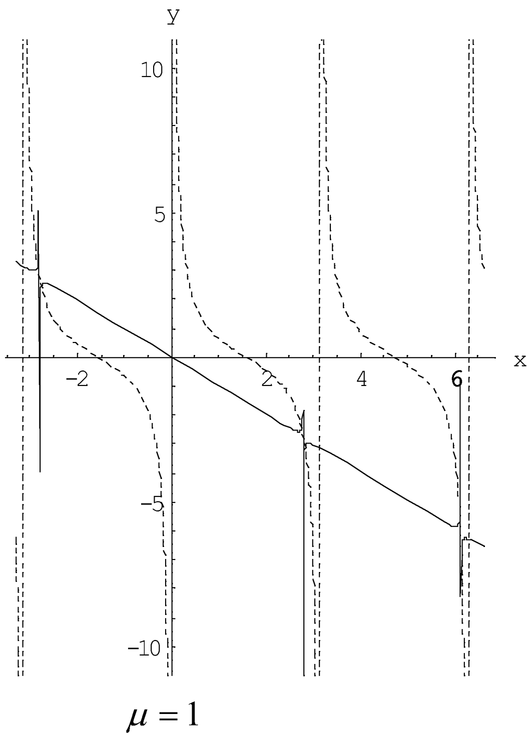
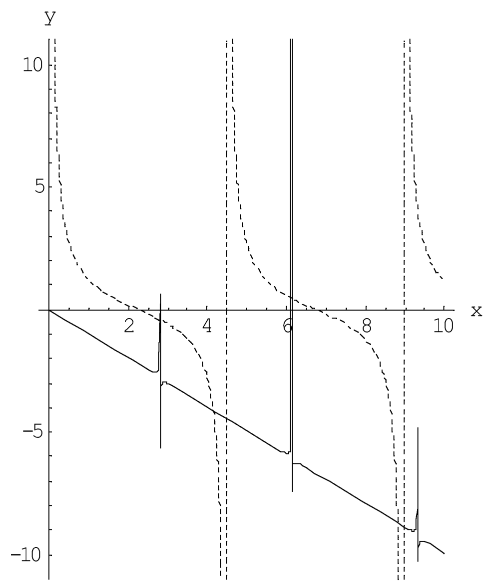


FIG. 4.

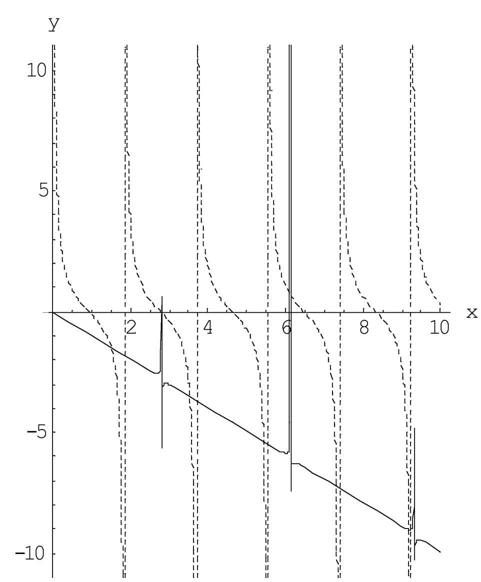
$- - - -$ $y = \cot x$, ———— $y = \frac{-xKh \cot x - (1-s)x^2 + s(kh)^2}{Kh(Kh \coth x + x)}$.

In the next section it will be shown by using Rouche's theorem that, apart from the aforesaid roots, there exist no other roots of the dispersion Eq. (1.6) in the complex k-plane.



$\mu = 0.7$

FIG. 5. – – – – – $y = \cot(0.7x)$, ——— $y = \frac{-xKh\cot x-(1-s)x^2+s(Kh)^2}{Kh(Kh\coth x+x)}$.



$\mu = 1.7$

FIG. 6. – – – – –y $= \cot(1.7x)$, ——— $y = \frac{-xKh\cot x-(1-s)x^2+s(Kh)^2}{Kh(Kh\coth x+x)}$.

3. Application of Rouche's theorem

Let us define

$$F(z) = (z - Kh\coth\mu z)(z - Kh\coth z), \tag{3.1}$$

$$G(z) = -s(z^2 - (Kh)^2), \tag{3.2}$$

where z is complex ($z = kh$). We consider the contour C of the square with vertices $(\pm(nq+\epsilon), \pm(nq+\epsilon))$ in the complex z-plane where n is a large integer and $\epsilon(>0)$ is sufficiently small. The contour C is chosen in such a way that it does not pass through any of the zeros of function $F(z)$. The equation $F(z)=0$ has four real roots and $2n(p+q)$ purely imaginary roots inside C. Therefore total number of roots of $F(z)=0$ inside C is $4+2n(p+q)$.

Now on the upper side of the square C,

$$z = x + i(nq+\epsilon)\pi,$$

where

$$-(nq+\epsilon)\pi \le x \le (nq+\epsilon)\pi.$$

Then on this side,

$$\left|\frac{F(z)}{G(z)}\right| = \left|\frac{[x+i(nq+\epsilon)\pi - Kh\coth(x+i(nq+\epsilon)\pi)]}{s[(x+i(nq+\epsilon)\pi)^2 - K^2]} \cdot \frac{[x+i(nq+\epsilon)\pi - Kh\coth\mu(x+i(nq+\epsilon)\pi)]}{1}\right| \tag{3.3}$$

$$= \left|\frac{\left[1 - \dfrac{Kh\coth(x+i(nq+\epsilon)\pi)}{n\left(\frac{x}{n} + i\left(q+\frac{\epsilon}{n}\right)\pi\right)}\right]\left[1 - \dfrac{Kh\coth\mu\left(x+i\left(nq+\epsilon\right)\pi\right)}{n\left(\frac{x}{n} + i\left(q+\frac{\epsilon}{n}\right)\pi\right)}\right]}{s\left[1 - \dfrac{K^2}{n^2\left(\frac{x}{n} + i\left(q+\frac{\epsilon}{n}\right)\pi\right)^2}\right]}\right|.$$

Thus as n is large, the value of $\left|\frac{F(z)}{G(z)}\right|$ on the upper side of C becomes $\frac{1}{s} > 1$.

Hence $|F(z)| > |G(z)|$ on the upper side of C. Similarly we can prove that $|F(z)| > |G(z)|$ on the other sides of the contour C. Therefore $|F(z)| > |G(z)|$ on the contour C.

Therefore by Rouche's theorem we find that $F(z)$ and $F(z)+G(z)$ have the same number of zeros inside C. This shows that the dispersion equation (1.6) has four real roots and an infinite number of purely imaginary roots and no other roots.

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