Effective mass Schrödinger equation and nonlinear algebras

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Abstract

Using supersymmetry we obtain solutions of Schrödinger equation with a position dependent effective mass exhibiting a harmonic oscillator like spectrum. We also discuss the underlying nonlinear algebraic symmetry of the problem.

1. Introduction

Quantum mechanical systems with a position dependent effective mass are useful in the study of various physical systems like electronic properties of semiconductors [1], quantum dots [2], liquid crystals [3], etc. and they have been studied widely during the last few years. Exact solutions of Schrödinger equations with an effective mass have also been obtained for a number of potentials [4].

Supersymmetry, on the other hand, has proved to be very useful in the study of quantum mechanical systems with a constant mass [5]. Recently it has been shown that supersymmetric methods are also useful in the study of effective mass Schrödinger equation and using supersymmetric methods new potentials with exactly known solutions have been obtained [6,7].

On the other hand nonlinear algebras arise in different contexts [8,9]. In particular it has been shown that in the case of constant mass Schrödinger equation there is a class of potentials (which are also supersymmetric partner of certain solvable potentials) have nonlinear algebraic symmetry [10,11]. Here we shall show that in the case of effective mass Schrödinger equation also there exists a similar class of exactly solvable potentials with nonlinear algebraic symmetry. More precisely we shall use supersymmetric techniques to obtain exact solutions of a potential which is isospectral with an effective mass harmonic oscillator. Subsequently it will be shown this potential has a nonlinear symmetry. We shall also briefly outline different methods of obtaining effective mass isospectral Hamiltonians. The organisation of the Letter is as follows: in Section 2, we present a brief review of the effective mass formalism for the harmonic oscillator; in Section 3 we construct the isospectral potential and discuss the symmetry underlying the model; finally Section 4 is devoted to conclusion.

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2. Harmonic oscillator in the effective mass formalism

When the mass depends on the position the kinetic energy operator can be constructed in three ways. In the present case we shall be following Levy-Leblond [12] and the Schrödinger equation is given by (we take $\hbar = 1$)

$$-\frac{d}{dx}\left(\frac{1}{2m(x)}\frac{d\psi(x)}{dx}\right) + V(x)\psi(x) = E\psi(x). \qquad (1)$$

The wave function $\psi(x)$ should be continuous across the mass discontinuity and it's derivative satisfies the condition:

$$\frac{1}{m(x)}\frac{d\psi(x)}{dx}\bigg|_{-} = \frac{1}{m(x)}\frac{d\psi(x)}{dx}\bigg|_{+}. \qquad (2)$$

We shall now obtain a potential $V(x)$ for which the spectrum is that of the standard harmonic oscillator. To do this it is necessary to factorise Eq. (1) and we consider the following operators [6]

$$A\psi = \frac{1}{\sqrt{2m}}\frac{d\psi}{dx} + W\psi, \qquad (3)$$

$$A^{\dagger}\psi = -\frac{d}{dx}\left(\frac{\psi}{\sqrt{2m}}\right) + W\psi, \qquad (4)$$

where A, $A^{\dagger}$ are the supercharges and the function $W(x)$ is called the superpotential [5]. Then we form the supersymmetric pair of isospectral Hamiltonians $H_{\pm}$:

$$\begin{aligned} H_{+} &= AA^{\dagger} \\ &= -\frac{1}{2m}\frac{d^2}{dx^2} - \left(\frac{1}{2m}\right)'\frac{d}{dx} - \left(\frac{W}{\sqrt{2m}}\right)' + W^2 \\ &\quad + \frac{2W'}{\sqrt{2m}} - \left(\frac{1}{\sqrt{2m}}\right)\left(\frac{1}{\sqrt{2m}}\right)'', \end{aligned} \qquad (5)$$

$$\begin{aligned} H_{-} &= A^{\dagger}A \\ &= -\frac{1}{2m}\frac{d^2}{dx^2} - \left(\frac{1}{2m}\right)'\frac{d}{dx} - \left(\frac{W}{\sqrt{2m}}\right)' + W^2. \end{aligned} \qquad (6)$$

The corresponding potentials $V_{\pm}$ are given by

$$\begin{aligned} V_{+} &= -\left(\frac{W}{\sqrt{2m}}\right)' + W^2 \\ &\quad + \frac{2W'}{\sqrt{2m}} - \left(\frac{1}{\sqrt{2m}}\right)\left(\frac{1}{\sqrt{2m}}\right)', \end{aligned} \qquad (7)$$

$$V_{-} = -\left(\frac{W}{\sqrt{2m}}\right)' + W^2. \qquad (8)$$

The spectral properties of the pair of Hamiltonians $H_{\pm}$ are given by (we assume that $E_0^{(-)} = 0$)

$$E_{n+1}^{(-)} = E_n^{(+)}, \qquad (9)$$

$$\begin{aligned} \psi_n^{(+)} &= \frac{1}{\sqrt{E_{n+1}^{(-)}}}A\psi_{n+1}^{(-)}, \\ \psi_{n+1}^{(-)} &= \frac{1}{\sqrt{E_n^{(+)}}}A^{\dagger}\psi_n^{(+)}. \end{aligned} \qquad (10)$$

We now turn to the construction of effective mass harmonic oscillator potentials. From Eqs. (5) and (6) it follows that

$$[A, A^{\dagger}] = \frac{2W'}{\sqrt{2m}} - \left(\frac{1}{\sqrt{2m}}\right)\left(\frac{1}{\sqrt{2m}}\right)''. \qquad (11)$$

Thus in order that $A^{\dagger}$ and A can be used in the same way as the standard harmonic oscillator creation and annihilation operator, respectively, we should have $[A, A^{\dagger}] = 1$. Then from Eq. (11) we get the following equation involving the superpotential $W(x)$ and the mass $m(x)$:

$$\frac{2W'}{\sqrt{2m}} - \left(\frac{1}{\sqrt{2m}}\right)\left(\frac{1}{\sqrt{2m}}\right)'' = 1. \qquad (12)$$

Eq. (12) can be integrated to give $W(x)$:

$$W(x) = \left(\frac{1}{2m}\right)' + \int^{x} \sqrt{2m(t)}\,dt. \qquad (13)$$

Thus for a given mass $m(x)$ the superpotential $W(x)$ can be determined from (13) and in this case the spectrum of $H_{\pm}$ are given by $E_{\pm} = (n + \frac{1}{2} \pm \frac{1}{2})$. The corresponding eigenfunctions can be obtained using Eq. (10). However it may be noted that due to Eq. (12) the wave functions may simply be obtained as $\psi_n^{-}(x) \sim (A^{\dagger})^n \psi_0^{-}(x)$.

3. Isospectral effective mass Hamiltonians and nonlinear algebras

In this section we shall construct Hamiltonians which are isospectral with $H_{\pm}$ and obtain the underlying nonlinear algebra. At this point, however, it is necessary to specify the form of the mass $m(x)$. Here we take the mass to be of the following form [6]

$$m(x) = \left(\frac{\alpha + x^2}{1 + x^2}\right)^2,$$
$$m(0) = \alpha^2,$$
$$m(\pm\infty) = 1. \qquad (14)$$

With the above choice of the mass function the superpotential can be obtained from Eq. (13) and is given by

$$W(x) = \frac{x}{\sqrt{2}} + \frac{\alpha - 1}{\sqrt{2}}\left[\tan^{-1} x + \frac{x}{(\alpha + x^2)^2}\right]. \qquad (15)$$

The specific form of the effective mass harmonic oscillator potential can now be obtained from Eqs. (7) and (8):

$$V_\pm(x) = \frac{1}{2}\left[x + (\alpha - 1)\tan^{-1} x\right]^2 + \frac{\alpha}{2(\alpha + x^2)^4}\left[3x^4 + (4 - 2\alpha)x^2 - \alpha\right] \pm \frac{1}{2}. \qquad (16)$$

Let us now consider a new superpotential $W_1(x)$ given by

$$W_1(x) = W(x) + v(x), \qquad (17)$$

where the function $v(x)$ will be determined later. The corresponding supersymmetric partner potential $U_+(x)$ is given by

$$U_+(x) = W_1^2 - \left(\frac{W_1}{\sqrt{2m}}\right)' + \frac{2W'}{\sqrt{2m}} - \left(\frac{1}{\sqrt{2m}}\right)\left(\frac{1}{\sqrt{2m}}\right)''. \qquad (18)$$

We now require that the potentials $V_+(x)$ and $U_+(x)$ are related by

$$U_+(x) = V_+(x) + \mu, \qquad (19)$$

where μ is a certain constant. The condition (19) implies that the spectrum (and eigenfunctions) of the potential $U_+(x)$ is the same as that of $V_+(x)$ except that the energy levels of $U_+(x)$ are shifted by the constant μ. Now using Eqs. (7), (17) and (18) we obtain from (19) a differential equation for $v(x)$:

$$\left(\frac{1}{\sqrt{2m}}\right)\frac{dv}{dx} + \left[2W - \left(\frac{1}{\sqrt{2m}}\right)'\right]v + v^2 = \mu. \qquad (20)$$

It is now necessary to solve Eq. (20) to determine the superpotential $W_1(x)$. In order to do this we choose the following ansatz for $v(x)$:

$$v(x) = \frac{2\delta f'(x)}{1 + \delta f^2(x)}, \qquad (21)$$

where δ is a constant (to be determined later) and the function $f(x)$ is given by

$$f(x) = \int^x \sqrt{2m(t)}\, dt. \qquad (22)$$

It can now be shown that with $v(x)$ given by (21), Eq. (20) will be satisfied if we choose $\delta = 1$ and $\mu = 2$. Thus in this case we have

$$U_+(x) = V_+(x) + 2 \qquad (23)$$

and if we denote the energy levels corresponding to $U_+(x)$ by $\mathcal{E}_n^+$ then

$$\mathcal{E}_n^+ = (n + 3), \quad n = 0, 1, 2, \ldots. \qquad (24)$$

We now consider the supersymmetric partner potential $U_-(x)$. This is given by

$$U_-(x) = W_1^2(x) - \left(\frac{W_1}{\sqrt{2m}}\right)' \neq V_-(x). \qquad (25)$$

By supersymmetry $U_-(x)$ has the same spectrum as $U_+(x)$ except for the zero energy state (see Eq. (9)). So denoting the energy levels of $U_-(x)$ by $\mathcal{E}_n^-$ we have

$$\mathcal{E}_0^- = 0, \qquad \mathcal{E}_{n+1}^- = \mathcal{E}_n^+ = (n + 3). \qquad (26)$$

The complete form of $U_-(x)$ is given by

$$U_-(x) = V_-(x) + \frac{4}{[1 + 2(x + (\alpha - 1)\tan^{-1} x)^2]} - \frac{8}{[1 + 2(x + (\alpha - 1)\tan^{-1} x)^2]^2} + 2. \qquad (27)$$

Let us now determine the symmetry of the potential $U_-(x)$. To do this we first have to obtain operators which would connect different levels of $U_-(x)$. Thus we define two operators $B^\dagger$ and B in the following way:

$$B^\dagger = A_1^\dagger A^\dagger A_1, \qquad B = A_1^\dagger A A_1, \qquad (28)$$

where the supercharges $A_1^\dagger$ and A_1 are given by

$$A_1\phi = \frac{1}{\sqrt{2m}}\frac{d\phi}{dx} + W_1\phi,$$

$$A_1^\dagger\phi = -\frac{d}{dx}\left(\frac{\phi}{\sqrt{2m}}\right) + W_1\phi. \tag{29}$$

The operators in Eq. (29) connects the eigenfunctions in the same way as in Eq. (10) with the eigenfunctions $\psi_n^\pm$ and energies $E_n^\pm$ replaced by $\phi_n^\pm$ and $\mathcal{E}_n^\pm$, respectively.

We now examine the action of the operators $B^\dagger$ and B on the eigenfunctions $\phi_n^\pm$ of the Hamiltonians $\mathcal{H}_\pm$:

$$\mathcal{H}_+ = -\frac{1}{2m}\frac{d^2}{dx^2} - \left(\frac{1}{2m}\right)'\frac{d}{dx} + U_+(x),$$
$$\mathcal{H}_- = -\frac{1}{2m}\frac{d^2}{dx^2} - \left(\frac{1}{2m}\right)'\frac{d}{dx} + U_-(x). \tag{30}$$

It may be noted that because of the condition (12) we have

$$A\psi_n^+ = \sqrt{n}\psi_{n-1}^+, \qquad A^\dagger\psi_n^+ = \sqrt{n+1}\psi_{n+1}^+. \tag{31}$$

Now using Eqs. (10) and (31) we get

$$\begin{aligned} B^\dagger\phi_{n+1}^- &= \sqrt{\mathcal{E}_{n+1}^+(n+1)\mathcal{E}_{n+1}^-}\phi_{n+2}^- \\ &= \sqrt{(n+2)(n+3)(n+4)}\phi_{n+2}^-, \\ B\phi_{n+1}^- &= \sqrt{\mathcal{E}_{n-1}^+ n\mathcal{E}_{n+1}^-}\phi_n^- \\ &= \sqrt{n(n+2)(n+3)}\phi_n^-. \end{aligned} \tag{32}$$

Thus the ground state of $U_-(x)$ can be obtained from the equation $A_1\phi_0^-(x) = 0$ while the excited states can be built up by repeated application of the creation operator $B^\dagger$ on $\phi_1^-(x)$:

$$\begin{aligned} \phi_{n+1}^-(x) &= (B^\dagger)^n\phi_1^-(x) \\ &= \sqrt{\frac{2!3!}{n!(n+2)!(n+3)!}}\phi_1^-(x). \end{aligned} \tag{33}$$

From (32) it follows that $\mathcal{H}_-$, B and $B^\dagger$ satisfy the following closed algebra:

$$\begin{aligned} [\mathcal{H}_-, B] &= -B, \\ [\mathcal{H}_-, B^\dagger] &= B^\dagger, \\ [B, B^\dagger] &= 3\mathcal{H}_- - 5\mathcal{H}_-^2. \end{aligned} \tag{34}$$

Thus unlike $U_+(x)$ or $V_\pm(x)$ the symmetry algebra underlying the potential $U_-(x)$ is a nonlinear (quadratic) one and is given by (34). However, since the ground state of $\mathcal{H}_-$ is annihilated by both B and $B^\dagger$ the algebra in (34) is realised over the excited states.

4. Conclusion

Here we have obtained a new potential $U_-(x)$ for which the effective mass Schrödinger equation is exactly solvable. It has also been shown that the underlying symmetry algebra is a quadratic one. This is similar to the situation in the constant mass case [10]. It may be noted that it is possible to proceed in a different way to obtain isospectral Hamiltonians. For example one could consider the equation $-(\frac{W}{\sqrt{2m}})' + W^2 = -(\frac{W_1}{\sqrt{2m}})' + W_1^2$ and solve it for W_1. Subsequently the symmetry behaviour of $U^+(x)$ may be studied. In this connection we note that exactly solvable effective mass Schrödinger equations (in particular the harmonic oscillator system, see Eq. (16)) can also be obtained using the point canonical transformations [13, 14]. We feel it would be interesting to analyse the symmetry of these exactly solvable models.

References

[1] G. Bastard, Wave Mechanics Applied to Semiconductor Heterostructures, Les Editions de Physique, Les Ulis, 1988.
[2] L.I. Serra, E. Lipparini, Europhys. Lett. 40 (1997) 667.
[3] M. Barranco, M. Pi, S.M. Gatica, E.S. Hernandez, J. Navarro, Phys. Rev. B 56 (1997) 8997.
[4] L. Dekar, L. Chetouani, T.F. Hammann, Phys. Rev. A 59 (1999) 107;
L. Dekar, L. Chetouani, T.F. Hammann, J. Math. Phys. 39 (1998) 2551;
A. de Souza Dutra, C.A.S. Almeida, Phys. Lett. A 275 (2000) 25;
B. Roy, P. Roy, J. Phys. A 35 (2002) 3961;
R. Koc, M. Koca, J. Phys. A 36 (2003) 8105;
R. Koc, M. Koca, E. Korcuk, J. Phys. A 35 (2002) L1;
C. Quesne, V.M. Tkachuk, J. Phys. A 37 (2004) 4267;
B. Bagchi, A. Banerjee, C. Quesne, V.M. Tkachuk, quant-ph/0412046;
G. Chen, Z. Chen, Phys. Lett. A 331 (2004) 312;
J. Yu, S.H. Dong, Phys. Lett. A 325 (2004) 194.
[5] F. Cooper, A. Khare, U. Shukhatme, Phys. Rep. 251 (1995) 267.
[6] A.R. Plastino, A. Rigo, M. Casas, F. Gracias, A. Plastino, Phys. Rev. A 60 (1999) 4398.
[7] V. Milanović, Z. Iković, J. Phys. A 32 (1999) 7001.
[8] Ya.I. Granovskii, I.M. Lutzenko, A.S. Zhedanov, Ann. Phys. 217 (1992) 1.
[9] J. de Boer, F. Harmsze, T. Tjin, Phys. Rep. 272 (1996) 139.
[10] G. Junker, P. Roy, Phys. Lett. A 232 (1997) 155.
[11] D.J. Fernandez C., V. Hussin, J. Phys. A 32 (2000) 3603.
[12] J.-M. Levy-Leblond, Phys. Rev. A 52 (1995) 1845.
[13] A.D. Alhaidary, Phys. Rev. A 66 (2002) 042116.
[14] B. Roy, P. Roy, quant-ph/0106028.