A simple test of exponentiality against NWBUE family of life distributions

M. Z. Anis[1,*,†] and M. Mitra[2]

[1] *Indian Statistical Institute, Kolkata, India*
[2] *B. E. College (Deemed University), Howrah, India*

SUMMARY

In this paper we propose a simple procedure to test the null hypothesis of exponentiality against the alternative that it belongs to the new worse then better than used in expectation (NWBUE) family. The test is shown to be consistent and the asymptotic distribution of the test statistic has been obtained. The performance of the test against various classes of alternatives has been studied by means of simulation.

KEY WORDS: hypothesis testing; NWBUE distribution; exponential distribution; characterization theorem; asymptotic normality

1. INTRODUCTION

The exponential distribution has been widely used in the theory of reliability and life testing. Typically, when one is dealing with failure times of items such as fuses, transistors, bulbs, etc., where failure is caused due to sudden shocks rather than wear and tear, the assumption of exponentiality is particularly justified. Testing for exponentiality of the failure time is, in effect, the same as testing whether the shocks responsible for the failure arrive according to a Poisson process.

Equally important in reliability theory is the concept of ageing. '*No ageing*' means the age of the component has no effect on the distribution of its residual lifetime. '*Positive ageing*' means that age has an adverse effect, in some probabilistic sense, on the residual lifetime. '*Negative ageing*' describes the opposite beneficial effect of ageing. If the same type of ageing persists throughout the entire lifetime of a unit, it is referred to as '*monotonic ageing*'.

However, in many practical situations, e.g. in many biological and mechanical systems, the effect of ageing appears to be non-monotonic. In biological systems, the newly born are more vulnerable to diseases and grow stronger after an initial 'burn-in' phase. Hence, the population experiences negative ageing (beneficial effect) initially. After a certain age, the trend reverses; old

*Correspondence to: M. Z. Anis, Indian Statistical Institute, SQC-OR Training & Promotional Unit, 203 Barrackpore Trunk Road, Kolkata, 700108 India.
† E-mail: zafar@isical.ac.in

age, fatigue set in and the units once again show an increase in their vulnerability. This phase of positive ageing (adverse effect) continues till the end. Similarly, mechanical systems (e.g. car, boiler, vessel, aeroplane) initially tend to become more reliable as they enter into service because of the coherent property among their components; but after some time point they tend to get weaker than before so that they are less reliable than a new one.

In contrast to monotone ageing, the modelling of non-monotonic ageing is a comparatively more recent phenomenon. If the ageing profile of a unit is studied through its failure rate function, then a typical approach to modelling non-monotonic ageing is through bathtub failure rate (BFR) models. There is a large body of literature dealing with BFR distributions, see for example References [1–6]. Another approach to model non-monotonic ageing uses the notion of mean residual life (MRL) function.

Definition 1.1

The MRL function at age $t > 0$ is defined as

$$m_F(t) = \begin{cases} E(X - t/X > t) & \text{for } \bar{F}(t) > 0 \\ 0 & \text{otherwise} \end{cases}$$

where F is a life distribution with finite first moment.

Guess *et al*. [7] have considered the MRL as a criterion of ageing and defined the following class of non-monotone ageing distribution.

Definition 1.2 (Guess et al. [7])

A life distribution F with a finite first moment is called an *increasing then decreasing mean residual life* (IDMRL) distribution if there exist a turning point $\tau \geqslant 0$ such that

$$m_F(s) \leqslant m_F(t) \quad \text{for } 0 \leqslant s \leqslant t < \tau$$

$$m_F(s) \geqslant m_F(t) \quad \text{for } \tau \leqslant s \leqslant t < \infty$$

The problem of testing the null hypothesis of exponentiality against alternatives belonging to a hierarchy of monotonic ageing classes has been treated extensively in the literature. The most familiar ageing families considered thus far in this context are increasing failure rate (IFR); increasing failure rate average (IFRA): new better than used (NBU): new better than used in expectation (NBUE) and harmonic new better than used in expectation (HNBUE). For definitions and further details see References [8, 9]. It is easy to show (see Reference [10]) that these families satisfy

$$\{\text{IFR}\} \subset \{\text{IFRA}\} \subset \{\text{NBU}\} \subset \{\text{NBUE}\} \subset \{\text{HNBUE}\}$$

Test procedures for IFR and IFRA alternatives are available in plenty in the literature, see for example References [11–15]. Hollander and Proschan [16] developed tests for decreasing mean residual life (DMRL) alternatives. Hollander and Proschan [17] considered the problem of testing exponentiality against NBU alternatives, and proposed a U-statistic J_n which has certain desirable properties based on a random sample of size n. Koul [18] proposed a class of tests based on scores $\{S_{ij}, 1 \leqslant i \leqslant j \leqslant n\}$ for the same problem. Koul [19] showed that a class of tests which are usually believed to be the tests of exponentiality vs IFR alternatives are in fact consistent against a much larger class of NBUE alternatives. Borges *et al.* [20] proposed another consistent test for testing exponentiality against NBUE alternatives. By appealing to Corollary 2.5 of Basu and Bhattacharjee [21], the test procedure of Borges *et al.* [20] can be extended to the HNBUE family, and this would result in a test which is consistent against the larger class of alternatives as well. This has already been observed by Kelfsjö [22] and Alzaid *et al.* [23]. Klefsjö [9] used an approach involving linear functions of order statistics to test exponentiality against HNBUE alternatives.

In contrast to the above, inference problems concerning non-monotonic ageing distributions have received comparatively less attention. The problem of estimating the change point of a bathtub model has been considered by Basu *et al.* [24] and Kulasekera and Lal Saxena [25]. Mitra and Basu [26] suggested strongly consistent estimators for change points in the context of BFR and IDMRL families of life distributions. The problem of testing exponentiality against the BFR-property has been investigated, among others, by Aarset [27]. Guess *et al.* [7] proposed test procedures for testing exponentiality against IDMRL alternatives. Kelfsjö [28] considered the problem of testing exponentiality against the situation when the life distribution changes from the NBUE to the NWUE-property. Our objective in this paper is to test exponentiality against the new worse then better than used in expectation (NWBUE) family of life distributions, introduced by Mitra and Basu [29]. It is worth mentioning that the NWBUE family includes all IDMRL and BFR life distributions.

The main contribution of this paper lies in proposing a test procedure to test the null hypothesis of exponentiality against the NWBUE family of alternatives. Our main objective has been to construct a relevant test statistic for the above testing problem and obtain the asymptotic distribution of the test statistic introduced. We have further proved that our test is consistent. These have been addressed in Section 3.

2. THE AGEING CLASS NWBUE

Definition 2.1 (Mitra and Basu [29])

A life distribution F having support on $[0, \infty)$ and finite mean μ is said to be NWBUE if there exists a point $x_0 \geqslant 0$ such that

$$m_F(x) = \begin{cases} \geqslant m_F(0) & \text{for } x < x_0 \\ \leqslant m_F(0) & \text{for } x \geqslant x_0 \end{cases}$$

where $m_F(\cdot)$ is the mean residual life function defined in Definition 1.1.

The point x_0 (which need not be unique) is referred to as a *change point* of the distribution function F. We shall write F is NWBUE (x_0) to denote that the life distribution F is NWBUE and x_0 is a change point of F.

Mitra and Basu [29] showed that the NWBUE family is a substantially large class in the sense that it contains the IDMRL class of distributions introduced by Guess *et al.* as well as all BFR distributions. Note that the NBUE family is also a special case of NWBUE distributions; here the change point is $x_0 = 0$, which is finite. Hence our test procedure is for testing against a wide class of alternatives. A rejection of the alternative hypothesis would imply, a fortiori, a rejection of BFR and IDMRL alternatives.

Note that for the exponential distribution, every point is a change point. However, we shall make the convention that for the exponential distribution the change point is $x_0 = 0$.

Mitra and Basu [29] have shown that the rth moment of F is dominated by the rth moment of G, where F is NWBUE (x_0), $x_0 < \infty$, with finite mean μ and G given by

$$G(x) = \begin{cases} 0 & x < x_0 \\ 1 - e^{-(x-x_0)/\mu} & x \geqslant x_0 \end{cases} \tag{1}$$

They obtain an interesting characterization of the exponential distribution within the family of NWBUE distributions. They have proved the following theorem.

Theorem 2.1 (Mitra and Basu [29])

Let F be NWBUE(x_0), $x_0 < \infty$, with finite mean μ and let G be as in Equation (1). Then F is the exponential distribution if $E_F X^r = E_G X^r$ for *some* $r > 1$.

It may be mentioned that Hawkins and Kochar [30] considered the problem of inference about the transition point in NBUE–NWUE or NWUE–NBUE models. The NWUE–NBUE class is essentially the NWBUE family introduced by Mitra and Basu [29]. The test statistic introduced by Hawkins and Kochar [30] is complicated and its asymptotics rely heavily on the theory of the empirical scaled total time on test process. In contrast, we develop a simple test statistic based on a characterization result (Theorem 2.1 above) in Reference [29]. We subsequently establish the asymptotic normality of the test statistic thus obtained and also prove the consistency of our test.

3. HYPOTHESIS TESTING

Let $\mathscr{E}$ be the class of exponential distributions, with distribution function

$$H(x) = 1 - e^{-\mu x}, \quad x \geqslant 0$$

where μ is any positive number (typically unknown).

Formally our problem is to test

$$H_0 : F \quad \varepsilon \quad \mathscr{E}$$
$$\text{vs } H_1 : F \quad \text{is} \quad \text{NWBUE}(x_0) - \mathscr{E}$$

It is clear from Theorem 2.1 that if $E_F X^r = E_G X^r$ for some $r > 1$, then F is necessarily exponential. So we may look at *any* $r > 1$, and develop a measure of deviation from exponentiality for NWBUE life distribution as follows:

Define

$$\Delta_r(F) := E_F X^r - E_G X^r$$

where $G(x)$ is as Theorem 2.1 above.

Since $E_F X^r \leqslant E_G X^r$, $\forall r \geqslant 1$ (see Corollary 3.5 of Mitra and Basu [29]), we have, using Theorem 2.1,

$$\Delta_r(F) = 0$$

if and only if F is exponential. Hence Δ_r (F) provides a measure of departure of F from the exponential distribution. We now replace the relevant moment of F by the corresponding sample moment to obtain a test statistic.

Hence, for the purpose of testing, we would essentially be examining how close $E_{F_n} X^r := (1/n)\sum_{i=1}^n X_i^r$ is to $E_G X^r$ for some $r > 1$. By virtue of Corollary 3.5 of Mitra and Basu [29] we know that the iid sequence X_1^r, X_2^r, $X_3^r, \ldots,$ has finite second moment. Hence, by appealing to the Central Limit Theorem, we get

$$\frac{\sqrt{n}((1/n)\sum_{j=1}^n X_j^r - E_F X^r)}{\sqrt{\mathrm{Var}_F(X^r)}} \xrightarrow{\mathrm{d}} N(0,1) \quad \text{as } n \to \infty$$

Under the null hypothesis $H_0 : F$ is exponential, we then have

$$\frac{\sqrt{n}((1/n)\sum_{j=1}^n X_j^r - \mu^r \Gamma(r+1))}{\mu^r \sqrt{[\Gamma(2r+1) - \{\Gamma(r+1)\}^2]}} \xrightarrow{\mathrm{d}} N(0,1) \quad \text{as } n \to \infty$$

i.e.

$$\frac{\sqrt{n}((1/n)\sum_{j=1}^n (X_j/\mu)^r - \Gamma(r+1))}{\sqrt{[\Gamma(2r+1) - \{\Gamma(r+1)\}^2]}} \xrightarrow{\mathrm{d}} N(0,1) \quad \text{as } n \to \infty$$

Now, we can replace μ by its consistent estimator $\bar{X}_n = (1/n)\sum_{j=1}^n X_j$, and using Slutsky's theorem obtain the following result:

Under H_0,

$$T_n = \frac{\sqrt{n}((1/n)\sum_{j=1}^n (X_j/\bar{X}_n)^r - \Gamma(r+1))}{\sqrt{[\Gamma(2r+1) - \{\Gamma(r+1)\}^2]}} \xrightarrow{\mathrm{d}} N(0,1) \quad \text{as } n \to \infty$$

Then, as the rth moment of F is dominated by the rth order moment of G, where F is NWBUE (x_0), $x_0 < \infty$, with finite mean μ and G is as in Equation (1), large negative values of the statistic would tend to reject the null hypothesis of exponentiality. More formally, a large sample level α-test rejects H_0 in favour of the NWBUE property if $T_n \leqslant -Z_\alpha$, the αth quantile of the standard normal distribution. Hence we have the following theorem:

Theorem 3.1

Under H_0, for $r > 1$, we have,

$$\sqrt{n}\left(\frac{1}{n}\sum_{j=1}^{n}\left(\frac{X_j}{\bar{X}_n}\right)^r - \Gamma(r+1)\right) \xrightarrow{\mathrm{d}} N(0,\sigma^2) \quad \text{as } n \to \infty$$

where $\sigma^2 = [\Gamma(2r+1) - \{\Gamma(r+1)\}^2]$.

Theorem 3.2

If F is NWBUE (x_0), $x_0 < \infty$, the test that rejects H_0 for large negative values of T_n is consistent.

Proof

This follows easily from the fact that $E_F X^r < E_G X^r$ for $r > 1$, where G is as in Equation (1). □

Remark

It is easy to see that the proposed test is scale-invariant.

4. A SIMULATION STUDY

In this section we do a simulation study to evaluate the performance of our test against various alternatives. We investigate how the performance varies with the parameter r; more precisely we study what values of r give better performance of the test for different alternatives. The simulation was done using IMSL library compiled on Sun platform.

We simulate observations from an exponential distribution and compute the test statistic T_n, for different large sample sizes ($n > 30$). Since the test is scale-invariant, we can take the scale parameter λ to be unity, without loss of generality, while performing the simulations. We check if this particular realization of the test statistic accepts or rejects the null hypothesis of exponentiality. (Naturally we would, in this case, expect H_0 to be accepted!). Next we repeat the whole procedure a suitably large number of times, say for $N = 5000$ times and observe the proportion of times the proposed test statistic takes the correct decision (of *not* rejecting H_0). This procedure has been repeated for different values of r, $r > 1$. It is observed that all the statistics $\{T_{n,r} : r > 1\}$ perform very well. Next, we do the same exercise using Weibull and Gamma distributions, since Weibull and Gamma are members of the class NWBUE $(x_0) - \mathscr{E}$. This time we observe the proportion of times our proposed test statistic rejects the null hypothesis (i.e. takes the correct decision). Thus we estimate the power of the test in different situations.

We have also investigated the power of the proposed test against the standard lognormal alternatives. It is well known that the lognormal distribution has an upside-down bathtub-shaped hazard rate (see for example Reference [31]) in contrast to the monotonic hazard rates of gamma and Weibull distributions. As before we observe the proportion of times our proposed test statistic rejects the null hypothesis (i.e. takes the correct decision). Thus we estimate the power of the proposed test.

The salient features observed based on the simulation studies are mentioned in the remarks given below.

Remark 1

The simulation study clearly indicates that the proposed test always accepts the null hypothesis if the data belong to the exponential distribution.

Remark 2

When the shape parameter α (for a Weibull or gamma alternative) is slightly different from one, it is expected that the test statistic would behave rather poorly; as it could easily confuse the observations to be coming from an exponential distribution ($\alpha = 1$).

Remark 3

For gamma alternatives, the test seems to perform excellently for moderate values of r in the range [2.1, 2.7]; while for Weibull alternatives, the test seems to perform best when $r \in [2.2, 2.6]$. For log-normal alternatives, the test seems to perform best when $r \in [2.1, 2.6]$. Thus a fairly conservative approach would be to choose a value of r close to 2.5 say in the range (2.4, 2.5) while performing the test. This may be taken as a general guideline.

Remark 4

It appears that the test statistic does not perform well for larger values of r as higher order moments are known to be unstable.

It should be noted that all the tests were performed at the 5 percent level of significance. Based on the simulation results, Figures 1–3 present graphs of the empirical power for different alternatives for selected values of the parameter r.

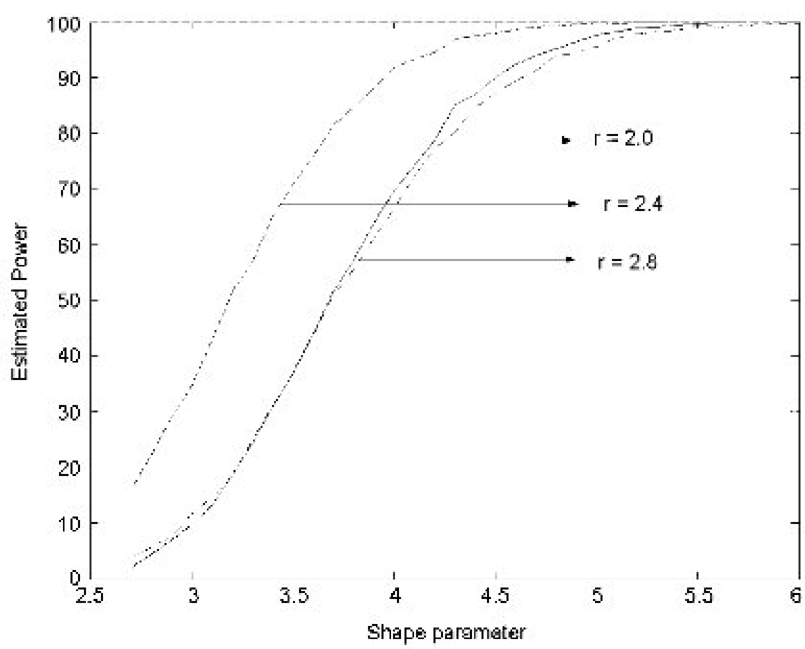

Figure 1. Plot of estimated power against shape parameter for Gamma distribution for specified values of r ($n = 100$).

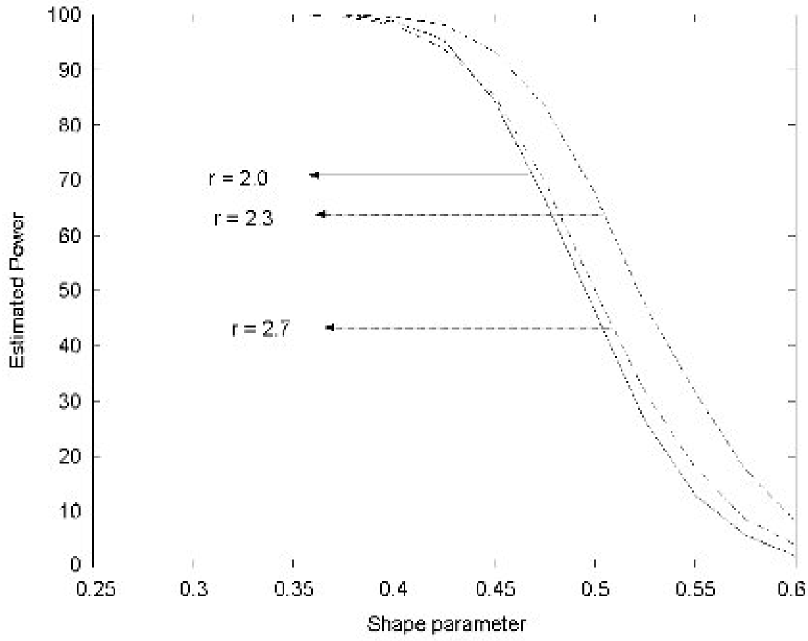

Figure 2. Plot of estimated power against shape parameter for lognormal distribution for specified values of r ($n = 100$).

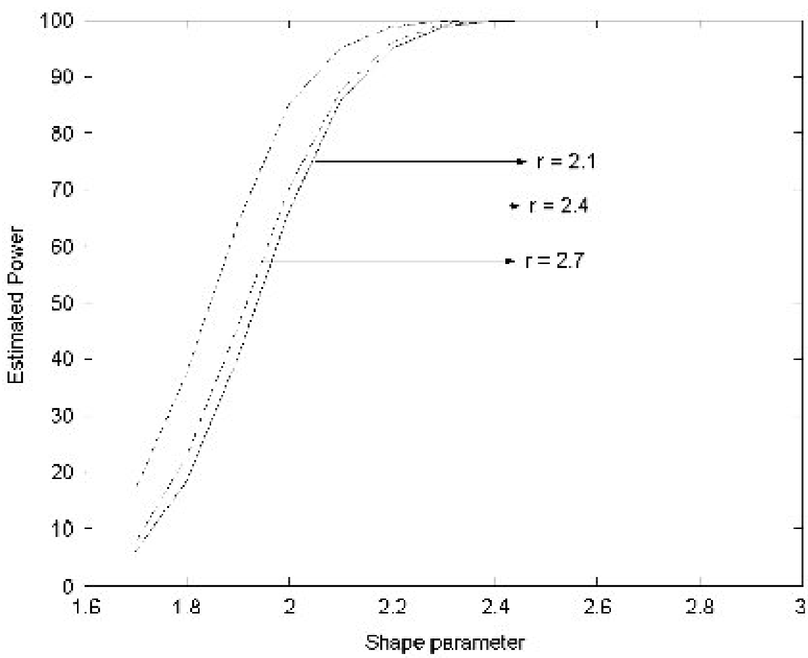

Figure 3. Plot of estimated power against shape parameter for Weibull distribution for specified values of r ($n = 100$).

ACKNOWLEDGEMENTS

The authors are grateful to an anonymous referee whose helpful comments substantially improved the presentation of the paper. The first author is indebted to his student Mr Chiranjit Mukherjee for his help in simulation.

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